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Approximate mixed additive and quadratic functional in 2-Banach spaces

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Abstract. In the paper we establish the general solution of the function equation $f(2x+y)+f(2x-y) = f(x+y)+f(x-y)+2f(2x)-2f(x)$ and investigate the Hyers-Ulam-Rassias stability of this equation in 2-Banach spaces.

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1. Introduction

In 1940, S. M. Ulam [8] gave a talk before the Mathematics club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

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for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \longrightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$

In 1941, D. H. Hyers [4] considered the case of approximately additive mappings $f : E \longrightarrow E^{'}$, where E and $E^{'}$ are Banach spaces and f satisfies Hyers inequality

$$\| f(x + y) - f(x) - f(y) \| \leq \varepsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \longrightarrow E^{'}$ is the unique additive mapping satisfying

$$\| f(x) - L(x) \| \leq \varepsilon.$$

In 1978, Th. M. Rassias [7] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.

In this paper, we deal with the next functional equation deriving from additive and quadratic functions:

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 2f(2x) - 2f(x) \quad (1.1)$$

It is easy to see that the function $f(x) = ax^2 + bx + c$ is a solution of the functional equation (1.1).

The main purpose of this paper is to establish the general solution of Eq. (1.1) and investigate the Hyers- Ulam- Rassias stability for Eq. (1.1).

We recall some basic facts concerning 2-Banach spaces and some preliminary results [2, 3].

Definition 1.1. Let X be a linear space over $\mathbb{R}$ with dim $X > 1$ and let $\| .,. \|: X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:

(1) $\| x, y \| = 0$ if and only if x and y are linearly dependent;
(2) $\| x, y \| = \| y, x \|$;
(3) $\| \alpha x, y \| = | \alpha | \| x, y \|$;
(4) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$

for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$. Then the function $\| .,. \|$ is called a 2-norm on X and the pair $(X, \| .,. \|)$ is called a linear 2-normed spaces. Sometimes the condition (4) called the triangle inequality.

We introduce a basic property of linear 2-normed spaces as follows.

Let $(X, \| .,. \|)$ be a linear 2-normed spaces, $x \in X$ and $\| x, y \| = 0$ for all $y \in X$. Suppose $x \neq 0$ and take y_1, y_2 linearly independent (so nonzero) in X. The condition (1) implies that x and y_1 are linearly dependent. Thus there exist $\alpha_1, \beta_1 \in \mathbb{R}$ such that $(\alpha_1, \beta_1) \neq (0,0)$ and $\alpha_1 x + \beta_1 y_1 = 0$, if $\beta_1 = 0$, we get $\alpha_1 \neq 0$. So we have $x = -\frac{\beta_1}{\alpha_1} y_1 = 0$, which is a contradiction. Thus we have $\beta_1 \neq 0$ and $y_1 = -\frac{\alpha_1}{\beta_1} x$. Similarly, there exist $\alpha_2, \beta_2 \in \mathbb{R}$ such that $\beta_2 \neq 0$ and $y_2 = -\frac{\alpha_2}{\beta_2} x$. Hence y_1 and y_2 are linearly dependent, which is a contradiction. Therefore we have the following lemma.

Lemma 1.2. *([6]) Let $(X, \| .,. \|)$ be a linear 2-normed space. If $x \in X$ and $\| x, y \| = 0$ for all $y \in X$, then $x = 0$.*

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a cauchy sequence if there are two points $y, z \in X$ such that y and z are linearly independent,

$$\lim_{m,n\to\infty} \| x_n - x_m, y \| = 0$$

and

$$\lim_{m,n\to\infty} \| x_n - x_m, z \| = 0.$$

Definition 1.4. A sequence $\{x_n\}$ in a linear 2-normed space X is called a convergent sequence if there is an $x \in X$ such that

$$\lim_{n\to\infty} \| x_n - x, y \| = 0$$

for all $y \in X$. If $\{x_n\}$ converges to x, write $x_n \longrightarrow x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

First we will quote some result by the authors in [5, 6], which will be applied later on.

Lemma 1.5. *If an even function $f : X \longrightarrow Y$ with $f(0) = 0$ satisfies (1.1) for all $x, y \in X$, X and Y will be real vector spaces, then f is quadratic.*

Lemma 1.6. *If an odd function $f : X \longrightarrow Y$ satisfies (1.1) for all $x, y \in X$, X and Y will be real vector spaces, then f is additive.*

Lemma 1.7. *For a convergent sequence $\{x_n\}$ in a linear 2-normed space X,*

$$\lim_{n\to\infty} \| x_n, y \| = \| \lim_{n\to\infty} x_n, y \|$$

for all $y \in X$.

Lemma 1.8. *Let $0 < p \leq 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} x_i^p.$$

Definition 1.9. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

In this paper, we investigate approximate mixed additive and quadratic function in 2-Banach spaces.

2. Main result

Throughout this paper, let X be a normed linear space and Y a 2-Banach space. In 1941, D. H. Hyers [4] obtained the first result on the stability of the Cauchy functional equation. In 1950, T. Aoki [1] generalized the Hyers result. It is the first result on the generalized Hyers-Ulam stability problem. In this section, we investigate the generalized Hyers-Ulam stability of the equation (1.1) in 2-Banach spaces.

Theorem 2.1. *Let* $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ *and* $p + q > 2$ *and let* $f : X \longrightarrow Y$ *with* $f(0) = 0$ *be a mapping satisfying*

$$\begin{aligned} \| Df(x,y), z \| &= \| f(2x+y) + f(2x-y) - f(x+y) - f(x-y) \\ &\quad -2f(2x) + 2f(x), z \| \\ &\leq \theta \| x \|^p \| y \|^q \| z \|^r \qquad (2.1) \end{aligned}$$

for all $x, y, z \in X$. *Then there is a unique quadratic mapping* $Q : X \longrightarrow Y$ *such that*

$$\| f(x) - Q(x), y \| \leq \frac{4+3^q}{(3^{p+q})(2^{p+q}-4)} \theta \| x \|^{p+q} \| y \|^r \qquad (2.2)$$

for all $x, y \in X$.

Proof. By replacing y by $x + y$ in (2.1), we get

$$\begin{aligned} &\| f(3x+y) + f(x-y) - f(2x+y) - f(y) - 2f(2x) + 2f(x), z \| \\ \leq\ &\theta \| x \|^{p+q} \| z \|^r + \theta \| x \|^p \| y \|^q \| z \|^r \qquad (2.3) \end{aligned}$$

for all $x, y, z \in X$. Replacing y by $-y$ in (2.3), we get

$$\begin{aligned} &\| f(3x-y) + f(x+y) - f(2x-y) - f(y) - 2f(2x) + 2f(x), z \| \\ \leq\ &\theta \| x \|^{p+q} \| z \|^r + \theta \| x \|^p \| y \|^q \| z \|^r \qquad (2.4) \end{aligned}$$

for all $x, y, z \in X$. It follows from (2.1), (2.3) and (2.4),

$$\begin{aligned} &\| f(3x+y) + f(3x-y) - 2f(y) - 6f(2x) + 6f(x), z \| \\ \leq\ &2\theta \| x \|^p \| y \|^q \| z \|^r + 2\theta \| x \|^{p+q} \| z \|^r \qquad (2.5) \end{aligned}$$

for all $x, y, z \in X$. By letting $y = 0$ and $y = 3x$ in (2.5), we get the inequalities

$$\| 2f(3x) - 6f(2x) + 6f(x), z \| \leq 2\theta \| x \|^{p+q} \| z \|^r, \qquad (2.6)$$

$$\| f(6x) - 2f(3x) - 6f(2x) + 6f(x), z \| \leq (2 + 3^q)\theta \| x \|^{p+q} \| z \|^r \qquad (2.7)$$

for all $x, z \in X$. It follows from (2.6) and (2.7),

$$\| f(6x) - 4f(3x), z \| \leq (4 + 3^q)\theta \| x \|^{p+q} \| z \|^r \tag{2.8}$$

for all $x, z \in X$. If we replace x by $\frac{x}{3}$ in (2.8), we get

$$\| f(2x) - 4f(x), z \| \leq \frac{4 + 3^q}{3^{p+q}} \theta \| x \|^{p+q} \| z \|^r \tag{2.9}$$

for all $x, z \in X$. If we replace x in (2.9) by $\frac{x}{2^{n+1}}$ and multiply both sides of (2.9) by 4^n, then we have

$$\| 4^{n+1} f(\frac{x}{2^{n+1}}) - 4^n f(\frac{x}{2^n}), z \| \leq \frac{(4 + 3^q)}{(3^{p+q})} \frac{2^{n(2-p-q)}}{2^{p+q}} \theta \| x \|^{p+q} \| z \|^r \tag{2.10}$$

for all $x, z \in X$ and all non-negative integers n. For all integer m and n with $n \geq m$, we get

$$\begin{aligned} & \| 4^{n+1} f(\frac{x}{2^{n+1}}) - 4^m f(\frac{x}{2^m}), z \| \\ \leq & \sum_{i=m}^{n} \frac{4 + 3^q}{(3^{p+q})(2^{p+q})} 2^{i(2-p-q)} \theta \| x \|^{p+q} \| z \|^r \end{aligned} \tag{2.11}$$

for all $x, z \in X$. So we get

$$\lim_{n,m\to\infty} \| 4^{n+1} f(\frac{x}{2^{n+1}}) - 4^m f(\frac{x}{2^m}), z \| = 0$$

for all $x, z \in X$. Thus the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is a 2-Banach space, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \longrightarrow Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. That is,

$$\lim_{n\to\infty} \| 4^n f(\frac{x}{2^n}) - Q(x), y \| = 0$$

for all $x, y \in X$. Now, we show that Q is quadratic.

By lemma (1.7) and (2.1), we get

$$\begin{aligned} \| DQ(x, y), z \| &= \| Q(2x + y) + Q(2x - y) - Q(x + y) - Q(x - y) \\ &\quad -2Q(2x) + 2Q(x), z \| \\ &= \lim_{n\to\infty} 4^n \| Df(\frac{x}{2^n}, \frac{y}{2^n}), z \| \\ &\leq \theta \| x \|^p \| y \|^q \| z \|^r \lim_{n\to\infty} 2^{n(2-p-q)} = 0 \end{aligned}$$

for all $x, y, z \in X$. By lemma (1.2),

$$Q(2x+y)+Q(2x-y)=Q(x+y)+Q(x-y)+2Q(2x)-2Q(x)$$

for all $x, y \in X$. Letting $m=0$ and passing the limit $n \to \infty$ in (2.11), we get

$$\begin{aligned} \| f(x)-Q(x), y \| &= \lim_{n\to\infty} \| f(x)-4^n f(\frac{x}{2^n}), y \| \\ &\leq \frac{4+3^q}{(3^{p+q})(2^{p+q}-4)} \theta \| x \|^{p+q} \| y \|^r \end{aligned}$$

for all $x, y \in X$.

Now, let $T : X \longrightarrow Y$ be another quadratic mapping satisfying (2.2). Then we have

$$\begin{aligned} \| Q(x)-T(x), y \| &= 4^n \| Q(\frac{x}{2^n})-T(\frac{x}{2^n}), y \| \\ &\leq 4^n [\| Q(\frac{x}{2^n})-f(\frac{x}{2^n}), y \| + \| f(\frac{x}{2^n})-T(\frac{x}{2^n}), y \|] \\ &\leq \frac{4+3^q}{(3^{p+q})(2^{p+q-1}-2)} 2^{n(2-p-q)} \theta \| x \|^{p+q} \| y \|^r \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in X$. By lemma (1.2), we can conclude that $Q(x)=T(x)$ for all $x \in X$. This proves the uniqueness of Q. □

Theorem 2.2. *Let $\theta \in [0,\infty)$, $p, q, r \in (0,\infty)$ and $p+q<2$ and let $f : X \longrightarrow Y$ with $f(0)=0$ be a mapping satisfying*

$$\begin{aligned} \| Df(x,y), z \| &= \| f(2x+y)+f(2x-y)-f(x+y)-f(x-y) \\ &\quad -2f(2x)+2f(x), z \| \\ &\leq \theta \| x \|^p \| y \|^q \| z \|^r \end{aligned} \tag{2.12}$$

for all $x, y, z \in X$. Then there is a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\| f(x)-Q(x), y \| \leq \frac{4+3^q}{(3^{p+q})(4-2^{p+q})} \theta \| x \|^{p+q} \| y \|^r$$

for all $x, y \in X$.

Proof. By the same argument as in the proof of Theorem 2.1, we get

$$\| f(2x)-4f(x), z \| \leq \frac{4+3^q}{3^{p+q}} \theta \| x \|^{p+q} \| z \|^r \tag{2.13}$$

for all $x, z \in X$. Replacing x by $2^n x$ and dividing 4^{n+1} in (2.13), we obtain

$$\| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x), z \| \leq \frac{4+3^q}{4(3^{p+q})} 2^{n(p+q-2)} \theta \| x \|^{p+q} \| z \|^r$$

for all $x, z \in X$ and all integer $n > 0$. For all integer m and n with $n \geq m$, we get

$$\| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^m} f(2^m x), z \| \leq \sum_{i=m}^{n} \frac{4+3^q}{4(3^{p+q})} 2^{i(p+q-2)} \theta \| x \|^{p+q} \| z \|^r$$

for all $x, z \in X$. So we get

$$\lim_{n,m \to \infty} \| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^m} f(2^m x), z \| = 0$$

for all $x, z \in X$. Thus the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence in Y . Since Y is a 2-Banach space, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. That is,

$$\lim_{n \to \infty} \| \frac{1}{4^n} f(2^n x) - Q(x), y \| = 0$$

for all $x, y \in X$.

The further part of the proof is similar to the proof of Theorem 2.1. □

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