Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **4**(1)(2015), 107-124

Studies on Sturm-Liouville boundary value problems for multi-term fractional differential equations

Xiaohui Yang [1], Shengpign Chen [2], Yuji Liu [3] and Xingyuan Liu [4]
[1] Department of Mathematics, Guangdong Police College, China
[2], [3] Guangdong University of Business Studies, China
[4] Department of Mathematics, Shaoyang University, China

Abstract. The Sturm-Liouville boundary value problem of the multi-order fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}[p(t)D_{0+}^{\beta}u(t)] + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ a\lim_{t\to 0} t^{1-\beta}u(t) - b\ \lim_{t\to 0}\ t^{1-\alpha}p(t)D_{0+}^{\beta}u(t) = 0, \\ c\ \lim_{t\to 1} u(t) + d\lim_{t\to 1} p(t)D_{0+}^{\beta}u(t) = 0 \end{cases}$$

is studied. Results on the existence of solutions are established. The analysis relies on a weighted function space and a fixed point theorem. An example is given to illustrate the efficiency of the main theorems.

Keywords: multi-order fractional differential equation, Sturm-Liouville boundary value problems, fixed-point theorem.

2000 Mathematics subject classification: xxxx, xxxx; Secondary xxxx.

1. Introduction

In recent years, many authors have studied the existence of solutions of boundary value problems for fractional differential equations with Riemann-Liouville fractional derivative or Caputo's fractional derivative, refer to Refs [1-16].

[1] Corresponding author: xiaohui_yang@sohu.com
Received: 18 February 2013
Revised: 09 December 2013
Accepted: 09 February 2015

In [14,15], the authors considered the existence and multiplicity of positive solutions of the following boundary value problem of nonlinear fractional differential equation

$$\begin{cases} \mathbf{D}_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\gamma}u(t)), \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \\ u(1) + u'(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, $\gamma \in (0, \alpha - 1$, and $\mathbf{D}_{0+}^{*}$ is the **Caputo's fractional derivative** of order $*$, and $f : [0, 1] \times [0, +\infty) \times R \to [0, +\infty)$ is continuous. By means of a fixed-point theorem on cones, existence and multiplicity results of positive solutions of (1.1) were obtained.

In [16], the authors studied the existence of solutions of the following more generalized boundary value problem of fractional differential equation

$$\begin{cases} \mathbf{D}_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in [0, T], \\ u(0) - u'(0) = \int_0^T g(s, y)ds, \\ u(T) + u'(T) = \int_0^T h(s, y)ds, \end{cases} \tag{1.2}$$

where $T > 0$, $1 < \alpha \leq 2$ is a real number, and $\mathbf{D}_{0+}^{\alpha}$ is the **Caputo's fractional derivative**, and $f, g, h : [0, T] \times R \to R$ is continuous.

In [17], authors studied the solvability of the following two-point boundary value problem for fractional $p-$Laplace differential equation

$$\begin{cases} \mathbf{D}_{0+}^{\beta}[\phi_p(\mathbf{D}_{0+}^{\alpha}u(t))] = f(t, u(t), \mathbf{D}_{0+}^{\alpha}u(t)), \quad t \in [0, 1], \\ u(0) = 0, \\ {}^cD_{0+}^{\alpha}u(1) = {}^c D_{0+}^{\alpha}u(0), \end{cases} \tag{1.3}$$

where $\mathbf{D}_{0+}^{*}$ denotes the Caputo fractional derivatives of order $*$, $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $\phi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator, $f : [0, 1] \times R^2 \to R$ is continuous. By using the coincidence degree theory, the existence of solutions for above fractional boundary value problem was obtained.

It is well known that properties of fractional differential equations with Riemann-Liouville fractional derivatives are different from those of fractional differential equations with Caputo's fractional derivatives [4]. To compare with [14-16], it is meaningful to define and study Sturm-Liouville boundary value problems for fractional differential equations with Riemmann-Liouville fractional derivatives.

In this paper, we discuss the existence of solutions of the following boundary value problem (BVP for short) for the nonlinear fractional differential equation with multi-term Riemmann-Liouville fractional derivatives

$$\begin{cases} D_{0+}^{\alpha}[p(t)D_{0+}^{\beta}u(t)] + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ a \lim_{t\to 0} t^{1-\beta}u(t) - b \lim_{t\to 0} t^{1-\alpha}p(t)D_{0+}^{\beta}u(t) = 0, \\ c \lim_{t\to 1} u(t) + d \lim_{t\to 1} p(t)D_{0+}^{\beta}u(t) = 0 \end{cases} \tag{1.4}$$

where $a,b,c,d \geq 0$, D_{0+}^{*} is the Riemann-Liouville fractional derivative of order $*$, $0 < \beta, \alpha \leq 1$ with $\alpha + \beta > 1$, $f : [0,1] \times R \to R$ satisfies the assumption (H1) (see Section 2), $p : (0,1) \to (0,\infty)$ satisfies the assumption (H2) (see Section 2), $q : (0,1) \to [0,\infty)$ satisfies the assumption (H3) (see Section 2).

We obtain the Green's function of BVP(1.4) and establish the existence results of solutions of BVP(1.4). An example is given to illustrate the efficiency of the main theorem.

We clarify the structure of sequential fractional differential equations. It is the lack of commutativity of the fractional derivatives that represents an interesting complication that does not arise in the integer-order setting. In the problem (1.4), we have a composition of two fractional derivatives, which gives rise to a sequential problem. Problem (1.4) is a natural generalized form of ordinary differential equation $[p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0$. In problem (1.4), we allow $\gamma \in (\beta, \alpha + \beta)$.

The remainder of this paper is as follows: in section 2, we present preliminary results. In section 3, the main theorems and their proof are given. In section 4, an example is given to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory that can be found in the literatures [4,7]. Denote the Gamma and Beta functions by

$$\Gamma(\alpha_1) = \int_0^{+\infty} s^{\alpha_1 - 1} e^{-s} ds, \quad \mathbf{B}(\alpha_2, \beta_2) = \int_0^1 (1-x)^{\alpha_2 - 1} x^{\beta_2 - 1} dx, \alpha_1 > 0, \alpha_2 > 0, \beta_2 > 0.$$

Definition 2.1[4]. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds,$$

provided that the right-hand side exists.

Definition 2.2[4]. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha \le n$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.1[4]. Let $n-1 < \alpha \le n$, $u \in C^0(0,\infty) \bigcap L^1(0,\infty)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \dots n$, and for $\alpha \ge 0, \mu > -1$, it holds that

$$I_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad D_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

Suppose that

(H1) $f : [0,1] \times R \to R$ satisfies that

(i) $u \to f(t, t^{\beta-1}u)$ is continuous for all $t \in [0,1]$,

(ii) $t \to f(t, t^{\beta-1}u)$ is measurable on $[0,1]$ for all $(u,v) \in R^2$ and

(iii) for each $r > 0$ there exists $M_r \ge 0$ such that

$$\left| f\left(t, t^{\beta-1}u\right)\right| \le M_r \text{ for all } t \in [0,1], \ |u|, |v| \le r.$$

(H2) $p : (0,1) \to (0,\infty)$ is continuous and there exists a number $M_0 > 0$ such that $p(t) \le \frac{M_0}{t^{1-\beta}(1-t)^{\alpha+\beta-1}}$ for all $t \in (0,1)$.

(H3) $q : (0,1) \to [0,\infty)$ is continuous and there exist $k > -1$ and $l > -\alpha$ with $k + l + 1 > 0$ such that $q(t) \le t^k(1-t)^l$ for all $t \in (0,1)$ (q may be singular at $t = 0$ and $t = 1$).

(H4) m is a positive integer, there exist nonnegative functions $\psi_0, \phi \in L^0(0,1)$, $\mu > 0$ such that

$$\left| q(t) f\left(t, t^{\beta-1}u\right) - \psi_0 \right| \le \psi(t)|u|^{\mu}, t \in (0,1), u \in R.$$

For our construction, we let

$$X = \left\{ x : (0,1] \to R \ \begin{array}{c} x \in C(0,1], \\ \text{the following limit exists} \\ \lim_{t\to 0} t^{1-\beta} x(t) \end{array} \right\}.$$

For $x \in X$, let

$$\|x\| = \sup_{t \in (0,1)} t^{1-\beta} |x(t)|.$$

Then X is a Banach space.

Denote

$$\sigma(s,t) = \int_s^t \frac{(t-w)^{\beta-1}(w-s)^{\alpha-1}}{p(w)} dw, \quad \Delta = (bc + ad)\Gamma(\beta) + ac\sigma(0,1).$$

Lemma 2.2. Suppose that $\Delta \neq 0$, $x \in X$ and (H1)-(H3) hold. Then u is a solution of

$$\begin{cases} D_{0^+}^{\alpha}[p(t)D_{0^+}^{\beta}u(t)] + q(t)f(t,x(t)) = 0, 0 < t < 1, \\ a\lim_{t\to 0} t^{1-\beta}u(t) - b\ \lim_{t\to 0}\ t^{1-\alpha}p(t)D_{0^+}^{\beta}u(t) = 0, \\ c\ \lim_{t\to 1} u(t) + d\lim_{t\to 1} p(t)D_{0^+}^{\beta}u(t) = 0, \end{cases} \tag{2.1}$$

if and only if

$$u(t) = \int_0^1 G(t,s)q(s)f(t,x(s))ds, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\begin{cases} \begin{matrix} -\delta\sigma(s,t) \\ +ad\Gamma(\beta)\sigma(0,t)(1-s)^{\alpha-1} + ac\sigma(0,t)\sigma(s,1) \\ +bd\Gamma(\beta)^2 t^{\beta-1}(1-s)^{\alpha-1} + bc\Gamma(\beta)t^{\beta-1}\sigma(s,1), \end{matrix} & s \leq t, \\ \begin{matrix} ad\Gamma(\beta)\sigma(0,t)(1-s)^{\alpha-1} + ac\sigma(0,t)\sigma(s,1) \\ +bd\Gamma(\beta)^2 t^{\beta-1}(1-s)^{\alpha-1} + bc\Gamma(\beta)t^{\beta-1}\sigma(s,1), \end{matrix} & t \leq s. \end{cases} \tag{2.3}$$

Proof. Since $x \in X$, we get $r = \|x\| < +\infty$. Then (H1) implies that there exists a nonnegative number M_r such that

$$|f(t,x(t))| = \left| f\left(t, t^{\beta-1}[t^{1-\beta}x(t)]\right)\right| \leq M_r \text{ for all } t \in (0,1).$$

If u is a solution of BVP(2.1), then we get

$$p(t)D_{0^+}^{\beta}u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)f(s,x(s))ds + c_1t^{\alpha-1}, t \in (0,1)$$

for some $c_1 \in R$. Then

$$D_{0^+}^{\beta}u(t) = -\tfrac{1}{p(t)}\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s))ds + c_1\tfrac{t^{\alpha-1}}{p(t)},$$

$$u(t) = \int_0^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-\tfrac{1}{p(s)}\int_0^s \tfrac{(s-w^{\alpha-1}}{\Gamma(\alpha)}q(w)f(w,x(w))dw + \tfrac{c_1s^{\alpha-1}}{p(s)}\right]ds + c_2t^{\beta-1},$$

$$t^{1-\alpha}p(t)D_{0^+}^{\beta}u(t) = -t^{1-\alpha}\int_0^t \tfrac{(t-s)^{\alpha-1}}{\gamma(\alpha)}q(s)f(s,x(s))ds + c_1.$$

Since

$$t^{1-\alpha}\left|\int_0^t (t-s)^{\alpha-1}q(s)f(s,x(s))ds\right| \leq t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^k(1-s))^l M_r ds$$

$$\leq t^{1-\alpha}\int_0^t (t-s)^{\alpha+l-1}s^k M_r ds = t^{1+k+l}M_r\int_0^1 (1-w)^{\alpha+l-1}w^k dw \to 0, t \to 0,$$

and

$$t^{1-\beta}\left|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-\frac{1}{p(s)}\int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}q(w)f(w,x(w))du+c_1\frac{s^{\alpha-1}}{p(s)}\right]ds\right|$$

$$\leq t^{1-\beta}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\frac{M_0}{s^{1-\beta}(1-s)^{\alpha+\beta-1}}\int_0^s(s-w)^{\alpha-1}w^k(1-w)^l M_r dwds$$

$$+c_1t^{1-\beta}\int_0^t(t-s)^{\beta-1}\frac{M_0s^{\alpha-1}}{s^{1-\beta}(1-s)^{\alpha+\beta-1}}ds$$

$$\leq M_rM_0t^{1-\beta}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}(t-s)^{1-\alpha-\beta}\int_0^s(s-w)^{\alpha+l-1}w^k dwds$$

$$+c_1M_0t^{1-\beta}\int_0^t(t-s)^{\beta-1}s^{\alpha+\beta-2}(t-s)^{1-\alpha-\beta}ds$$

$$\leq M_rM_0t^{1-\beta}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{-\alpha}s^{\alpha+\beta+k+l-1}ds\mathbf{B}(\alpha+l,k+1)$$

$$+c_1M_0t^{1-\beta}\int_0^t(t-s)^{-\alpha}s^{\alpha+\beta-2}ds$$

$$=M_rM_0t^{k+l+1}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\mathbf{B}(1-\alpha,\alpha+\beta+k+l)\mathbf{B}(\alpha+l,k+1)$$

$$+c_1M_0t\mathbf{B}(1-\alpha,\alpha+\beta-1)\to 0, t\to 0.$$

From the boundary conditions in (2.1), we get

$$ac_2-bc_1=0,$$

$$c_1\left[-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))dwds\right.$$

$$\left.+\frac{c_1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds+c_2\right]$$

$$+d\left[-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s))ds+c_1\right]=0.$$

It follows that

$$c_1 = \frac{\frac{ad}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds}{bc+ad+\frac{ac}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}$$

$$+\frac{\frac{ac}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))dwds}{bc+ad+\frac{ac}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds},$$

$$c_2 = \frac{\frac{bd}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s),D_{0+}^{\gamma}x(s))ds}{bc+ad+\frac{ac}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}$$

$$+\frac{\frac{bc}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))duds}{bc+ad+\frac{ac}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}.$$

Therefore,

$$u(t) = -\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))dwds$$

$$+\int_0^t(t-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\frac{ad\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}\right.$$

$$\left.+\frac{ac\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))dwds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}\right]$$

$$+\frac{t^{\beta-1}}{\Gamma(\alpha)}\left[\frac{bd\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s),D_{0+}^{\gamma}x(s))ds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}\right.$$

$$\left.+\frac{bc\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))duds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}\right].$$

Since

$$\int_0^t(t-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-w)^{\alpha-1}q(w)f(w,x(w))dwds$$

$$=\int_0^t\int_u^t(t-s)^{\beta-1}(s-w)^{\alpha-1}\frac{1}{p(s)}dsq(w)f(w,x(w))dw$$

$$=\int_0^t\int_s^t\frac{(t-w)^{\beta-1}(w-s)^{\alpha-1}}{p(w)}dwq(s)f(s,x(s))ds$$

$$=\int_0^t\sigma(s,t)q(s)f(s,x(s))ds,$$

we get

$$u(t) = -\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \sigma(s,t)q(s)f(s,x(s))ds$$

$$+\frac{\sigma(0,t)}{\Gamma(\alpha)\Gamma(\beta)}\frac{ad\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds+ac\int_0^1\sigma(s,1)q(s)f(s,x(s))ds}{\delta}$$

$$+\frac{t^{\beta-1}}{\Gamma(\alpha)}\frac{bd\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds+bc\int_0^1\sigma(s,1)q(s)f(s,x(s))ds}{\delta}$$

$$= \frac{1}{\delta\Gamma(\alpha)\Gamma(\beta)}\int_0^t\left[-\delta\sigma(s,t) + ad\Gamma(\beta)\sigma(0,t)(1-s)^{\alpha-1} + ac\sigma(0,t)\sigma(s,1)\right.$$

$$\left.+bd\Gamma(\beta)^2t^{\beta-1}(1-s)^{\alpha-1} + bc\Gamma(\beta)t^{\beta-1}\sigma(s,1)\right]q(s)f(s,x(s))ds$$

$$+\frac{1}{\delta\Gamma(\alpha)\Gamma(\beta)}\int_t^1\left[ad\Gamma(\beta)\sigma(0,t)(1-s)^{\alpha-1} + ac\sigma(0,t)\sigma(s,1)\right.$$

$$\left.+bd\Gamma(\beta)^2t^{\beta-1}(1-s)^{\alpha-1} + bc\Gamma(\beta)t^{\beta-1}\sigma(s,1)\right]q(s)f(s,x(s))ds$$

$$= \int_0^1 G(t,s)q(s)f(s,x(s))ds.$$

Then u satisfies (2.2). Here G is defined by (2.3). It is easy to prove that $u \in X$.

Reciprocally, let u satisfy (2.2). It is easy to show that $u \in X$, and

$$a\lim_{t\to 0}t^{1-\beta}u(t) - b\lim_{t\to 0}t^{1-\alpha}p(t)D_{0^+}^{\beta}u(t) = 0,\ c\lim_{t\to 1}u(t) + d\lim_{t\to 1}p(t)D_{0^+}^{\beta}u(t) = 0.$$

Furthermore, we have $D_{0^+}^{\alpha}[p(t)D_{0^+}^{\beta}u(t)] + q(t)f(t,x(t)) = 0$. Then $u \in X$ is a solution of BVP(2.1). The proof is complete.

Lemma 2.3. Suppose that $\Delta > 0$, and (H2) holds. Then

$$t^{1-\beta}|G(s,t)| \le L(1-s)^{\alpha-1},\ s,t \in (0,1), \tag{2.4}$$

where

$$L = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Delta}\left[\Delta M_0\mathbf{B}(1-\alpha,\alpha+\beta-1) + ad\Gamma(\beta)M_0\mathbf{B}(1-\alpha,\alpha+\beta-1)\right.$$

$$\left.+acM_0^2\mathbf{B}(1-\alpha,\alpha+\beta-1)^2 + bd\Gamma(\beta)^2 + bc\Gamma(\beta)M_0\mathbf{B}(1-\alpha,\alpha+\beta-1)\right].$$

Proof. From (2.3), one sees for $0 < s \leq t < 1$ that

$$\begin{aligned}
t^{1-\beta}\sigma(s,t) &= t^{1-\beta}\int_s^t \frac{(t-u)^{\beta-1}(u-s)^{\alpha-1}}{p(u)}du \\
&= t^{1-\beta}\int_0^{t-s} \frac{(t-s-u)^{\beta-1}v^{\alpha-1}}{p(s+v)}dv \\
&= t^{1-\beta}(t-s)^{\alpha+\beta-1}\int_0^1 \frac{(1-w)^{\beta-1}w^{\alpha-1}}{p(s+w(t-s))}dw \\
&\leq t^{1-\beta}(t-s)^{\alpha+\beta-1}\int_0^1 \frac{M_0(1-w)^{\beta-1}w^{\alpha-1}}{[s+w(t-s)]^{1-\beta}[1-s-w(t-s)]^{\alpha+\beta-1}}dw \\
&\leq t^{1-\beta}(t-s)^{\alpha+\beta-1}\int_0^1 \frac{M_0(1-w)^{\beta-1}w^{\alpha-1}}{[wt]^{1-\beta}[(1-w)(t-s)]^{\alpha+\beta-1}}dw \\
&= M_0\int_0^1 (1-w)^{-\alpha}w^{\alpha+\beta-2}dw \\
&= M_0\mathbf{B}(1-\alpha,\alpha+\beta-1).
\end{aligned}$$

Hence, for $0 < s \leq t < 1$, we get

$$\begin{aligned}
& t^{1-\beta}|G(s,t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Delta}\Big[\Delta t^{1-\beta}|\sigma(s,t)| + ad\Gamma(\beta)t^{1-\beta}|\sigma(0,t)|(1-s)^{\alpha-1} \\
& +act^{1-\beta}|\sigma(0,t)||\sigma(s,1)| + bd\Gamma(\beta)^2(1-s)^{\alpha-1} + bc\Gamma(\beta)\sigma(s,1)\Big] \\
\leq\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Delta}\big[\Delta M_0\mathbf{B}(1-\alpha,\alpha+\beta-1) + ad\Gamma(\beta)M_0\mathbf{B}(1-\alpha,\alpha+\beta-1)(1-s)^{\alpha-1} \\
& +acM_0^2\mathbf{B}(1-\alpha,\alpha+\beta-1)^2 + bd\Gamma(\beta)^2(1-s)^{\alpha} + bc\Gamma(\beta)M_0\mathbf{B}(1-\alpha,\alpha+\beta-1)\big] \\
\leq\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Delta}\big[\Delta M_0\mathbf{B}(1-\alpha,\alpha+\beta-1) + ad\Gamma(\beta)M_0\mathbf{B}(1-\alpha,\alpha+\beta-1) \\
& +acM_0^2\mathbf{B}(1-\alpha,\alpha+\beta-1)^2 + bd\Gamma(\beta)^2 + bc\Gamma(\beta)M_0\mathbf{B}(1-\alpha,\alpha+\beta-1)\big](1-s)^{\alpha-1} \\
=\ & L(1-s)^{\alpha-1}.
\end{aligned}$$

For $0 < t \leq s < 1$, we can prove similarly that

$$t^{1-\beta}|G(s,t)| \leq L(1-s)^{\alpha-1}.$$

The proof of (2.4) is completed.

Now, we define the operator T on X, by

$$(Tx)(t) = \int_0^1 G(t,s)q(s)f(s,x(s))ds\,.$$

Lemma 2.4. Suppose that $\Delta > 0$, **(H1)-(H3)** hold. Then $T : X \to X$ is completely continuous.

Proof. To complete the proof, we must prove that $T : X \to X$ is well defined, T is continuous and T maps any bounded subsets of X to relative compact sets of X [15]. We divide the proof into four steps.

Step 1. We prove that $T : X \to X$ is well defined.

For $x \in X$, we have

$$\sup_{t\in(0,1)} t^{1-\beta}|x(t)| = r < \infty.$$

Hence from (H1) there exists a nonnegative function $\phi_r \in L^1(0,1)$ such that

$$|q(s)f(s,x(s))| = \left|q(t)f\left(t, t^{\beta-1}t^{1-\beta}x(t)\right)\right| \leq \phi_r(t), t \in (0,1).$$

From

$$(Tx)(t) = -\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-u)^{\alpha-1}q(u)f(u,x(u))duds$$

$$+\int_0^t(t-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\frac{ad\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}\right.$$

$$\left.+\frac{ac\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-u)^{\alpha-1}q(u)f(u,x(u))duds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}\right]$$

$$+\frac{t^{\beta-1}}{\Gamma(\alpha)}\frac{bd\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds}$$

$$+\frac{t^{\beta-1}}{\Gamma(\alpha)}\frac{bc\int_0^1(1-s)^{\beta-1}\frac{1}{p(s)}\int_0^s(s-u)^{\alpha-1}q(u)f(u,x(u))duds}{(bc+ad)\Gamma(\beta)+ac\int_0^1(1-s)^{\beta-1}\frac{s^{\alpha-1}}{p(s)}ds},$$

and (H2), we find that (Tu) is continuous on $(0,1]$ and there exists the limit

$$\lim_{t\to 0} t^{1-\beta}(Tx)(t).$$

Hence $Tx \in X$. Then $T : X \to X$ is well defined.

Step 2. We prove that T is continuous.

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence such that $y_n \to y_0$ in X. Then

$$r = \sup_{n=0,1,2,\cdots} ||y_n|| < \infty.$$

So there exists a nonnegative $\alpha-$well function ϕ_r such that

$$|q(t)f(t,y_n(t))| \leq \phi_r(t)$$

holds for $t \in (0,1), n = 0,1,2,\cdots$. Then for $t \in (0,1)$, we have from Lemma 2.3 that

$$t^{1-\beta}|(Ty_n)(t) - (Ty_0)(t)|$$

$$= \left|\int_0^1 t^{1-\beta}G(s,t)q(s)f(s,y_n(s))ds - \int_0^1 t^{1-\beta}G(s,t)q(s)f(s,y_0(s))ds\right|$$

$$\leq \int_0^1 t^{1-\beta}G(s,t)|q(s)f(s,y_n(s),D_{0+}^{\beta}y_n(t)) - q(s)f(s,y_0(s))|ds$$

$$\leq L\int_0^1 (1-s)^{\alpha-1}|q(s)f(s,y_n(s)) - f(s,y_0(s))|ds$$

$$\leq 2L\int_0^1 (1-s)^{\alpha-1}\phi_r(s)ds.$$

Since f is continuous, by the Lebesgue dominated convergence theorem, we get $||Ty_n - Ty_0|| \to 0$ as $n \to \infty$. Then T is continuous.

Step 3. Let $M = \{y \in X : ||y|| \leq r\}$. We prove that TM is bounded.

It suffices to show that there exists a positive number $L > 0$ such that for each $x \in M = \{y \in X : ||y|| \leq r\}$, we have $||Ty|| \leq L$.

By the assumption, there exists a nonnegative $\alpha-$well function $\phi_r \in L^1(0,1)$ such that

$$|f(t,y(t))| \leq \phi_r(t), t \in (0,1).$$

By the definition of T, for $y \in M$, we get

$$\begin{aligned} t^{1-\beta}|(Ty)(t)| &= \left|\int_0^1 t^{1-\beta}G(t,s)q(s)f(s,y(s))ds\right| \\ &\leq \int_0^1 L(1-s)^{\alpha-1}|q(s)f(s,y(s))|ds \\ &\leq L\int_0^1 (1-s)^{\alpha-1}\phi_r(s)ds. \end{aligned}$$

It follows that there exists $L > 0$ such that $||Ty|| \leq L$ for each $y \in \{y \in X : ||y|| \leq r\}$. Then T maps bounded sets into bounded sets in X.

Step 4. Let $M = \{y \in X : ||y|| \leq r\}$. Prove that $t^{1-\beta}TM$ is equicontinuous on $[0,1]$.

Let $t_1, t_2 \in (0,1]$ with $t_1 < t_2$ and $x \in M = \{y \in X : ||y|| \leq r\}$. By the assumption, there exists a $\alpha-$well function $\phi_r \in L^1(0,1)$ such that

$$|q(t)f(t,x(t))| \leq \phi_r(t), t \in (0,1).$$

Note

$$(Tx)(t) = -\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \sigma(s,t)q(s)f(s,x(s))ds$$

$$+\frac{\sigma(0,t)}{\Gamma(\alpha)\Gamma(\beta)}\frac{ad\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds+ac\int_0^1\sigma(s,1)q(s)f(s,x(s))ds}{\delta}$$

$$+\frac{t^{\beta-1}}{\Gamma(\alpha)}\frac{bd\Gamma(\beta)\int_0^1(1-s)^{\alpha-1}q(s)f(s,x(s))ds+bc\int_0^1\sigma(s,1)q(s)f(s,x(s))ds}{\delta}.$$

For $s \in (0, t_1]$, one sees from (7) that

$$|t_2^{1-\beta}\sigma(s,t_2) - t_1^{1-\beta}\sigma(s,t_1)|$$

$$= \left|t_2^{1-\beta}\int_s^{t_2}\frac{(t_2-u)^{\beta-1}(u-s)^{\alpha-1}}{p(u)}du - t_1^{1-\beta}\int_s^{t_1}\frac{(t_1-u)^{\beta-1}(u-s)^{\alpha-1}}{p(u)}du\right|$$

$$= \left|\int_s^{t_2}\frac{[t_2^{1-\beta}(t_2-u)^{\beta-1}-t_1^{1-\beta}(t_1-u)^{\beta-1}](u-s)^{\alpha-1}}{p(u)}du\right|$$

$$+\left|t_1^{1-\beta}\int_{t_1}^{t_2}\frac{(t_1-u)^{\beta-1}(u-s)^{\alpha-1}}{p(u)}du\right|$$

$$= \left|\int_0^{t_2-s}\frac{[t_2^{1-\beta}(t_2-s-v)^{\beta-1}-t_1^{1-\beta}(t_1-s-v)^{\beta-1}]v^{\alpha-1}}{p(s+v)}dv\right|$$

$$+\left|t_1^{1-\beta}\int_{t_1-s}^{t_2-s}\frac{(t_1-s-v)^{\beta-1}v^{\alpha-1}}{p(v+s)}dv\right|$$

$$= \int_0^1\frac{[t_2^{1-\beta}(t_2-s)^{\alpha+\beta-1}(1-w)^{\beta-1}-t_1^{1-\beta}(t_1-s-w(t_2-s))^{\beta-1}(t_2-s)^{\alpha}]w^{\alpha-1}}{p(s+w(t_2-s))}dw$$

$$+t_1^{1-\beta}(t_1-s)^{\alpha+\beta-1}\int_1^{\frac{t_2-s}{t_1-s}}\frac{(1-w)^{\beta-1}w^{\alpha-1}}{p(w(t_1-s)+s)}dw$$

$$\le M_0\int_0^1\frac{[t_2^{1-\beta}(t_2-s)^{\alpha+\beta-1}(1-w)^{\beta-1}-t_1^{1-\beta}(t_1-s-w(t_2-s))^{\beta-1}(t_2-s)^{\alpha}]w^{\alpha-1}}{t_2^{1-\beta}(t_2-s)^{\alpha+\beta-1}}dw$$

$$+M_0t_1^{1-\beta}(t_1-s)^{\alpha+\beta-1}\int_1^{\frac{t_2-s}{t_1-s}}\frac{(1-w)^{\beta-1}w^{\alpha-1}}{t_1^{1-\beta}(t_1-s)^{\alpha+\beta-1}}dw$$

$$\le M_0\int_0^1\left|1-\frac{t_1^{1-\beta}}{t_2^{1-\beta}}\left(\frac{\frac{t_1-s}{t_2-s}-w}{1-w}\right)^{\beta-1}\right|(1-w)^{\beta-1}w^{\alpha-1}dw$$

$$+M_0\int_1^{\frac{t_2-s}{t_1-s}}(1-w)^{\beta-1}w^{\alpha-1}dw$$

$$\to 0 \text{ uniformly in } (0,t_1] \text{ as } t_2\to t_1.$$

We consider three cases:

Case 1. for $s\in(0,t_1]$, from (2.3) that

$$t_1^{1-\beta}G(s,t_1)-t_2^{1-\beta}G(s,t_2)$$

$$=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\Big[\delta[t_2^{1-\beta}\sigma(s,t_2)-t_1^{1-\beta}\sigma(s,t_1)]$$

$$+ad\Gamma(\beta)(1-s)^{\alpha-1}[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]$$

$$+ac\sigma(s,1)[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]\Big]$$

Case 2. For $s\in[t_1,t_2]$, we have

$$t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\Big[\delta t_2^{1-\beta}\sigma(s,t_2)$$

$$+ad\Gamma(\beta)(1-s)^{\alpha-1}[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]$$

$$+ac\sigma(s,1)[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]\Big]$$

Case 3. For $s\in[t_2,1]$, we have

$$t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\Big[ad\Gamma(\beta)(1-s)^{\alpha-1}[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]$$

$$+ac\sigma(s,1)[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]\Big]$$

Then from $t^{1-\beta}G(t,s)\le L(1-s)^{\alpha-1}$, we get

$$|t_1^{1-\beta}(Ty)(t_1)-t_2^{1-\beta}(Ty)(t_2)|$$

$$=\Big|\int_0^{t_1}[t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)]q(s)f(s,y(s))ds$$

$$+\int_{t_1}^{t_2}[t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)]q(s)f(s,y(s))ds$$

$$+\int_{t_2}^{1}[t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)]q(s)f(s,y(s))ds\Big|$$

$$\le\int_0^{t_1}|t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|q(s)|f(s,y(s))|ds$$

$$+\int_{t_1}^{t_2}|t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|q(s)|f(s,y(s))|ds$$

$$+\int_{t_2}^{1}|t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|q(s)|f(s,y(s))|ds$$

$$\leq \int_0^{t_1} |t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|\phi_r(s)ds$$

$$+\int_{t_1}^{t_2} |t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|\phi_r(s)ds$$

$$+\int_{t_2}^{1} |t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|\phi_r(s)ds$$

$$\leq \int_0^{t_1} |t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|\phi_r(s)ds$$

$$+2L\int_{t_1}^{t_2}(1-s)^{\alpha-1}\phi_r(s)ds$$

$$+\int_{t_2}^{1} |t_1^{1-\beta}G(t_1,s)-t_2^{1-\beta}G(t_2,s)|\phi_r(s)ds$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\int_0^{t_1}\Big[\delta[t_2^{1-\beta}\sigma(s,t_2)-t_1^{1-\beta}\sigma(s,t_1)]$$

$$+ad\Gamma(\beta)(1-s)^{\alpha-1}[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]$$

$$+ac\sigma(s,1)[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]\Big]\phi_r(s)ds$$

$$+2L\int_{t_1}^{t_2}(1-s)^{\alpha-1}\phi_r(s)ds$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\int_{t_2}^{1}\Big[ad\Gamma(\beta)(1-s)^{\alpha-1}[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]$$

$$+ac\sigma(s,1)[t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)]\Big]\phi_r(s)ds$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\int_0^{t_1}\Big[\delta|t_2^{1-\beta}\sigma(s,t_2)-t_1^{1-\beta}\sigma(s,t_1)|$$

$$+ad\Gamma(\beta)|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|$$

$$+acM_0|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|\Big](1-s)^{\alpha-1}\phi_r(s)ds$$

$$+2L\int_{t_1}^{t_2}(1-s)^{\alpha-1}\phi_r(s)ds$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\int_{t_2}^{1}\Big[ad\Gamma(\beta)|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|$$

$$+acM_0|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|\Big](1-s)^{\alpha-1}\phi_r(s)ds$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\Big[\delta\,|t_2^{1-\beta}\sigma(s,t_2)-t_1^{1-\beta}\sigma(s,t_1)|\Big|_{t_1,t_2\in[s,1]}$$

$$+ad\Gamma(\beta)|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|$$

$$+acM_0|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|\Big]\int_0^1(1-s)^{\alpha-1}\phi_r(s)ds$$

$$+2L\int_{t_1}^{t_2}(1-s)^{\alpha-1}\phi_r(s)ds$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)\delta}\Big[ad\Gamma(\beta)|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|$$

$$+acM_0|t_1^{1-\beta}\sigma(0,t_1)-t_2^{1-\beta}\sigma(0,t_2)|\Big]\int_0^1(1-s)^{\alpha-1}\phi_r(s)ds$$

Therefore, $t^{1-\beta}TM$ is equicontinuous on $[0,1]$.

From Steps 3 and 4, the Arzela-Ascoli theorem implies that $T(M)$ is relatively compact for each bounded set $M \subset X$. Thus, the operator $T : X \to X$ is completely continuous.

3. Main theorems

Now, we prove the main result. Denote

$$\Psi(t) = \int_0^1 G(s,t)\psi_0(s)ds$$

and L be defined in Section 2.

Theorem 3.1. Suppose that (H1)-(H4) hold. Then BVP(1.4) has at least one solution if

$$\sup_{y\in[0,\infty)} \frac{y}{(y+||\Psi||)^\mu} > L\int_0^1 (1-s)^{\alpha-1}\psi(s)ds. \tag{3.1}$$

Proof. From (3.1), we can choose $r > 0$ such that

$$\frac{r}{(r+||\Psi||)^\mu} > L\int_0^1 (1-s)^{\alpha-1}\psi(s)ds. \tag{3.2}$$

Denote

$$M_r = \{x \in X : ||x-\Psi|| \le r\}.$$

For $x \in M_r$, we have

$$||x|| \le ||x-\Psi|| + ||\Psi|| \le r + ||\Psi||.$$

Furthermore,

$$|q(t)f(t,x(t)) - \psi_0(t)| \le \psi(t)\left|t^{1-\beta}x(t)\right|^\mu.$$

So (3.2) implies that

$$\begin{aligned}
t^{1-\beta}|(Tx)(t)-\Psi(t)| &= \left|t^{1-\beta}\int_0^1 G(t,s)[q(s)f(s,x(s))-\psi_0(s)]ds\right| \\
&\leq \int_0^1 t^{1-\beta}|G(t,s)||q(s)f(s,x(s))-\psi_0(s)|ds \\
&\leq L\int_0^1 (1-s)^{\alpha-1}|q(s)f(s,x(s))-\psi_0(s)|ds \\
&\quad + \int_t^1 (1-s)^{\alpha-1}|q(s)f(s,x(s))-\psi_0(s)|ds \\
&\leq L\int_0^1 (1-s)^{\alpha-1}\psi_1(s)\left|s^{1-\beta}x(s)\right|^{\mu} ds \\
&\leq L\int_0^1 (1-s)^{\alpha-1}\psi(s)ds||x||^{\mu} \\
&\leq L\int_0^1 (1-s)^{\alpha-1}\psi(s)ds[r+||\Psi||]^{\mu} \\
&\leq r.
\end{aligned}$$

It follows that $Tx \in M_r$. So $T(M_r) \subseteq M_r$ and Schauder fixed point theorem implies that T has a fixed point $x \in M_r$. This x is a solution of BVP (1.4). The proof is complete.

Remark 3.1. Since

$$\sup_{y\in[0,\infty)} \frac{y}{(y+||\Psi||)^{\mu}} = \begin{cases} 1, & \mu = 1, \\ \infty, & \mu \in (0,1), \\ \frac{(\mu-1)^{\mu-1}}{\mu^{\mu}||\Psi||^{\mu-1}}, & \mu > 1, \end{cases}$$

we have the following corollary:

Corollary 3.1. Suppose that (H1)-(H3) hold. Then BVP(1.3) has at least one solution if one of the followings conditions holds:

(i) $\mu \in (0,1)$.

(ii) $\mu = 1$ and $L\int_0^1 (1-s)^{\alpha-1}\psi(s)ds < 1$.

(iii) $\mu > 1$ and $\frac{(\mu-1)^{\mu-1}}{\mu^{\mu}||\Psi||^{\mu-1}} > L\int_0^1 (1-s)^{\alpha-1}\psi(s)ds$.

4. An example

In this section, we give an example to illustrate the main theorem.

Example 4.1. Let$\lambda > 0$. Consider the following BVP

$$\begin{cases} D_{0^+}^{\frac{1}{2}}[t^{\frac{1}{2}} D_{0^+}^{\frac{1}{2}} u(t)] + (1-t)^{\frac{1}{4}}\, t^{-\frac{1}{2}} + \nu t^{\frac{3}{4}}[u(t)]^{\mu} = 0, \quad t \in (0,1], 1<\alpha<2, \\ \lim_{t\to 0} t^{\frac{1}{2}} u(t) - \lim_{t\to 0} D_{0^+}^{\frac{1}{2}} u(t) = 0, \\ \lim_{t\to 1} u(t) + \lim_{t\to 1} D_{0^+}^{\frac{1}{2}} u(t) = 0, \end{cases} \tag{4.1}$$

where $\nu > 0$ is a constant.

Corresponding to BVP(4.1), we find that $\beta = \frac{3}{4}$, $\alpha = \frac{1}{2}$, $a = b = c = d = 1$, $p(t) = t^{\frac{1}{2}}$ and

$$f(t,x) = (1-t)^{\frac{1}{4}}\, t^{-\frac{1}{2}} + \nu t^{\frac{3}{4}} x^{\mu}.$$

One can show that f satisfies (H1) and p satisfies (H2) with $M_0 = 1$, q satisfies (H3).

It is easy to see that

$$f\left(t, t^{-\frac{1}{2}} x\right) = (1-t)^{\frac{1}{4}}\, t^{-\frac{1}{2}} + \nu t^{\frac{3}{4}-\frac{1}{2}\mu} x^{\mu}.$$

Choose $\psi_0(t) = (1-t)^{\frac{1}{4}}\, t^{-\frac{1}{2}}$ and $\psi(t) = \nu t^{\frac{3}{4}-\frac{1}{2}\mu}$. Then both ψ_0 and ψ are nonnegative measurable functions. So (H4) holds with ψ_0, ψ and μ defined above.

By direct computation, we find

$$\sigma(0,1) = \int_0^1 \frac{(1-u)^{\frac{3}{4}-1} u^{\frac{1}{2}-1}}{u^{-\frac{1}{2}}} du = \frac{4}{3},$$

$$\Delta = (bc + ad)\Gamma(\beta) + ac\sigma(0,1) = 2\Gamma(3/4) + 2,$$

$$L = \tfrac{1}{\Gamma(1/2)\Gamma(3//4)\Delta}\left[\Delta M_0 \mathbf{B}(1/2,1/4) + \Gamma(3/4)\mathbf{B}(1/2,1/4) \right.$$
$$\left. + \mathbf{B}(1/2,1/4)^2 + \Gamma(3/4)^2 + \Gamma(3/4)\mathbf{B}(1/2,1/4)\right],$$

and

$$||\Psi|| = \sup_{t\in(0,1)} t^{\frac{1}{2}} \Psi(t) = \sup_{t\in(0,1)} t^{\frac{1}{2}} \int_0^1 G(s,t)\psi_0(s)ds \le L \int_0^1 (1-s)^{-\frac{1}{2}} \psi_0(s)ds.$$

By Corollary 3.1, we get BVP(4.1) has at least one solution if one of the followings holds:

(i) $\mu \in (0,1)$.

(ii) $\mu = 1$ and $\nu L \int_0^1 (1-s)^{-\frac{1}{2}} s^{\frac{1}{4}} ds = \nu L \mathbf{B}(1/2, 5/4) < 1$.

(iii) $\mu > 1$ and

$$\begin{aligned} \frac{(\mu-1)^{\mu-1}}{\mu^{\mu}} &> \nu L \int_0^1 (1-s)^{-\frac{1}{2}} s^{\frac{1}{4}} ds \left[L \int_0^1 (1-s)^{-\frac{1}{2}} \psi_0(s) ds\right]^{\mu-1} \\ &= \nu L^{\mu} \mathbf{B}(1/2,5/4)\mathbf{B}(3/4,1/2)^{\mu-1}. \end{aligned}$$

References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl Math, vol. 109: 973-1033, (2010).

[2] A. Arara, M. Benchohra, N. Hamidi, and J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal., vol. 72: 580-586, (2010).

[3] D. Guo, and J. Sun, it Nonlinear Integral Equations, Shandong Science and Technology Press, Jinan, 1987 (in Chinese).

[4] A. A. Kilbas, O.I. Marichev, and S. G. Samko, *Fractional Integral and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.

[5] A. A. Kilbas, and J.J. Trujillo, *Differential equations of fractional order: methods, results and problems-I*, Applicable Analysis, vol. 78: 153–192, (2001).

[6] A. Arara, M. Benchohra, N. Hamidi, and J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Analysis, vol. 72: 580-586, (2010).

[7] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, Wiley, New York, 1993.

[8] S. Z. Rida, H.M. El-Sherbiny, and A. Arafa, *On the solution of the fractional nonlinear Schrodinger equation*, Physics Letters A, vol. 372: 553-558, (2008).

[9] Bashir Ahmad, J. Nieto, Riemann-Liouville fractional differential equations with fractional boundary conditions, Fixed Point Theory, to appear.

[10] S. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl. vol. 252: 804–812, (2000).

[11] E. Kaufmann, E. Mboumi, *Positive solutions of a boundary value problem for a nonlinear fractional differential equation*, Electronic Journal of Qualitative Theory of Differential Equations, vol. 3: 1-11, (2008).

[12] R. Dehghant and K. Ghanbari, *Triple positive solutions for boundary value problem of a nonlinear fractional differential equation*, Bulletin of the Iranian Mathematical Society, vol. 33: 1-14, (2007).

[13] Y. Liu, *Positive solutions for singular FDES*, U.P.B. Sci. Series A, vol. 73: 89-100, (2011).

[14] S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional differential equation*, Electron. J. Diff. Eqns. vol. 36: 1-12, (2006).

[15] Y. Zhao, S. Sun, Z. Han, M. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Applied Mathematics and Computation, vol. 217: 6950-6958, (2011).

[16] M. Benchohra, J. Graef, S. Hamani, *Existence results for boundary value problems with nonlinear frational differential equations*, Applicable Analysis, vol. 87: 851-863, (2008).

[17] X. Tang, C. Yan, Q. Liu, *Existence of solutions of two-point boundary value problems for fractional p-Laplace differential equations at resonance*, Journal of Applied Mathematics and Computing, vol. 41: 119-131, (2013).