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Numerical integration using spline quasi-interpolants

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Abstract. In this paper, quadratic rules for obtaining approximate solution of definite integrals as well as single and double integrals using spline quasi-interpolants will be illustrated. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

Keywords: Spline quasi-interpolants, Gregory rules, Numerical integration, Double integral.

2000 Mathematics subject classification: 65D30; 65D99.

1. Introduction

Single and double integration of a function on bounded interval is an important operation for many physical and engineering problems. Most procedures for approximating the value of definite integral use polynomials that approximate the function. Therefore, numerical approximation techniques are used for the evaluation of the definite integral. There are several methods for numerical double integration of a function, (see e.g. [4,9]).
Quasi-interpolants (abbr. QIs) have been studied in the literature [5,6,10]

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in order to be employed in widespread applications in mechanics, engineering and scientific computations. This method is given by

$$Qf = \sum_{k \in J} \mu_k(f) B_k,$$

where $\{B_j, j \in J\}$ is the B-spline basis of some space of splines, on a bounded interval $I = [a, b]$ endowed with some partition X_n of I in n subintervals. The coefficients μ_j are local linear functionals which are in universal of one of the following types:
(i) differential type: $\mu_j(f)$ is a linear combination of values of derivatives of f, at some point in $supp(B_j)$ (see [1,2]). The related quasi-interpolant is called a differential quasi-interpolant.
(ii) integral type: $\mu_j(f)$ is a linear combination of weighted mean values of f, i.e. of quantities $\int f w_j$ where w_j can be, for example, a linear combination of B-splines (see [1,2]). The associated quasi-interpolant is called an integral quasi-interpolant.
(iii) discrete type: $\mu_j(f)$ is a linear combination of discrete values of f at some points in the neighbourhood of $supp(B_j)$ (see [3]). The associated quasi-interpolant is called a discrete quasi-interpolant.
The main advantage of QIs is that they have a direct construction without solving any system of linear equations.

The paper is organized into four sections. In section 2, we consider QIs and obtain approximate solution of definite integrals as well as single and double integrals. In section 3, we explain the Gregory rules of the same order. Finally, we give some numerical examples which illustrate the accuracy and the implementation of the method.

2. Quadrature rules based on a quadratic spline quasi-interpolants for calculating single and double integrals

Let $X_n := \{x_k,\ 0 \le k \le n\}$ be the uniform partition of the interval $I = [a, b]$ into n equal subintervals, i.e. $x_k := a + kh$, with $h = \frac{b-a}{n}$ and $0 \le k \le n$. We consider the space $S_2 = S_2(I, X_n)$ of quadratic splines of class C^1 on this partition. Its canonical basis is formed by the $n+2$ normalized B-splines, $\{B_k,\ k \in J\}$ with $J := \{1, 2, \cdots, n+2\}$. The support of each B_k is the interval $[x_{k-3}, x_k]$ if we add multiple knots at the endpoints. We consider the following quadratic spline quasi-interpolant (abbr. dQI) defined in [5] by

$$Q_2 f = \sum_{k \in J} \mu_k(f) B_k, \tag{2.1}$$

where $\mu_k(f)$ are given by

$$\mu_1(f) = f_1,\ \mu_{n+2}(f) = f_{n+2},$$

$$\mu_2(f) = -1/3f_1 + 3/2f_2 - 1/6f_3 = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3,$$

$$\mu_{n+1}(f) = -1/6f_n + 3/2f_{n+1} - 1/3f_{n+2} = \beta_3 f_n + \beta_2 f_{n+1} + \beta_1 f_{n+2}, \quad (2.2)$$

and for $3 \le j \le n$

$$\mu_j(f) = -1/8f_{j-1} + 5/4f_j - 1/8f_{j+1} = \gamma_1 f_{j-1} + \gamma_2 f_j + \gamma_3 f_{j+1}, \quad (2.3)$$

with $f_i = f(t_i)$, $t_1 = a$, $t_{n+2} = b$, and $t_i = a + (i - 3/2)h,\ 2 \le i \le n+1$. By integrating this dQI, the author in [7] constructs a new and simple quadrature rule of convergence order $O(h^4)$. He also gives error estimate for smooth functions.

Here, we will construct quadrature rules based on this type of dQI and of convergence order $O(h^l)$, with $l \ge 4$ (see [7]). For $j = 2, \dots, n+1$, we retain the same values of $\mu_j(f)$, given by (2.2), (2.3) and for $j = 1$ and $j = n+2$, we set

$$\mu_1(f) = \sum_{i=1}^{m} \alpha_i f_i,$$

$$\mu_{n+2}(f) = \sum_{i=1}^{m} \alpha_i f_{n+3-i}, \quad (2.4)$$

where m is an odd integer such that $3 \le m \le n+2$, and $(\alpha_1, \alpha_2, \dots, \alpha_m)$ are real parameters. We consider the quadrature rule defined by

$$\mathcal{I}_{Q_2}(f) := \int_l Q_2 f(x)dx. \quad (2.5)$$

$$E_m = \mathcal{I}^m_{Q_2}(f) - \int_a^b f(x)dx,$$

be the error of the quadrature rule (2.5) on function f with $I = [a, b]$. From $\int_a^b B_j = \frac{1}{d+1}(x_{j+1} - x_{j-d})$ we can write:

$$\int_I B_1(x)dx = \int_I B_{n+2}(x)dx = h/3,$$

$$\int_I B_2(x)dx = \int_I B_{n+1}(x)dx = 2h/3,$$

$$\int_I B_k(x)dx = h, \qquad 3 \le k \le n.$$

Lemma 2.1. *Let m be an odd integer with $3 \leq m \leq n+2$, and let*

$$S = \sum_{j=3}^{n} (\gamma_1 t_{j-1}^{m-1} + \gamma_2 t_j^{m-1} + \gamma_3 t_{j+1}^{m-1}).$$

Then S is a polynomial function of degree m in the variable n. More precisely,

$$S = \sum_{j=0}^{m} \theta_j^{(m)} n^j, \tag{2.6}$$

where

$$\theta_0^{(m)} = \sum_{l=1}^{3} \gamma_l (\frac{1}{m}((l-7/2)^m + (1/2-l)^m) + 1/2((l-7/2)^{m-1} + (1/2-l)^{m-1})$$
$$+ \sum_{i=1}^{\frac{m-1}{2}} \frac{\tilde{B}_{2i}}{2i!}(m-1)\dots(m-2i+1)((l-7/2)^{m-2i} + (1/2-l)^{m-2i})),$$
$$\theta_j^{(m)} = \sum_{l=1}^{3} \gamma_l (\frac{1}{m} C_m^j (l-7/2)^{m-j} + 1/2 C_{m-1}^j (l-7/2)^{m-1-j}$$
$$+ \sum_{i=1}^{(\frac{m-j-1}{2})} \frac{\tilde{B}_{2i}}{2i!}(m-1)\dots(m-2i+1) C_{m-2i}^j (l-7/2)^{m-2i-j}), \; j = 1, \dots, m,$$

where C_m^j are the binomial coefficients, $\tilde{B}_{2i}$ are the Bernoulli numbers and $[x]$ denotes the integer part of x.

Proof. Refer to [8]. □

Lemma 2.2. *For m odd and $3 \leq m \leq n+2$, we have*

$$E_m = \sum_{j=0}^{m-1} \lambda_j^{(m)} n^j,$$

where

$$\lambda_0^{(m)} = 2(\frac{1}{3}\sum_{i=1}^{m} \alpha_i t_i^{m-1} + \frac{2}{3}\sum_{i=1}^{3} \beta_i t_i^{m-1}) + \theta_0^m,$$
$$\lambda_j^{(m)} = \frac{1}{3}\sum_{i=1}^{m} \alpha_i C_{m-1}^j (-t_i)^{m-1-j} + \frac{2}{3}\sum_{i=1}^{3} \beta_i C_{m-1}^j (-t_i)^{m-1-j} + \theta_j^{(m)}.$$

According to the above lemma, we deduce that imposing $E_m = 0$ for all n is equivalent to solving the following linear system on $(\alpha_1, \alpha_2, \dots, \alpha_m)$:

$$\lambda_j^{(m)} = 0, \qquad j = 0, \dots, m-1. \tag{2.7}$$

Proof. Refer to [8]. □

We calculate the parameters $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ by solving the linear system (2.7) and we replace them in the expressions of the functional coefficients (2.4) for obtaining the quadrature rules $\mathcal{I}_{Q_2}^m$. This quadrature formula can be written as

$$\mathcal{I}_{Q_2}^m(f) = h\sum_{i=1}^{m} v_i^{(m,2)}(f_i + f_{n+3-i}) + h\sum_{i=m+1}^{n+2-m} f_i.$$

In Table 5, we list correction weights $v_i^{(m,2)}, i = 1, \ldots, m$, for $m = 5, 7, 9, 13, 17, 21$. Now, we can obtain the double integration formula, as follows:

$$x_1 = a,\ x_{n+2} = b,\ x_i = a + (i - 3/2)h_x,$$

$$y_1 = c,\ y_{n+2} = d,\ y_i = c + (i - 3/2)h_y,$$

$$h_x = \frac{b-a}{n}, h_y = \frac{d-c}{n'},$$

where x_i and y_i are grid points. Let

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b g(x)dx,$$

such that

$$\begin{aligned} g(x) &= \int_c^d f(x,y)dy \\ &= h_y(\sum_{j=1}^{m'} v_j^{(m',2)}(f(x,y_j) + f(x,y_{n'+3-j}))) + h_y \sum_{j=m'+1}^{n'+2-m'} f(x,y_j), \end{aligned}$$

Then by integrating we have

$$\int_a^b g(x)dx = h_y \sum_{j=1}^{m'} v_j^{(m',2)} \int_a^b f(x,y_j)dx$$
$$+ h_y \sum_{j=1}^{m'} v_j^{(m',2)} \int_a^b f(x, y_{n'+j-3})dx + h_y \sum_{j=m'}^{n'+2-m'} \int_a^b f(x,y_j)dx$$
$$= h_y \sum_{j=1}^{m'} v_j^{(m',2)} h_x[\sum_{i=1}^{m} v_i^{(m,2)}(f(x_i,y_j) + f(x_{n+3-i},y_j)) + \sum_{i=m+1}^{n+2-m} f(x_i,y_j)]$$
$$+ h_y \sum_{j=1}^{m'} v_j^{(m',2)}[h_x \sum_{i=1}^{m} v_i^{(m,2)}(f(x_i,y_{n'+3-j}) + f(x_{n+3-i},y_{n'+3-j}))$$
$$+ h_x \sum_{i=m+1}^{n+2-m} f(x_i,y_{n'+3-j})]$$
$$+ h_y \sum_{i=m'+1}^{n'+2-m'}[h_x \sum_{i=1}^{m} v_i^{(m,2)}(f(x_i,y_j) + f(x_{n+3-i},y_j)) + h_x \sum_{i=m+1}^{n+2-m} f(x_i,y_j)],$$

Finally, we deduce the double integrating formula, as follows:

$$\mathcal{I}_{Q_2}^{m,m'}(f(x,y)) = h_x h_y (\sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} f(x_i,y_j)$$
$$+ \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} f(x_{n+3-i},y_j) + \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} f(x_i,y_j)$$
$$+ \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} f(x_i,y_{n'+3-j})$$
$$+ \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} f(x_{n+3-i},y_{n'+3-j})$$
$$+ \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} f(x_i,y_{n'+3-j}) + \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} f(x_{n+3-i},y_j))$$
$$+ \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} f(x_i,y_j) + \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} f(x_{n+3-i},y_j). \quad (2.8)$$

3. Gregory rules

Gregory rules are expressible in the form

$$GR_{k,n}(f) = h\{\sum_{i=0}^{n} f_{n,i}+\sum_{i=0}^{k}[(-1)^{i+1}\sum_{j=i}^{k}\binom{j}{i}\mathcal{L}_{j+1}](f_{n,i}+f_{n,n-i})\}, \quad (3.1)$$

where $f_{n,i} = f(x_{n,i}), h = \frac{b-a}{n}$, $x_{n,i} = a + ih, i = 0, 1, \ldots, n$. The Laplace coefficients $\{\mathcal{L}_j,\ j = 1, 2, \ldots\}$ can be computed from the recursive formula

$$\sum_{v=1}^{\mu} \frac{\mathcal{L}_v}{\mu - v + 1} = \frac{1}{\mu + 1}, \quad \mu = 1, 2, \ldots$$

Now, we can obtain the double integration formula, as follows:

$$x_{n,i} = a + ih_x,\ i = 0, 1, \ldots, n,$$

$$y_{n',j} = c + jh_y,\ j = 0, 1 \ldots, n',$$

where $y_{n',j}$ and $x_{n,i}$ are grid points. Let

$$\int_a^b \int_c^d f(x, y)dydx = \int_a^b g(x)dx$$

such that

$$\begin{aligned} g(x) &= \int_c^d f(x, y)dy \\ &= h_y(\sum_{j=0}^{n'} f_{n',j} + \sum_{j=0}^{m'-1}[(-1)^{j+1}\sum_{k=j}^{m'-1}\mathcal{L}_{k+1}\binom{k}{j}(f_{n',j} + f_{n',n'-j})]) \end{aligned}$$

with $f_{n,i} = f(x_{n,i}, y)$ and $f_{n',j} = f(x, y_{n',j})$. Then by integrating we have

$$\begin{aligned} \int_a^b g(x)dx &= h_y \sum_{j=0}^{n'} \int_a^b f_{n',j}dx \\ &+ h_y \sum_{j=0}^{m'-1}[(-1)^{j+1}\sum_{k=j}^{m'-1}\mathcal{L}_{k+1}\binom{k}{j}\int_a^b f_{n',j}dx] \\ &+ h_y \sum_{j=0}^{m'-1}[(-1)^{j+1}\sum_{k=j}^{m'-1}\mathcal{L}_{k+1}\binom{k}{j}\int_a^b f_{n',n'-j}dx], \end{aligned}$$

$$\int_a^b g(x)dx = h_y \sum_{j=0}^{n'} [h_x(\sum_{i=0}^{n} f(x_{n,i}, y_{n',j})$$
$$+ \sum_{i=0}^{m-1} [(-1)^{i+1} \sum_{k'=i}^{m-1} \mathcal{L}_{k'+1} \binom{k'}{i}](f(x_{n,i}, y_{n',j}) + f(x_{n,n-i}, y_{n',j}))))]$$
$$+ h_y \sum_{j=0}^{m'-1} ([(-1)^{j+1} \sum_{k=j}^{m'-1} \mathcal{L}_{k+1} \binom{k}{j}] h_x(\sum_{i=0}^{n} f(x_{n,i}, y_{n',j})$$
$$+ \sum_{i=0}^{m-1} [(-1)^{i+1} \sum_{k'=i}^{m-1} \mathcal{L}_{k'+1} \binom{k'}{i}](f(x_{n,i}, y_{n',j}) + f(x_{n,n-i}, y_{n',j}))))])$$
$$+ h_y \sum_{j=0}^{m'-1} ([(-1)^{j+1} \sum_{k=j}^{m'-1} \mathcal{L}_{k+1} \binom{k}{j}] h_x(\sum_{i=0}^{n} f(x_{n,i}, y_{n',n'-j})$$
$$+ \sum_{i=0}^{m-1} [(-1)^{i+1} \sum_{k'=i}^{m-1} \mathcal{L}_{k'+1} \binom{k'}{i}](f(x_{n,i}, y_{n',n'-j}) + f(x_{n,n-i}, y_{n',n'-j}))))]),$$

Finally, we deduce the double integrating formula, as follows:

$$GR_{m-1,m'-1,n,n'} = h_x h_y (\sum_{j=0}^{n'} \sum_{i=0}^{n} f(x_{n,i}, y_{n',j})$$
$$+ \sum_{j=0}^{n'} \sum_{i=0}^{m-1} \sum_{k'=i}^{m-1} (-1)^{i+1} \binom{k'}{i} \mathcal{L}_{k'+1}(f(x_{n,i}, y_{n',j}) + f(x_{n,n-i}, y_{n',j}))$$
$$+ \sum_{j=0}^{m'-1} \sum_{k=j}^{m'-1} \sum_{i=0}^{n} (-1)^{j+1} \mathcal{L}_{k+1} \binom{k}{j} (f(x_{n,i}, y_{n',j}) + f(x_{n,i}, y_{n',n'-j}))$$
$$+ \sum_{j=0}^{m'-1} \sum_{k=j}^{m'-1} \sum_{i=0}^{m-1} \sum_{k'-i}^{m-1} (-1)^{i+j+2} \mathcal{L}_{k+1} \binom{k}{j} \mathcal{L}_{k'+1} \binom{k'}{i} f(x_{n,i}, y_{n',j})$$
$$+ \sum_{j=0}^{m'-1} \sum_{k=j}^{m'-1} \sum_{i=0}^{m-1} \sum_{k'=i}^{m-1} (-1)^{i+j+2} \mathcal{L}_{k+1} \binom{k}{j} \mathcal{L}_{k'+1} \binom{k'}{i} f(x_{n,n-i}, y_{n',j})$$
$$+ \sum_{j=0}^{m'-1} \sum_{k=j}^{m'-1} \sum_{i=0}^{m-1} \sum_{k'-i}^{m-1} (-1)^{i+j+2} \mathcal{L}_{k+1} \binom{k}{j} \mathcal{L}_{k'+1} \binom{k'}{i} f(x_{n,i}, y_{n',n'-j})$$
$$+ \sum_{j=0}^{m'-1} \sum_{k=j}^{m'-1} \sum_{i=0}^{m-1} \sum_{k'=i}^{m-1} (-1)^{i+j+2} \mathcal{L}_{k+1} \binom{k}{j} \mathcal{L}_{k'+1} \binom{k'}{i} f(x_{n,n-i}, y_{n',n'-j})).$$

4. EXAMPLE

In this section, we compare the errors of integration for the rules $\mathcal{I}_{Q_2}^{m}$ and $GR_{m-1,n}$, also errors of double integration for the rules $\mathcal{I}_{Q_2}^{m,m'}$ and $GR_{m-1,m'-1,n,n'}$, for different values of corrections m.

Example 4.1. Consider $f(x) = \cos(201x) + \sin(200x)$. In Tables 1 and 2, numerical results are presented for the rules $\mathcal{I}_{Q_2}^{m}$ and $GR_{m-1,n}$ respectively. Column 1 of each table contains the number of nodes discretizing the interval [0,1]. The columns 2-6 in Table 1 (respectively Table 2) contain the relative errors for the rule $\mathcal{I}_{Q_2}^{m}$ (respectively, for the rule $GR_{m-1,n}$) for various number of corrections m.

Example 4.2. Consider $f(x) = \frac{1}{(x^2+y^2)^{1/2}}$. In Tables 3 and 4, numerical results are presented for the rules $\mathcal{I}_{Q_2}^{m,m'}$ and $GR_{m-1,m'-1,n,n'}$ respectively.

Table 1: The error $\left|\int_0^1 f(x)dx - \mathcal{I}_{Q_2}^{m}(f)\right|$.

$m=21$	$m=17$	$m=13$	$m=9$	$m=7$	$m=5$	n
$4.09E-04$	$7.83E-05$	$1.04E-05$	$3.80E-05$	$1.50E-05$	$6.89E-05$	128
$2.55E-10$	$1.21E-09$	$1.22E-09$	$4.17E-08$	$2.73E-07$	$9.96E-07$	256
$*$	$4.76E-15$	$8.76E-14$	$5.43E-11$	$6.55E-10$	$5.03E-09$	512
$*$	$*$	$*$	$2.08E-14$	$5.67E-13$	$2.36E-11$	1024
$*$	$*$	$*$	$*$	$2.11E-15$	$1.17E-12$	2048

Table 2: The error $\left|\int_0^1 f(x)dx - GR_{m-1,n}(f)\right|$.

$m=21$	$m=17$	$m=13$	$m=9$	$m=7$	$m=5$	n
$2.76E-02$	$9.21E-03$	$3.30E-03$	$1.33E-03$	$6.88E-04$	$6.72E-04$	128
$3.20E-08$	$9.47E-08$	$5.07E-07$	$1.80E-06$	$7.01E-06$	$1.53E-05$	256
$3.07E-15$	$3.87E-13$	$4.77E-11$	$3.02E-09$	$1.97E-08$	$9.33E-08$	512
$*$	$*$	$2.22E-15$	$1.23E-12$	$2.04E-11$	$1.50E-10$	1024
$*$	$*$	$*$	$*$	$4.80E-14$	$1.52E-11$	2048

Table 3: The error $\left|\int_1^4 \int_2^5 f(x,y)dydx - \mathcal{I}_{Q_2}^{m,m'}(f(x,y))\right|$.

$m=m'=13$	$m=m'=9$	$m=m'=7$	$m=m'=5$	$n=n'$
$3.9285E-08$	$3.9285E-08$	$3.9285E-08$	$3.9285E-08$	128
$3.9285E-08$	$3.9285E-08$	$3.9285E-08$	$3.9285E-08$	256
$3.9285E-08$	$3.9285E-08$	$3.9285E-08$	$3.9285E-08$	512

Table 4: The error $\left|\int_1^4\int_2^5 f(x,y)dydx - GR_{m-1,m'-1,n,n'}(f(x,y))\right|$.

$m=m'=13$	$m=m'=9$	$m=m'=7$	$m=m'=5$	$n=n'$
$3.92853E-08$	$3.92855E-08$	$3.92855E-08$	$3.92855E-08$	128
$3.92855E-08$	$3.92855E-08$	$3.92855E-08$	$3.92855E-08$	256
$3.92855E-08$	$3.92855E-08$	$3.92855E-08$	$3.92855E-08$	512

From the above examples and our other numerical experiments, we observe that the numerical behavior of the corrected rules is in good agreement with the asymptotic estimates.

Table 5: Quadrature weights $v_i^{(m,2)}$.

$m=5$	$m=7$	$m=9$	$m=13$	$m=17$	$m=21$
0.1307936	0.1374149	0.1400901	0.1414888	0.1411373	0.1402697
0.8359375	0.8190165	0.8109525	0.8060753	0.8076111	0.8118026
1.0449652	1.0698175	1.0870467	1.1014595	1.0941489	1.0695184
0.9861458	0.9603402	0.9321887	0.8977344	0.9269492	1.05033436
1.0021577	1.0177210	1.0478285	1.1060041	1.0200106	0.5567149
	0.9950634	0.9742957	0.9037215	1.0944433	2.4318011
	1.0006252	1.0095197	1.0720481	0.7458750	2.3010254
		0.9978581	0.9571099	1.3940013	6.9810893
		1.0002198	1.0196830	0.5574191	7.8014726
			0.9933074	1.3805591	11.6754457

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