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A New Characterization of The $2k$- Adic-Type Cantor-Like Sets and $2k$- adic-Type Cantor Functions

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Abstract. In this paper, a new representation of the rescaling Cantor sets of type $2k-$ adic-type Cantor-Like sets are presented. The necessary and sufficient condition is given for an element on $[0, 1]$ to be in the $2k-$ adic-type Cantor-Like sets. Two methods are proposed to the construction of the type $2k-$ adic-type Cantor-Like functions via functions and iteration functions. Finaly, some results of convergent for the $2k-$ adic-type Cantor-Like functions are discussed in our work.

Keywords: Cantor set, Cantor-like sets, the generalization of Smith-Volterra Cantor set, C_λ- Cantor sets, and rescaling Cantor sets of type $2k-$ adic-type Cantor-Like sets.

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1. Introduction

Many authors were interested in given a novel different representations methods for Cantor ternary set ($C.T.S$) such as: Cantor-like sets ($C.L.S.'s$) ([1], [2], [3], [4], [5]) C_λ- Cantor sets ($0 < \lambda \leq 1$), ([1], [6], [7]), the generalization of Smith-Volterra Cantor set ($G.S.V.C.S$) and Smith-Volterra Cantor sets ($S.V.C$ ($n \in \mathbf{N}, n \geq 3$)), ([8], [9], [10]).

In ([4], [9]) used a sequence of positive numbers $\{u_\zeta\}_{\zeta=0}^{\infty}$ such that $\sum_{\zeta=0}^{\infty} u_\zeta 2^\zeta = u, 0 < u < 1$, to construct C.L.S.'s. Another construction of C.L.S.'s was based on the term homogeneous perfect set, [11]. In their paper, they introduced the notion of homogeneous perfect sets as a generalization of Cantor type sets with some results for computing the dimension of Cantor type sets. Also, Besicovitch and Taylor in [4], studied the generalization of C.L.S.'s construction. Their construction can be explained based on a closed set P contained in real closed interval $[0,1]$ and $\Lambda = (0,1)$, then $\Lambda - P$ is an open interval and composed of an enumerable sequence of open intervals. The length of these intervals are a sequence of positive numbers, $\{w_\zeta\}_{\zeta=1}^{\infty}$. If

$$\sum_{\zeta=1}^{\infty} w_\zeta = 1$$

then P has zero measure. Finally, [5] C.L.S.'s can be defined by using two sequences $\{d_\zeta\}_{\zeta=1}^{\infty}$ and $\{q_\zeta\}_{\zeta=1}^{\infty}$, where, $\{d_\zeta\}_{\zeta=1}^{\infty}$ be a sequence of positive integers such that $d_\zeta > 2$ for each $\zeta, \zeta \in N$, and $\{q_\zeta\}_{\zeta=1}^{\infty}$ be a sequence of positive integers satisfy the relation:

$$0 < d_\zeta q_\zeta < q_{\zeta-1}$$

with

$$\nabla_\zeta = \frac{1}{d_\zeta - 1}(q_{\zeta-1} - q_\zeta), \forall\zeta.$$

The above relation used to construct Cantor ternary set ($C.T.S$), C_λ-Cantor sets ($0 < \lambda \leq 1$), the generalization of Smith-Volterra Cantor set (G.S.V.C.S.) and Smith-Voltera Cantor sets (S.V.C. ($n \in \mathbf{N}$, $n \geq 3$)) as follow:

C.T.S.:

$$d_\zeta = 2, q_0 = 1,$$

and

$$q_\zeta = \frac{1}{3^\zeta}, \forall\zeta \geq 1. \nabla_0 = 1, \nabla_\zeta = 2.3^{-\zeta} \forall\zeta \geq 1.$$

$C_\lambda-$ Cantor sets:

$$d_\zeta = 2, q_0 = 1, q_\zeta = 2^{-\zeta}(1-(1-(2/3)^\zeta)\lambda), \nabla_0 = 1,$$

and

$$\nabla_\zeta = 2^{-\zeta}((1-\lambda+2(2/3)^\zeta)\lambda), \forall\zeta \geq 1.$$

G.S.V.C.S.:

$$d_\zeta = 2, q_0 = 1,$$

and

$$q_\zeta = 2^{-\zeta}(1-e) - 2^{-\zeta+1}ge(\frac{1-(2g)^{\zeta-1}}{1-2g}),$$

where,

$$0 < g < 1, 0 < e < 1, 0 < e < 1-2g,$$

$$\nabla_\zeta = 2^{-\zeta}(1-e) - \frac{ge}{1-2g}(2^{-\zeta+1} - (1-g)g^{\zeta-2}), \forall\zeta, \zeta \geq 1.$$

S.V.C. ($n \in \mathbf{N}$, $n \geq 3$):

$$d_\zeta = 2, q_0 = 1,$$

and

$$q_\zeta = 2^{-\zeta}(1-e) - 2^{-\zeta+1}ge(\frac{1-(2g)^{\zeta-1}}{1-2g}), \forall\zeta, \zeta \geq 1.$$

In this construct, their Cantor-Lebesgue function was determined in [5].

Note that $k-$ adic-type Cantor-Like sets are another variation and generalization of Cantor set. Tsang, [12] was introduced the generalization of the $k-$ adic-type Cantor-Like sets called rescaling Cantor sets (R.C.S.'s).

In this paper, because of $k-$ adic-type Cantor-Like sets construction are not necessary to include in the C.L.S.'s construction, [5], even in their Cantor-Lebesgue function, we will propose another method to construct R.C.S.'s of type $2k-$ adic-type Cantor-Like sets with their $2k-$ adic-type Cantor functions.

2. $k-$Adic-Type Cantor-Like Sets

A generalization of the $k-$ adic-type Cantor-Like sets, was given in the following construction: let $n \in \mathbf{N}$, with $n > 2$, $\hbar \in \mathbf{N}, 1 < \hbar < n$ and begin with the unite interval M_0= $[0, 1]$. Dividing M_0 into sub intervals of equal length and remove $\hbar$-open intervals of the division leaving us the first and last intervals obtaining M_1^n. Next, dividing each of the remaining $n - \hbar$ sub intervals in M_1^n into $n-$divisions of equal length and remove the same $\hbar$- open intervals of the division corresponding to the intervals removed in the first iteration obtaining M_2^n. Continue dividing each sub interval in $n-$ sub interval and removing $\hbar$ of them in the same method. Then

$$M^n = \bigcap_{l=1}^{\infty} M_l^n.$$

M^n is called $k-$ adic-type Cantor-Like sets. Also, M^n can be defined directly from the $k-$ adic expansion of the real numbers in [0,1]:

$M^n = \left\{\omega \in [0,1] : \omega = \sum_{l=1}^{\infty} \frac{\tau_l}{n^l}, \tau_l \in \{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_\hbar\}\right\}$, and $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_\hbar\} \subseteq \{0, 1, 2, \ldots, n-1\}, \sigma_1 = 0, \sigma_\hbar = n-1.$

All $n-$adic-type Cantor-like sets had measure zero, because of for $j \geq 1$, the measure $\mu(M_l^n) = \left(\frac{n-\hbar}{n}\right)^{ji}$ and

$$\mu(M^n) = \lim_{l\to\infty} \mu(M_l^n) = 0.$$

3. Main result

In [13], it is introduced a new form representation of Cantor set and used them to design a complete orthonormal system on [0,1] which is equivalent to the Walsh system. In this section, we will extend this form to include $2k-$ adic-type Cantor-Like sets and $2k-$ adic-type Cantor functions.

Let $k \in \mathbf{N} = \{1, 2, 3, ...\}$ and the set of all even natural numbers can be written as: $A = \{2k, k \in \mathbf{N}\}$, then the $2k-$ adic-type Cantor-Like sets can be generated by:

For each $k : k \geq 2$, and for each $\gamma \in \mathbf{N} \cup 0$, construct the closed intervals:

$$M_{2^\gamma+r}^{2k} = \left[\frac{(2k-1)z}{(2k)^\gamma}, \frac{(2k-1)z+1}{(2k)^\gamma}\right], 0 \leq r < 2^\gamma$$

where, $z = \sum_{l=0}^{\infty} z_l \,(2k)^l$, and z_l being the dyadic coefficients of r in binary system:$r = \sum_{l=0}^{\infty} r_l 2^l, r_l \in \{0, 1\}$. Then the $2k-$adic-type Cantor-like sets are:

$$M^{2k} = \cap_{\gamma=0}^{\infty} \cup_{r=0}^{2^\gamma-1} M_{2^\gamma+r}^{2k}, 0 \leq r < 2^\gamma.$$

The removed middle open intervals in the above generating are defined as follows:

$$\theta^{2k} = \cup_{\gamma=0}^{\infty} \cup_{r=0}^{2^\gamma-1} \theta_{2^{d-1}+r}^{2k}$$

where,

$\theta_{2^{d-1}+r}^{2k} = \left(\frac{(2k)(2k-1)z+1}{(2k)^\gamma}, \frac{(2k)(2k-1)z+(2k-1)}{(2k)^\gamma}\right), 0 \leq r < 2^{d-1}, 1 \leq d \leq \gamma.$

In the non-stationary Moran setting, the Hausdorff dimension is given by:

$$dim\left(M^{2k}\right) = \lim_{k\to\infty} \frac{\sum_{i=1}^{k} \ln G_i}{-\sum_{i=1}^{k} \ln \varepsilon_i}$$

where, G_i is the number of intervals retained at step i, and ε_i is the construction ratio at step i. In the special self-similar case, when $G_i = G$ and $\varepsilon_i = \varepsilon$, this reduces to:

$$dim\left(M^{2k}\right) = \frac{\ln G}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

Theorem 3.1. *An element $y \in [0,1]$ belongs to the $2k-$ adic-type Cantor-Like sets if it has a sequence $\langle y_i \rangle$,$i \in \mathbf{N}$, where $y_i \in \{0, 2k-1\}$, $\forall k \geq 2$. That is $y = \sum_{i=1}^{\infty} \frac{y_i}{(2k)^2}$, where, $y_i \neq 1, \forall i$.*

Proof. By construction of the $2k-$ adic-type Cantor-Like sets, observe that removing middle third $\left(\frac{1}{2k}, \frac{2k-1}{2k}\right)$corresponds to removing all numbers in [0,1] with unique $2k-$ adic-type Cantor-Like sets presentation $\langle y_1, y_2, \ldots \rangle, y_1 = 1$, thus removing middle third $\left(\frac{1}{(2k)^2}, \frac{2k-1}{(2k)^2}\right)$ corresponds to removing all numbers in [0,1] with unique $2k-$ adic-type Cantor-Like sets presentation $\langle y_1, y_2, \ldots \rangle, y_1 = 0, y_2 = 1$. By the same as for the middle third $\left(\frac{(2k)^2-2}{(2k)^2}, \frac{(2k)^2+1}{(2k)^2}\right)$, we remove those real in [0,1], where, $2k-$ adic-type Cantor-Like sets representation have $y_1 = 2k - 1$,$y_2 = 1$, and also on the many steps, removing middle third corresponds to removing all numbers in [0,1] with the unique $2k-$ adic-type Cantor-Like sets representation $y_i = 0, 2k-1$,$i \in \mathbf{N}$. □

Illustrative: consider the first iteration of 8-adic-type Cantor-like sets:
First iteration: $\left[0, \frac{1}{8}\right] \cup \left[\frac{7}{8}, 1\right]$
Second iteration: $\left[0, \frac{1}{64}\right] \cup \left[\frac{7}{64}, \frac{1}{8}\right] \cup \left[\frac{7}{8}, \frac{57}{64}\right] \cup \left[\frac{63}{64}, 1\right]$
Third iteration: $\left[0, \frac{1}{512}\right] \cup \left[\frac{5}{512}, \frac{1}{64}\right] \cup \left[\frac{7}{64}, \frac{57}{512}\right] \cup \left[\frac{63}{512}, \frac{1}{8}\right] \cup \left[\frac{7}{8}, \frac{449}{512}\right] \cup \left[\frac{455}{512}, \frac{57}{64}\right] \cup \left[\frac{63}{64}, \frac{505}{512}\right] \cup \left[\frac{511}{512}, 1\right]$.

We must convert each of these end points to base 8:
$0 = (0.0)_8, 1/8 = (0.0\bar{7}), 7/8 = (0.7)_8, 1/64 = (0.00\bar{7})_8, 7/64 = (0.07)_8,$ $57/64 = (0, 70\bar{7})63/64 = (0.77)_8, 1/512 = (0.000\bar{7})_8, 7/512 = (0.007)_8,$ $57/512 = (0.070\bar{7})_8, 63/512 = (0.077)_8, 449/512 = (0.700\bar{7})_8, 455/512 =$ $(0.707)_8$, $505/512 = (0.770\bar{7})_8, 511/512 = (0.777)_8$.

Theorem(1) may be viewed as generalization of the Cantor set representation theorem stated in [15]. It should be emphasized that the proof presented in this paper follows a different approach and employs a distinct methodology from that adopted in [15], while still building on the same underlying idea.

Note that $2k-$adic-type Cantor-like sets can be included with a monotone non decreasing continuous function called the $2k-$adic-type Cantor-Lebesgue function and introduced by the following processing: for each closed interval Γ. Let

$$B_\Gamma(t) = \frac{1}{|\Gamma|}\int_0^t \chi_\Gamma(x)dx$$

where,

$$\chi_\Gamma(x) = \begin{cases} 0 & x \notin \Gamma \\ 1 & x \in \Gamma \end{cases}$$

χ_Γ and $|\Gamma|$ denotes the characteristic function and the length of the interval Γ, respectively. For each $\gamma \in \mathbf{N} \cup 0$, let

$$F_\gamma^{2k}(x) = \frac{1}{2^\gamma}\sum_{r=0}^{2^\gamma-1} H^{2k}_{M^{2k}_{2^\gamma+r}}(x)$$

where,

$$H^{2k}_{M^{2k}_{2^\gamma+r}}(x) = \frac{1}{\left|M^{2k}_{2^\gamma+r}\right|}\int_0^x \chi_{M^{2k}_{2^\gamma+r}}(t)dt = (2k)^\gamma \int_0^x \chi_{M^{2k}_{2^\gamma+r}}(t)dt =$$

$$\begin{cases} 0, & x < \frac{(2k-1)z}{(2k)^\gamma} \\ (2k)^\gamma - (2k-1)z, & x \in M^{2k}_{2^\gamma+r} \\ 1, & x > \frac{(2k-1)z+1}{(2k)^\gamma}. \end{cases}$$

In general, $F_0^{2k}(x) = x$, $x \in [0,1]$ and for $\gamma \in N$

$$F_\gamma^{2k}(x) =$$

$$\begin{cases} 2^{-\gamma}((2k)^\gamma x - (2k-1)z + r) & x \in M^{2k}_{2^\gamma+r}, 0 \le r < 2^\gamma \\ 2^{-d}(2r+1), & x \in \theta^{2k}_{2^{d-1}+r}, 0 \le r < 2^{d-1}, 1 \le d \le \gamma. \end{cases}$$

F_γ^{2k} is called the $2k-$adic-type Cantor-Lebesgue functions.

Theorem 3.2. *Any real valued functions:*$F_0^4(x), F_1^4(x), F_2^4(x), F_3^4(x)$ *on* M_0 *that satisfies:*

$$F_0^4(x) = x, x \in \mathrm{M}_1^4$$

$$F_1^4(x) = \begin{cases} 2x, & x \in \mathrm{M}_2^4 \\ 1/2, & x \in \theta_1^4 \\ 2x-1, & x \in \mathrm{M}_3^4 \end{cases}$$

$$F_2^4(x) = \begin{cases} 4x, & x \in \mathrm{M}_4^4 \\ 1/4, & x \in \theta_2^4 \\ 4x-1/2, & x \in \mathrm{M}_5^4 \\ 4x-5/2, & x \in \mathrm{M}_6^4 \\ 3/4, & x \in \theta_3^4 \\ 4x-3, & x \in \mathrm{M}_7^4 \end{cases}$$

$$F_3^4(x) = \begin{cases} 8x, & x \in \mathrm{M}_8^4 \\ 1/8, & x \in \theta_4^4 \\ 1/4(32x-1), & x \in \mathrm{M}_9^4 \\ 1/4(32x-5), & x \in \mathrm{M}_{10}^4 \\ 3/8, & x \in \theta_5^4 \\ 1/2(16x-3), & x \in \mathrm{M}_{11}^4 \\ 1/2(16x-11), & x \in \mathrm{M}_{12}^4 \\ 5/8, & x \in \theta_6^4 \\ 1/4(32x-23), & x \in \mathrm{M}_{13}^4 \\ 1/4(32x-27), & x \in \mathrm{M}_{14}^4 \\ 7/8, & x \in \theta_7^4 \\ 1/2(16x-14), & x \in \mathrm{M}_{15}^4 \end{cases}$$

are the $4-$*adic-type Cantor-Lebesgue functions.*

Proof. It is clear that from construction of the $2k-$ adic-type Cantor-like sets, the closed and open intervals of the $4-$ adic-type Cantor-like sets can be evaluated by:
For $\gamma = 0$, $M_1^4 = [0, 1]$;
while if $\gamma = 1$, we have:
$0 \leq r < 2 : r = \{0, 1\}$:
$r = 0 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (0)$, this implies that $z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{0} 0. \left(4\right)^l = 0$, then $M_2^4 = [0, 1/4]$
$r = 1 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1)$, this implies that $z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{0} 1. \left(4\right)^l = 1$, then $M_3^4 = [3/4, 1]$,
if $\gamma = 2$, we get:
$0 \leq r < 4 : r = \{0, 1, 2, 3\}$:
$r = 0 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (0)$, this implies that $z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{0} 0. \left(4\right)^l = 0$, then $M_4^4 = [0, 1/16]$
$r = 1 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1)$, this implies that $z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{0} 1. \left(4\right)^l = 1$, then $M_5^4 = [3/16, 1/4]$,
$r = 2 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{1} 1.2^l \equiv (1, 0)$, this implies that$z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{1} z_l. \left(4\right)^l = 4$, then $M_6^4 = [12/16, 13/16]$,
$r = 3 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{1} 0.2^l \equiv (1, 1)$, this implies that$z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{1} z_l. \left(4\right)^l = 5$, then $M_7^4 = [15/16, 1]$
if $\gamma = 3$, we get:
$0 \leq r < 8 : r = \{0, 1, 2, 3, 4, 5, 6, 7\}$:
$r = 0 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (0)$, this implies that$z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$
$= \sum_{l=0}^{0} 0. \left(4\right)^l = 0$, then $M_8^4 = [0, 1/64]$
$r = 1 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1)$, this implies that$z = \sum_{l=0}^{\infty} z_l \left(2k\right)^l$

$= \sum_{l=0}^{0} 1.\,(4)^l = 1$, then $M_3^4 = [3/64, 1/16]$,
$r = 2 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (1,0)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 0.\,(4)^l = 4$, then $M_{10}^4 = [12/64, 13/64]$
$r = 3 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1,1)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 1.\,(4)^l = 5$, then $M_{11}^4 = [15/64, 1/4]$,
$r = 4 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{1} 1.2^l \equiv (1,0,0)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l = \sum_{l=0}^{2} z_l.\,(4)^l = 16$, then $M_{12}^4 = [3/4, 49/64]$,
$r = 5 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{2} 0.2^l \equiv (1,0,1)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l = \sum_{l=0}^{2} z_l.\,(4)^l = 17$, then $M_{13}^4 = [51/64, 52/64]$
$r = 6 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{2} 0.2^l \equiv (1,1,0)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l = \sum_{l=0}^{2} 0.\,(4)^l = 20$, then $M_{14}^4 = [60/64, 61/64]$
$r = 7 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{2} 0.2^l \equiv (1,1,1)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l = \sum_{l=0}^{2} 0.\,(4)^l = 21$, then $M_{15}^4 = [63/64, 1]$.
Corresponding the open interval of the 4−adic-type Cantor-like sets can be evaluated by:
If $\gamma = 1$, then $d = 1$ and $0 \le r < 1 : r = \{0\}$:
$r = 0 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (0)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 0.\,(4)^l = 0$, then $\theta_1^4 = (1/4, 3/4)$.
While, if $\gamma = 2$, then $d = 2$ and $0 \le r < 2 : r = \{0,1\}$:
$r = 0 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (0)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 0.\,(4)^l = 0$, then $\theta_2^4 = (1/16, 3/16)$
$r = 1 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 1.\,(4)^l = 1$, then $\theta_3^4 = (13/16, 15/16)$.
If $\gamma = 3$, then $d = 3$ and $0 \le r < 4 : r = \{0,1,2,3\}$:
$r = 0 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 0.2^l \equiv (0)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 0.\,(4)^l = 0$, then $\theta_4^4 = (1/64, 3/64)$
$r = 1 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1)$, this implies that $z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 1.\,(4)^l = 1$, then $\theta_5^4 = (13/64, 15/64)$
$r = 2 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1,0)$, this implies that$z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 1.\,(4)^l = 4$, then $\theta_6^4 = (49/64, 51/64)$
$r = 3 = \sum_{l=0}^{\infty} r_l 2^l = \sum_{l=0}^{0} 1.2^l \equiv (1,1)$, this implies that$z = \sum_{l=0}^{\infty} z_l\,(2k)^l$
$= \sum_{l=0}^{0} 1.\,(4)^l = 5$, then $\theta_7^4 = (61/64, 63/64)$.
Now, we evaluate$F_\gamma^4(x), \forall \gamma : \gamma = 0,1,2,3$:
When, $\gamma = 0$, we obtain

$$H_{M_1^4}^4(x) = \frac{1}{|M_1^4|} \int_0^x \chi_{M_1^4}(t)dt = \int_0^x \chi_{M_1^4}(t)dt = \int_0^x 1.dt = x$$

$$F_0^{2k} = x, x \in [0,1].$$

If $\gamma = 1$, then, we have:

$$H^4_{M^4_2}(x) = \frac{1}{|M^4_2|}\int_0^x \chi_{M^4_2}(t)dt = 4\int_0^x \chi_{M^4_2}(t)dt = \begin{cases} 0, & x < 0 \\ 4x, & x \in M^4_2 \\ 1, & x > 1/4 \end{cases}$$

$$H^4_{M^4_3}(x) = \frac{1}{|M^4_3|}\int_0^x \chi_{M^4_3}(t)dt = 4\int_{3/4}^x \chi_{M^4_3}(t)dt = \begin{cases} 0, & x < 3/4 \\ 4x-3, & x \in M^4_3 \\ 1, & x > 1 \end{cases}$$

$$F^{2k}_1(x) = \begin{cases} 2x, & x \in M^4_2 \\ 2x-1, & x \in M^4_3 \\ 1/2, & x \in \theta^4_1. \end{cases}$$

When $\gamma = 2$, then, we get:

$$H^4_{M^4_4}(x) = \frac{1}{|M^4_4|}\int_0^x \chi_{M^4_4}(t)dt = 16\int_0^x \chi_{M^4_4}(t)dt = \begin{cases} 0, & x < 0 \\ 16x, & x \in M^4_4 \\ 1, & x > 1/16 \end{cases}$$

$$H^4_{M^4_5}(x) = \frac{1}{|M^4_5|}\int_0^x \chi_{M^4_5}(t)dt = 16\int_{3/16}^x \chi_{M^4_5}(t)dt = \begin{cases} 0, & x < 3/16 \\ 16x-3, & x \in M^4_5 \\ 1, & x > 1/4 \end{cases}$$

$$H^4_{M^4_6}(x) = \frac{1}{|M^4_6|}\int_0^x \chi_{M^4_6}(t)dt = 16\int_{3/4}^x \chi_{M^4_6}(t)dt = \begin{cases} 0, & x < 3/4 \\ 16x-12, & x \in M^4_6 \\ 1, & x > 13/16 \end{cases}$$

$$H^4_{M^4_7}(x) = \frac{1}{|M^4_7|}\int_0^x \chi_{M^4_7}(t)dt = 16\int_{15/16}^x \chi_{M^4_7}(t)dt = \begin{cases} 0, & x < 15/16 \\ 16x-15, & x \in M^4_7 \\ 1, & x > 1. \end{cases}$$

Then we have

$$F^4_2(x) = \begin{cases} 4x, & x \in M^4_4 \\ 1/4, & x \in \theta^4_2 \\ 4x-1/2, & x \in M^4_5 \\ 4x-5/2, & x \in M^4_6 \\ 3/4, & x \in \theta^4_3 \\ 4x-3, & x \in M^4_7. \end{cases}$$

By using the same processing above, we obtain $F^4_3(x)$. □

Recently, we used 4−adic-type Cantor-like sets to generate a system of orthogonal functions with respect to the measure on 4−adic-type Cantor-like sets and applied them on transforms information over communication channel which were used as carrier signals, [14].

$F_{\gamma}^{2k}(x)$ can be constructions in the iterative construction for the $2k-$adic-type Cantor-Lebesgue functions. This is shown in the bellow theorem.

Theorem 3.3. *Define the functions $F_0^{2k}, F_1^{2k}, F_2^{2k}, \ldots : [0,1] \to R$ recursively: let $F_0^{2k}(x) = x$ for $x \in [0,1]$ and for $\gamma \in N$ define*

$$F_{\gamma}^{4}(x) = \begin{cases} 1/2F_{\gamma-1}^{2k}(2kx), & if x \in [0, 1/2k] \\ 1/2, & if x \in (1/2k, (2k-1)/2k) \\ 1/2F_{\gamma-1}^{2k}(2kx - 2k + 1) + 1/2, & if x \in [(2k-1)/2k, 1]. \end{cases}$$

Then the following results are true: (1) The sequence of functions $\{F_{\gamma}^{2k}\}_{\gamma=0}^{\infty}$ converges uniformly to a limit $F^{2k} : [0,1] \to R$. The limit $F^{2k}(x)$, known as the $2k-$adic-type Cantor-Lebesgue functions.
(2)F^{2k} are continuous increasing functions on [0,1], with $F_{\gamma}^{2k}(0) = 0, F_{\gamma}^{2k}(1) = 1$. That satisfies$\frac{d}{dx}F^{2k}(x) = 0$, $\forall x$, in the open set [0,1]/ M^{2k}.
(3) The sequence of functions$\{F_{\gamma}^{2k}\}_{\gamma=0}^{\infty}$ defined above converges uniformly to the Devil's staircase:

$$\left\|F_{\gamma}^{2k} - D\right\|_{\infty} \to 0 \quad as \quad \gamma \to \infty.$$

Note that D is the Devil's staircase.
(4) The $2k-$adic-type Cantor-Lebesgue functions satisfy:$F^{2k}\left(\mathrm{M}^{2k}\right) = [0,1]$.

Proof. (1): $\forall x \in [0,1]$ and $\gamma \in \mathbf{N}$. Note that $\left|F_1^{2k}(x) - F_0^{2k}\right| \leq \frac{1}{2}$ and $\left|F_2^{2k}(x) - F_1^{2k}\right| \leq \frac{1}{4}\, \forall x \in [0,1]$. Assume, for $\gamma \geq 2$, that $\left|F_{\gamma}^{2k}(x) - F_{\gamma}^{2k}\right| \leq \frac{1}{2^{\gamma}}$, $x \in [0,1]$. Now, consider the expression $\left|F_{\gamma+1}^{2k}(x) - F_{\gamma}^{2k}\right|$. We will apply the three-part piecewise recursive definition of F_{γ}^{2k} to prove the induction step.
Case(1): If $0 \leq x < \frac{1}{2k}$, then
$\left|F_{\gamma+1}^{2k}(x) - F_{\gamma}^{2k}(x)\right| = \left|\frac{1}{2}F_{\gamma}^{2k}(2kx) - \frac{1}{2}F_{\gamma-1}^{2k}(2kx)\right|$
$= \frac{1}{2}\left|F_{\gamma}^{2k}(2kx) - F_{\gamma-1}^{2k}(2kx)\right| \leq \frac{1}{2^{\gamma+1}}$.
Case(2): If $\frac{1}{2k} \leq x < \frac{2k-1}{2k}$, then

$$\left|F_{\gamma+1}^{2k}(x) - F_{\gamma}^{2k}(x)\right| = \left|\frac{1}{2}F_{\gamma}^{2k}(x) - F_{\gamma-1}^{2k}(x)\right| = 0 \leq \frac{1}{2^{\gamma+1}}.$$

Case(3): If $\frac{2k-1}{2k} \leq x < 1$, then

$$\left|F_{\gamma+1}^{2k}(x) - F_{\gamma}^{2k}(x)\right| =$$

$$\left|\tfrac{1}{2}F_{\gamma}^{2k}(2kx - (2k-1)) + \tfrac{1}{2}) - \left[\tfrac{1}{2}F_{\gamma-1}^{2k}(2kx - (2k-1)) + \tfrac{1}{2})\right]\right| =$$

$$\frac{1}{2}\left|F_{\gamma}^{2k}(2kx - (2k-1)) - F_{\gamma-1}^{2k}(2kx - (2k-1))\right| \leq \frac{1}{2^{\gamma+1}}.$$

For all $\gamma \in \mathbf{N}$, let $1 \leq s < \gamma$. Then

$$\left|F_s^{2k}(x) - F_\gamma^{2k}(x)\right| =$$

$$\left|(F_s^{2k}(x) - F_{s+1}^{2k}(x)) + (F_{s+1}^{2k}(x) - F_{s+2}^{2k}(x) + \ldots + (F_{\gamma-1}^{2k}(x) - F_\gamma^{2k}(x))\right| \leq$$

$$\left|F_s^{2k}(x) - F_{s+1}^{2k}(x)\right| + \left|(F_{s+1}^{2k}(x) - F_{s+2}^{2k}(x))\right| + \ldots + \left|F_{\gamma-1}^{2k}(x) - F_\gamma^{2k}(x)\right| \leq$$

$\frac{1}{2^{s+1}} + \frac{1}{2^{s+2}} + \ldots + \frac{1}{2^\gamma}$

$$< \frac{1}{2^{s+1}} + \frac{1}{2^{s+2}} + \ldots = \frac{1}{2^s}.$$

Thus, given any $\epsilon > 0$, we can find ω such that $\frac{1}{2^\omega} < \epsilon$. Also for every $\gamma > s > \omega$, then by the previous calculation, we get

$$\left|F_s^{2k}(x) - F_\gamma^{2k}(x)\right| < \frac{1}{2^s} \leq \frac{1}{2^\omega} < \epsilon.$$

By Cauchy's sense for uniform convergence, the sequence $\left\{F_\gamma^{2k}\right\}_{\gamma=0}^\infty$ of continuous functions converges uniformly to a limit function.
(2)Since each F_γ^{2k} is continuous and sequence $\left\langle F_\gamma^{2k}\right\rangle \to F^{2k}$ uniformly, it follows that F^{2k} is continuous. Because $F_\gamma^{2k}(0) = 0$, $F_\gamma^{2k}(1) = 1$ $\forall\gamma$, we have $F^{2k}(0) = 0$, and $F^{2k}(1) = 1$. Let $0 \leq \beta < \alpha < 1$. Since F_γ^{2k} are increasing, $F_\gamma^{2k}(\beta) \leq F_\gamma^{2k}(\alpha)$. By using the order limit theorem $F^{2k}(\beta) \leq F^{2k}(\alpha)$, and also F^{2k} are increasing. Because F^{2k} are constant on each open intervals whose union form [0,1]$/M^{2k}$. Then

$$\frac{d}{dx}F^{2k}(x) = 0, \forall x, \text{in the open set}[0,1]/M^{2k}.$$

(3): Let

$$\left\|F_{\gamma+1}^{2k} - F_\gamma^{2k}\right\|_\infty = \sup_{x\in[0,1]}\left|F_{\gamma+1}^{2k}(x) - F_\gamma^{2k}(x)\right| =$$

$$\frac{1}{2k-1}\sup_{x\in[0,1]}\left|F_\gamma^{2k}(x) - F_{\gamma-1}^{2k}(x)\right| = \frac{1}{2k-1}\left\|F_\gamma^{2k} - F_{\gamma-1}^{2k}\right\|_\infty.$$

Thus,

$$\left\|F_{\gamma+1}^{2k} - F_\gamma^{2k}\right\|_\infty \leq \frac{1}{2k-1}\left\|F_\gamma^{2k} - F_{\gamma-1}^{2k}\right\|_\infty \leq \ldots \leq \frac{1}{2^\gamma}\left\|F_1^{2k} - F_0^{2k}\right\|_\infty$$

$$= \frac{1}{4k\,(2k-1)^\gamma}.$$

Next, given a a sequence $\{\lambda_\gamma\}_{\gamma=0}^\infty$ in a metric space (X, d), if there exists a sequence $\{\varepsilon\}_{\gamma=0}^\infty \subset [0, \infty)$, satisfying the two conditions:
(i) $d(\lambda_\gamma, \lambda_{\gamma+1}) \leq \varepsilon_\gamma$for each$\gamma \geq 0$, and
(ii)$\sum_{\gamma=0}^\infty < \infty$, then the sequence $\{\lambda_\gamma\}_{\gamma=0}^\infty$ is a Cauchy sequence.
In our work, such sequence exists $\{\varepsilon_\gamma\}_{\gamma=0}^\infty = \frac{1}{2k(2k-1)^{\gamma+1}} \in (0, \infty]$, and

$\sum_{\gamma=0}^{\infty} \varepsilon_\gamma = \frac{1}{2k} \sum_{\gamma=0}^{\infty} \frac{1}{2k(2k-1)^{\gamma+1}} < \infty$(since geometric series with ratio $\frac{1}{2k} < 1$). Thus the sequence of functions $\left\{F_\gamma^{2k}\right\}_{\gamma=0}^{\infty}$ is a Cauchy sequence therefore it is convergent.

(4):First, we compute

$$F^{2k}([0,1]) = F^{2k}\left(M^{2k}\right) \cup F^{2k}\left([0,1]/M^{2k}\right).$$

Since $F^{2k}\left([0,1]/M^{2k}\right)$ is countable (taking the values $2^{-d}(2r+1)$), then it's measure is zero.
$\mu\left(F^{2k}\left(M^{2k}\right) \cup F^{2k}\left([0,1]/M^{2k}\right)\right) \leq \mu\left(F^{2k}\left(M^{2k}\right)\right) + \mu\left(F^{2k}\left([0,1]/M^{2k}\right)\right)$ and $\mu\left(F^{2k}\left([0,1]\right)\right) = 1$, therefore $\mu\left(F^{2k}\left(M^{2k}\right)\right) \geq 1$. Because

$$\left(F^{2k}\left(M^{2k}\right)\right) \subset [0,1] \Rightarrow \mu\left(F^{2k}([0,1])\right) \leq \mu\left([0,1]\right) = 1,$$

therefore $\mu\left(F^{2k}\left(M^{2k}\right)\right) = 1$. Since M^{2k} are compact and F^{2k} are continuous, we get $\left(F^{2k}\left(M^{2k}\right)\right)$ are compact (every continuous image of compact set is compact also it is closed and bounded) $\Rightarrow \left(F^{2k}\left(M^{2k}\right)\right)$ are closed and bounded. We proceed via contraction and since

$$(0,1)/\left(F^{2k}\left(M^{2k}\right)\right) \neq \phi.$$

Then because $(0,1)/\left(F^{2k}\left(M^{2k}\right)\right)$ is open, it follow that there exist $x \in (0,1)/\left(F^{2k}\left(M^{2k}\right)\right)$ and an open interval (ρ,σ), $x \in (\rho,\sigma)$, satisfy $x \in (\rho,\sigma) \subset (0,1)/\left(F^{2k}\left(M^{2k}\right)\right)$. Therefore, we have

$$F^{2k}\left(M^{2k}\right) \cup (\rho,\sigma) = \mu\left(F^{2k}\left(M^{2k}\right)\right) + \mu\left((\rho,\sigma)\right) = 1 + \rho - \sigma \qquad (3.1)$$

since $(\rho,\sigma) \cap F^{2k}\left(M^{2k}\right) = \phi$.
On other hand $F^{2k}\left(M^{2k}\right) \cup (\rho,\sigma) \subset [0,1]$, yields: $\mu\left(F^{2k}\left(M^{2k}\right)\right) \cup (\rho,\sigma) \leq \mu([0,1]) = 1$, which contradicts eq.(3.1). We have shown that $\mu\left(F^{2k}\left(M^{2k}\right)\right) = [0,1]$, and since $F^{2k}(0) = 0, F^{2k}(1) = 1, \{0,1\} \subset M^{2k}$, we conclude that $F^{2k}\left(M^{2k}\right) = [0,1]$. □

Theorem 3.4. *The $2k-$ adic-type Cantor-Lebesgue functions F^{2k} are an increasing continuous functions:$F^{2k} : [0,1] \to [0,1]$. It's derivative exists on the open set $\theta_{2^{d-1}+r}^{2k}$, $0 \leq r < 2^{d-1}, 1 \leq d \leq \gamma$ and $\left(F^{2k}(x)\right)' = 0$, for each $x \in \theta_{2^{d-1}+r}^{2k}$.*

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