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DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING A NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION WITH PROPORTIONAL DELAYS

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ABSTRACT. In this article Differential Transformation Method (DTM) has been used to solve neutral functional-differential equations with proportional delays. The method can simply be applied to many linear and nonlinear problems and is capable of reducing the size of computational work while still providing the series solution with fast convergence rate. Exact solutions can also be obtained from the known forms of the series solutions. The results show that the method is effective, suitable, easy, practical and accurate.

Keywords : Differential transformation method, Neutral functional - Differential equations with proportional delay.
Classification 34K28, 39A13, 33E30

1. INTRODUCTION

Consider the following neutral functional-differential equation with proportional delays,

$$(y(t) + \alpha(t)\, y(p_m t))^m = \zeta y(t) + \sum_{k=0}^{m-1} \beta_k(t)\, y^k(p_k t) + h(t), \quad t \geq 0, \tag{1.1}$$

with initial conditions

$$\sum_{k=0}^{m-1} c_{ik} y^k(0) = \eta_i, \quad i = 0, 1, \ldots, m-1. \tag{1.2}$$

Here, $\alpha \; and \; \beta_k \; (k = 0, 1, \ldots, m-1)$ are given analytical functions, and $\zeta, p_k, c_{ik}, \eta_i$ denote given constants with $0 < p_k < 1 \; (k = 0, 1, \ldots, m)$. Neutral functional-differential equations with proportional delays represent a particular class of delay differential equation. Chen et al. applied the variational iteration method for solving a

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neutral functional differential equation with proportional delays[1]. Such functional-differential equations play an important role in the mathematical modeling of real world phenomena [2]. Wang et al. obtained approximate solutions for neutral delay differential equations by continuous Runge-Kutta methods [3] and one-leg θ-methods [4, 5].Obviously, most of these equations cannot be solved exactly. It is therefore necessary to design efficient numerical methods to approximate their solutions. Ishiwata et al. used the rational approximation method [6] and the collocation method [7] to compute numerical solutions of delay differential equations with proportional delays. Hu et al. [8] applied linear multistep methods to compute numerical solutions for neutral delay differential equations. Karakoç et al. obtained solutions of delay differential equations by using differential transform method [9]. The aim of this paper is to extend the differential transform method to solve neutral functional-differential equations with proportional delays and numerical results will be compared with that of analytical ones. This paper is planned as follows: in Section 2, DTM is introduced and some basic mathematical operations are found; in Section 3, DTM is applied to five initial value problems. Finally, in Section 4, the conclusion is given.

2. DIFFERENTIAL TRANSFORMATION METHOD

As in [10-20], the basic definition of the differential transformation method is given as follows:

Definition 2.1. *If $y\left(t\right)$ is analytic in the domain$T$, then it will be differentiated continuously with respect to time t,*

$$\frac{\partial^k y\left(t\right)}{\partial t^k} = \varphi\left(t,k\right),\ for\ all\ t \in T \tag{2.1}$$

for $t = t_i$, then $\varphi\left(t,k\right) = \varphi\left(t_i,k\right)$, where k belongs to set of nonnegative integers, denoted as the K-domain. Consequently, Eq. (2.1) can be rewritten as

$$Y\left(k\right) = \varphi\left(t_i,k\right) = \left[\frac{\partial^k y\left(t\right)}{\partial t^k}\right]_{t=t_i},\ y\left(t\right)\ at\ t = t_i \tag{2.2}$$

where $Y\left(k\right)$ is called the spectrum of $y\left(t\right)\ at\ t = t_i$,

Definition 2.2. *If $y\left(t\right)$ can be described by it is erased. Taylor$'sseries, then y(t)$ can be shown as*

$$y\left(t\right) = \sum_{k=0}^{\infty}\left[\frac{\left(t-t_i\right)^k}{k!}\right]Y\left(k\right). \tag{2.3}$$

Eq. (2.3) is called the inverse of $y\left(t\right)$, with the symbol D denoting the differential transformation process. Upon combining (2.2) and (2.3), we attain

$$y\left(t\right) = \sum_{k=0}^{\infty}\left[\frac{\left(t-t_i\right)^k}{k!}\right]Y\left(k\right) \equiv D^{-1}Y(k). \tag{2.4}$$

Using the differential transformation, a differential equation in the domain of interest can be transformed to an algebraic equation in the K -domain and the $y\left(t\right)$ can be *obtained by finite-term it is erased.Taylor's series plus a remainder, as*

$$y(t)=\sum_{k=0}^{n}\left[\frac{(t-t_i)^k}{k!}\right]Y(k)+R_{n+1}(t) \tag{2.5}$$

From the definitions (2.2) and (2.4), it is easy to obtain the following mathematical operations:

Table 1 *Operations of the one dimensional differential transform*

Original function	*Transformed function*
$u(x)=f(x)\mp g(x)$	$U(k)=F(k)\mp G(k)$
$u(x)=\xi f(x)$	$U(k)=\xi F(k)$
$u(x)=\frac{df(x)}{dx}$	$U(k)=(k+1)F(k+1)$
$u(x)=\frac{d^n f(x)}{dx^n}$	$U(k)=\frac{(k+n)!}{k!}F(k+n)$
$u(x)=x^r$	$U(k)=\delta(k-r)=\begin{cases}1, & k=r,\\ 0, & otherwise\end{cases}$
$u(x)=f(x)g(x)$	$U(k,h)=\sum_{m=0}^{k}F(m)G(k-m)$

Theorem 2.1(see [10]) *if* $y(x)=y_1(x)y_2(x)\dots y_{n-1}(x)y_n(x)$ *, then*

$$Y(k)=\sum_{k_{n-1=0}}^{k}\sum_{k_{n-2=0}}^{k_{n-1}}\cdots\sum_{k_{2=0}}^{k_3}\sum_{k_{1=0}}^{k_2}Y_1(k_1)Y_2(k_2-k_1)\dots Y_{n-1}(k_{n-1}-k_{n-2})Y_n(k-k_{n-1}).$$

Theorem 2.2(see [11]) *if* $y(x)=g(x+a)$*, then*

$$Y(k)=\sum_{s_{1=k}}^{N}\begin{pmatrix}s_1\\ k\end{pmatrix}a_1^{s_1-k}G(s_1)\quad for\ N\to\infty.$$

Theorem 2.3(see [9]) *if* $y(x)=g\left(\frac{x}{a}\right)$ *, then*

$$Y(k)=\sum_{s_{1=k}}^{N}\frac{(-1)^{s_1-k}(a-1)^{s_1-k}}{a_{s_1}}\begin{pmatrix}s_1\\ k\end{pmatrix}a_1^{s_1-k}G(s_1)\quad for\ N\to\infty$$

3. APPLICATIONS

In this section, we will present three examples. To illustrate the method for linear and non-linear systems of delay differential equations.

Example 3.1. *We initially consider following first order neutral functional differential equation with proportional delay[1-4]*

$$y'(t)=-y(t)+\frac{1}{2}y\left(\frac{t}{2}\right)+\frac{1}{2}u'\left(\frac{t}{2}\right). \tag{3.1}$$

with initial condition,

$$y(0)=1. \tag{3.2}$$

applying the differential transform of (3.1) and (3.2) , then

$$(k+1)\,Y\,(k+1) = -Y\,(k) + \frac{1}{2^{k+1}}Y\,(k) + \frac{(k+1)}{2^{k+1}}Y\,(k+1)\,. \tag{3.3}$$

$$Y\,(0) = 1. \tag{3.4}$$

Substituting (3.4) in (3.3), the series following solution form can be obtained

$$y\,(t) = \sum_{k=0}^{\infty} Y\,(k)\,t^k = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right). \tag{3.5}$$

this series has the closed solution form e^{-t}, *which is the exact solution of neutral functional differential equation with proportional delay.*

Example 3.2. *We consider following first order neutral functional differential equation with proportional delay[1-4]*

$$y'\,(t) = -y\,(t) + 0.1y\,(0.8t) + 0.5u'\,(0.8t) + (0.32t - 0.5)\,e^{-0.8t} + \mathrm{e}^{-\mathrm{t}}. \tag{3.6}$$

with initial condition,

$$y\,(0) = 0. \tag{3.7}$$

applying the differential transform of (3.6) and (3.7) , then

$$(k+1)\,Y\,(k+1) = -Y\,(k) + \frac{1}{10\left(\frac{5}{4}\right)^k}Y\,(k) + \frac{(k+1)}{2\left(\frac{5}{4}\right)^k}Y\,(k+1)$$

$$+0.32\sum_{r=0}^{k} \frac{Dirac\,(r-1)\,(-0.8)^{k-r}}{(k-r)!} - \frac{(-0.8)^k}{2k!} + \frac{(-1)^k}{k!}. \tag{3.8}$$

$$Y\,(0) = 0. \tag{3.9}$$

Substituting (3.9)in (3.8), the series following solution form can be obtained

$$y\,(t) = \sum_{k=0}^{\infty} Y\,(k)\,t^k = \left(t - t^2 + \frac{t^3}{2!} - \frac{t^4}{3!} + \frac{t^5}{4!} + \ldots\right). \tag{3.10}$$

this series has the closed solution form te^{-t}, *which is the exact solution of neutral functional differential equation with proportional delay.*

Example 3.3. *We consider following the second order neutral functional differential equation with proportional delay[1-4]*

$$y''(t) = \frac{3}{4}y\,(t) + y\left(\frac{t}{2}\right) + y'\left(\frac{t}{2}\right) + \frac{1}{2}y''\left(\frac{t}{2}\right) - t^2 - t + 1. \tag{3.11}$$

with initial condition,

$$y(0) = 0, y'(0) = 0. \tag{3.12}$$

applying the differential transform of (3.11) and (3.12) , then

$$
\begin{aligned}
&(k+1)(k+2)Y(k+2) = -\frac{3Y\left(k\right)}{4} + \frac{1}{2^k}Y\left(k\right) + \frac{(k+1)}{2^{k+1}}Y\left(k+1\right) \\
&+\frac{\left(k+1\right)\left(k+2\right)}{2^{k+1}}Y\left(k+2\right) - Dirac\left(k-2\right) \\
&-Dirac\left(k-1\right) + Dirac(k). && (3.13) \\
&Y(0) = 0. && (3.14)
\end{aligned}
$$

Substituting (3.14) in (3.13), the series following solution form can be obtained

$$y\left(t\right) = \sum_{k=0}^{\infty} Y\left(k\right) t^k = t^2. \tag{3.15}$$

this series has the closed solution form t^2, which is the exact solution of neutral functional differential equation with proportional delay.

Example 3.4. *We consider following third- order neutral functional differential equation with proportional delay[1-4]*

$$y'''\left(t\right) = y\left(t\right) + y'\left(\frac{t}{2}\right) + y''\left(\frac{t}{3}\right) + \frac{1}{2}y'''\left(\frac{t}{4}\right) - t^4 - \frac{t^3}{2} - \frac{4t^2}{3} + 21t. \tag{3.16}$$

with initial condition,

$$y\left(0\right) = 0, y'\left(0\right) = 0,\ y''\left(0\right) = 0. \tag{3.17}$$

applying the differential transform of (3.16) and (3.17) , then

$$
\begin{aligned}
&\left(k+1\right)\left(k+2\right)\left(k+3\right)Y\left(k+3\right) = Y\left(k\right) + \frac{\left(k+1\right)}{2^k}Y\left(k\right) \\
&+\frac{\left(k+1\right)\left(k+2\right)}{3^k}Y\left(k+2\right) + \frac{\left(k+1\right)\left(k+2\right)\left(k+3\right)}{2^{2k+1}}Y\left(k+3\right) \\
&-Dirac\left(k-4\right) - 0.5Dirac\left(k-3\right) - \frac{4Dirac\left(k-2\right)}{3} \\
&+21Dirac(k-1). && (3.18) \\
&Y\left(0\right) = 0, Y\left(1\right) = 0,\ Y\left(2\right) = 0. && (3.19)
\end{aligned}
$$

Substituting (3.19) in (3.18), the series following solution form can be obtained

$$y\left(t\right) = \sum_{k=0}^{\infty} Y\left(k\right) t^k = t^4. \tag{3.20}$$

this series has the closed solution form t^4, which is the exact solution of neutral functional differential equation with proportional delay.

Example 3.5. *We consider following second- order neutral functional differential equation with proportional delay[1-4]*

$$y''(t) = y'\left(\frac{t}{2}\right) - \frac{1}{2}y''\left(\frac{t}{2}\right) + 2. \tag{3.21}$$

with initial condition,

$$y\left(0\right) = 1, y'\left(0\right) = 0. \tag{3.22}$$

applying the differential transform of (3.21)and (3.22) , then

$$\begin{aligned}
&(k+1)\left(k+2\right)\left(k+3\right)Y\left(k+3\right) = Y\left(k\right) + \frac{(k+1)}{2^k}Y\left(k\right) \\
&+\frac{(k+1)\left(k+2\right)}{3^k}Y\left(k+2\right) + \frac{(k+1)\left(k+2\right)\left(k+3\right)}{2^{2k+1}}Y\left(k+3\right) \\
&-Dirac\left(k-4\right) - 0.5Dirac\left(k-3\right) \\
&-\frac{4Dirac\left(k-2\right)}{3} + 21Dirac(k-1). &(3.23) \\
&Y(0) = 1,\ Y(1) = 0. &(3.24)
\end{aligned}$$

Substituting (3.24) in (3.23), the series following solution form can be obtained

$$y\left(t\right) = \sum_{k=0}^{\infty} Y\left(k\right)t^k = 1 + t^2. \tag{3.25}$$

this series has the closed solution form$1 + t^2$, which is the exact solution of neutral functional differential equation with proportional delay.

4. CONCLUSION

In this paper, we used the Differential Transformation Method (DTM) for finding the exact and approximate solutions of neutral functional-differential equations with proportional delays. DTM can be applied to many complicated linear and strongly nonlinear ordinary or partial differential equations and systems of partial differential equations and does not require linearization, discretization or perturbation. The obtained results show that this method is powerful and meaningful for solving neutral functional-differential equations with proportional delays.

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