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Constrained bi-matrix games with intuitionistic fuzzy goals

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Abstract. In this paper a constrained bi-matrix game with intuitionistic fuzzy goals is studied. The Nash-equilibrium solution is defined for the problem, and some necessary and sufficient conditions for the existence of equilibrium solution are stated and proved. Then, a mathematical programming model is given, and it is proved that its optimal solution is an equilibrium solution of this bi-matrix game. Finally, two numerical examples are presented to illustrate the proposed model.

Keywords: Bi-matrix game, Constrained bi-matrix game, Intuitionistic fuzzy set, Intuitionistic fuzzy goal, Nash equilibrium solution.

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1. Introduction

The study of game problems in uncertain environments is one of the topics that has attracted the attention of researchers. In recent years, many researchers have focused on matrix and bi-matrix games in uncertain environments, some of which are reviewed here, briefly.

Nayak and Pal [33] considered bi-matrix games with interval payoffs and defined a Nash equilibrium solution for pure strategies based on interval number ranking methods for such games. Li [22] developed a simple and efficient linear programming method for solving matrix games in which payoffs are expressed as intervals. Brikaa et al [11] considered a multi-objective programming approach for a constrained matrix game with payoffs of fuzzy numbers. Vijay et al [41] considered a two-player zero-sum game with fuzzy goals and payoffs and solved it using a ranking function. Vijay et al [42] introduced a generalized model for a two-player zero-sum game with fuzzy objectives and fuzzy outcomes through a fuzzy relation approach and showed that it is equivalent to two semi-infinite optimization problems. Bector et al [5] studied a bi-matrix game with fuzzy payoffs and fuzzy goals, and showed that a two-player zero-sum matrix game with fuzzy objectives is equivalent to a primal-dual pair of fuzzy linear programming problems. Mi et al [28] investigated probabilistic linguistic information as inputs of a two-player zero-sum game with triangular membership functions. Seikh et al [37] investigated multi-objective matrix games with hesitant triangular fuzzy pay-offs and proposed a new method based on the concept of weighted average. Jana and Roy [20] considered a two-person game with hesitant fuzzy payoffs and its application to multiple attribute decision making. For further research, one can see, for example, [6–10, 12–14, 16–19, 21, 26, 27, 29, 35, 44].

Fuzzy sets are commonly employed for the mathematical modeling of uncertainty, where the membership degree of an element in a set is a real number in the interval [0, 1], and its non-membership degree is simply the complement with respect to 1. In other words, the sum of an element's membership degree and non-membership degree equals 1 — for instance, representing how strongly someone accepts or rejects the proposition "Ali is young." However, there are cases in which uncertainty arises not only from acceptance or rejection, but also from hesitancy or doubt resulting from lack of information. Consider, for example, a legislative vote in a parliament. Besides supporters and opponents, there may be individuals who remain neutral or undecided — those who abstain or fail to cast a definitive "yes" or "no"—. Clearly, if 90 percent of representatives participate in the vote and 70 percent of those vote in favor, one cannot assert that 70 percent of all representatives support the bill and 30 percent oppose it. Because the position of the 10 percent who abstained or did not vote remains unknown. Thus, a degree of indeterminacy or hesitation emerges, stemming from doubt, incomplete information, lack of information, or uncertainty.

Such kinds of uncertainty are common in many real-world contexts — in fields such as management, economics, law, finance, etc. — and cannot be adequately captured by classical fuzzy sets. Instead, "intuitionistic fuzzy sets," which assign to each element three parameters — membership degree, non-membership degree, and hesitation (uncertainty) degree — provide an appropriate mathematical framework for representing and modeling these forms of uncertainty. Such uncertainty may appear in many real-world decision-making problems, generally, and game-theoretic problems, especially. Some researches on game problems in intuitionistic environment are reviewed here briefly.

Li and Tu [25] proposed a new method with regard to the acceptance rate for bi-matrix games with general intuitionistic fuzzy payoffs. Naqvi [31] presented a comprehensive study of matrix games where payoffs are determined through linguistic interval-valued intuitionistic fuzzy sets. Xing and Qiu [43] considered a matrix game with triangular intuitionistic fuzzy payoffs, and showed that a solution can be obtained by solving a primal-dual pair of intuitionistic fuzzy linear programming problems by applying the accuracy function. Singla et al [39] dealt with solving an intuitionistic fuzzy bi-matrix game involving multiple options. Agarwal and Chandra [1] solved a matrix game with intuitionistic fuzzy payoffs using security strategies. They considered the membership and non-membership functions in two forms, triangular and trapezoidal, and found an efficient solution. Li et al [23], Li et al [24] and Seikh et al [38] converted a matrix game with payoffs of Atanassov's triangular intuitionistic fuzzy numbers to bi-objective fuzzy linear programming models and solved them using a different ranking function. Nan et al [30] transformed matrix games with trapezoidal intuitionistic fuzzy payoffs into bi-objective linear programming models and solved them using a ranking function. Aggarwal et al [2] proposed an application of linear programming with intuitionistic fuzzy sets to a matrix game with an intuitionistic fuzzy goal. Verma and Aggarwal [40] presented a study on matrix games with payoffs represented by Linguistic intuitionistic fuzzy Numbers (LIFN). Dong and Wan [15] showed that the second type intuitionistic fuzzy matrix games (T2IVIFS) can solve the game problem more effectively and flexibly than the two types (IVIFS) and T2IFS. They also gave an example of the development of the new energy vehicle industry.

Nayak and Pal [34] studied a bi-matrix game with intuitionistic fuzzy

goals and proposed a mathematical programming model to obtain a solution of the game. However, they considered a special case of intuitionistic fuzzy numbers: in their research, the non-membership degrees of the intuitionistic fuzzy goals are the complement of the membership degrees with respect to one i.e. the uncertainty degree which is defined as 1-(membership degree+non-membership degree) is zero. Therefore, the fuzzy goals in their research are actually ordinary fuzzy numbers. Furthermore, sometimes in real word game problems the players have some limitations to choose their strategies (e.g. because of budget limitation). So the problem is a constrained game.The constrained bi-matrix games with intuitionistic fuzzy goals has not been treated in prior literature to the best of the authors' knowledge. So, we propose a method to find a solution for this type of game.
The remainder of the paper is organized as follows: In Section 2, some preliminaries about intuitionistic fuzzy sets, intuitionistic fuzzy optimization, and game theory are given. In Section 3, the constrained bi-matrix game with intuitionistic fuzzy goals is studied. In Section 4, two numerical examples are solved using the proposed method. Finally, Section 5 is devoted to concluding remarks.

2. Preliminaries

In this section, we review some basic notions of intuitionistic fuzzy sets, intuitionistic fuzzy optimization, and bi-matrix game.

2.1. **Intuitionistic fuzzy sets.** Let X be a nonempty set. An intuitionistic fuzzy set (IFS) $\tilde{A}$ in X is represented by $\{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle | x \in X\}$, where the functions $\mu_{\tilde{A}}, \nu_{\tilde{A}} : X \to [0,1]$ are respectively, the membership and non-membership functions of $\tilde{A}$. For every element $x \in X$, $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ are respectively the membership and non-membership degree of the element x to the set $\tilde{A}$ and $0 \leq \mu_{\tilde{A}}(x)+\nu_{\tilde{A}}(x) \leq 1$. Furthermore, $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ is called the intuitionistic fuzzy index or uncertainty degree of x in $\tilde{A}$. Obviously, we have $\pi_{\tilde{A}} : X \to [0,1]$.
Clearly, if $\pi_(x) = 0$ for all $x \in X$, Then $\tilde{A}$ is an ordinary fuzzy set. Therefore, the intuitionistic fuzzy sets are generalization of ordinary fuzzy sets.
For two IFSs, $\tilde{A}$ and $\tilde{B}$, three notions $\subset$, $\cup$ and $\cap$ are defined as follows [3]:

(1) $\tilde{A} \subset \tilde{B}$ whenever $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x) \quad \forall x \in X$,
(2) $\tilde{A} \cup \tilde{B} = \{(x, min(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)), max(\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)))|x \in X\}$,
(3) $\tilde{A} \cap \tilde{B} = \{(x, max(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)), min(\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)))|x \in X\}$.

2.2. **Intuitionistic fuzzy optimization.** An optimization problem can be written as follows:

$$\begin{aligned} \max \quad & f(x) \\ s.t. \quad & x \in X = \{x \in \mathbb{R}^n | g(x) \leq 0, h(x) = 0\} \end{aligned}$$

where $g(x) = (g_1(x), g_2(x), ..., g_{k_1}(x))$ and $h(x) = (h_1(x), h_2(x), ..., h_{k_2}(x))$. 0 means a vector whose elements are all zero.

Sometimes, instead of solving the above problem, we want to solve an intuitionistic fuzzy version of the problem. A simple case of such a problem which we need in this paper is an intuitionistic fuzzy linear programming ($IFLP$) problem as follows:

$$\begin{aligned} (IFLP) : \widetilde{\max} \quad & z = cx \\ s.t. \quad & Nx \geq b \\ & \sum_{i=1}^{n} x_i = 1 \\ & x \geq 0. \end{aligned}$$

In ($IFLP$) problem, $c = (c_1, c_2, ..., c_n)$ and $x = (x_1, x_2, ..., x_n)^T$ are n-dimensional row and column vectors respectively, $N \in \mathbb{R}^{p \times n}$ and $b = (b_1,\ b_2,\ \ldots\ ,\ b_p)^T$. $\widetilde{\max}$ is the intuitionistic version of max, which means that the objective function should be greater than or equal to some real number, z_0 as much as possible, in an intuitionistic sense.

By considering the aspiration level z_0 for the objective function of the problem ($IFLP$), it can be rewritten as the following intuitionistic fuzzy decision making ($IFDM$) problem:

$$\begin{aligned} (IFDM) : \ & \text{Find } x \text{ such that} \\ & cx \,\widetilde{\geq}\, z_0 \\ & Nx \geq b \\ & R^T x = 1 \\ & x \geq 0, \end{aligned}$$

where R is an n-dimensional column matrix whose elements are all one. The inequality $\widetilde{\geq}$ is the intuitionistic version of $\geq$. To manipulate the intuitionistic fuzzy inequality $\widetilde{\geq}$, we can use some membership and non-membership functions such as linear, exponential, hyperbolic and so on, which are used in the case of usual fuzzy inequalities [36]. Suppose that the intuitionistic fuzzy goals of the players could be modeled by linear membership and non-membership functions. The linear membership and non-membership functions of intuitionistic fuzzy inequality $cx \,\widetilde{\geq}\, z_0$

are defined as follows:

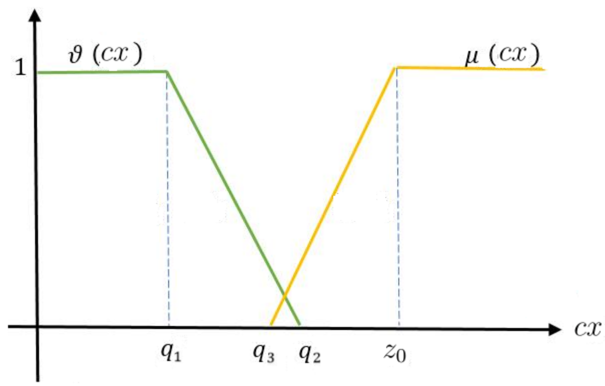

FIGURE 1. Membership and non-membership functions of intuitionistic fuzzy inequality $cx \overset{\sim}{\geq} z_0$

$$\mu(cx) = \begin{cases} 0 & cx \leq q_3 \\ \frac{cx-q_3}{z_0-q_3} & q_3 \leq cx \leq z_0 \\ 1 & cx \geq z_0 \end{cases}$$

and

$$\nu(cx) = \begin{cases} 1 & cx \leq q_1 \\ \frac{q_2-cx}{q_2-q_1} & q_1 \leq cx \leq q_2 \\ 0 & cx \geq q_2 \end{cases}$$

in which in order to guarantee the conditions $0 \leq \mu(z) + \nu(z) \leq 1$ for each z, we must have $q_1 \leq q_3$ and $q_2 \leq z_0$.

The values z_0 and q_3 are respectively the smallest value of cx whose membership degree is 1, which is completely acceptable, and the largest value of cx whose membership degree is zero, which may be unacceptable. Similarly, q_1 is the largest value of cx whose non-membership degree is one which is completely unacceptable, and q_2 is the smallest value of cx whose non-membership degree is zero, which may be acceptable. These four values are given by experts.

To solve the problem $(IFDM)$, one must simultaneously maximize the membership degree and minimize the non-membership degree. Therefore, we face to a bi-objective programming problem. Which maximize $\mu(z)$ and minimaze $\nu(z)$, are equivalently maximize $-\nu(z)$. This problem, can be converted tothe following single-objective problem using weighted-sum method [36] with eqaul weights for $\mu(z)$ and $-\nu(z)$:

$$
\begin{aligned}
\max \quad & \mu(cx) - \nu(cx) \\
s.t. \quad & Nx \geq b \\
& \mu(cx) \geq \nu(cx) \qquad (2.1) \\
& \mu(cx) \geq 0 \qquad (2.2) \\
& \nu(cx) \geq 0 \qquad (2.3) \\
& R^T x = 1 \\
& x \geq 0.
\end{aligned}
$$

The constraint $\mu(cx) \geq \nu(cx)$ is added to the model to prevent obtaining the undesirable solutions, in which the membership degree is less than the non-membership degree. Because a point with membership degree less than nonmembership degree is often not a good choice. However, if the decision maker could accept the solution which violate this condition, this constraint can be removed from the problem.
Note that according to the constraints (2.1) and (2.3), the constraint (2.2) is guaranteed. Therefore, it is redundant, and can be removed from the model.

2.3. **Bi-matrix game problem.** In this subsection, we introduce some basic notions of bi-matrix games. Consider a bi-matrix game with two Players 1 and 2. Suppose the number of strategies of Player 1 and 2 are m and n, respectively. We denote the set of pure strategies of Players 1 and 2 by $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$, respectively. If Player 1 chooses the pure strategy $i \in I$ and Player 2 chooses the pure strategy $j \in J$, the payoff of Players 1 and 2 will be a_{ij} and b_{ij}, respectively. This game is defined by the matrices $A, B \in \mathbb{R}^{m\times n}$ where $A = [a_{ij}]$ and $B = [b_{ij}]$. The aim of the two players is to maximize their payoffs. The players choose their strategies rationally, Player 1 seeks to maximize the minimum amount he can get. In other words, he chooses a strategy to $maximize \min_{j\in J} a_{ij}$ over I. Similarly, Player 2 chooses a strategy to $maximize \min_{i\in I} b_{ij}$ over J. A mixed strategy of Player 1, which is a probability distribution over the set of his pure strategeis, is a vector $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$ such that $x \geq 0$ and $\sum_{i=1}^m x_i = 1$. A mixed strategy of Player 2 is defined similarly. The sets of mixed strategies of Players 1 and 2, denoted by X and Y respectively, are

$$
\begin{aligned}
X &= \{x \in \mathbb{R}^m | R_1^T x = 1, \quad x \geq 0\}, \\
Y &= \{y \in \mathbb{R}^n | R_2^T y = 1, \quad y \geq 0\}.
\end{aligned}
$$

Here, R_1^T and R_2^T are m-dimensional and n-dimensional row vectors with components of one, respectively.

If Player 1 chooses the mixed strategy x and Player 2 chooses the mixed strategy y, the expected payoffs of Players 1 and 2 are $x^T Ay$ and $x^T By$, respectively, which they seek to maximize. We denote by $BiG(X, Y, A, B)$ the bi-matrix game whose payoff matrices of Players 1 and 2 are A and B and the sets of mixed strategies of Players 1 and 2 are X and Y, respectively.

Definition 2.1. [4] The pair of strategies $(x^*, y^*) \in X \times Y$ which satisfies

$$\begin{cases} x^T Ay^* \le x^{*T} Ay^* & \forall x \in X \\ x^{*T} By \le x^{*T} By^* & \forall y \in Y \end{cases}$$

is called a Nash equilibrium solution for the game $BiG(X, Y, A, B)$. x^* and y^* are the equilibrium strategies for Players 1 and 2, respectively.

Each bi-matrix game $BiG(X, Y, A, B)$ has at least one Nash equilibrium solution is mixed strategies [32] which can be obtained by solving a mathematical programming problem, given in the following theorem.

Theorem 2.2. *[26] Let $BiG(X, Y, A, B)$ be a bi-matrix game. The pair (x^*, y^*) is a Nash equilibrium solution of $BiG(X, Y, A, B)$ if and only if it is an optimal solution to the following mathematical programming problem:*

$$\begin{aligned} (P_1): \quad \max_{x,y,a,b} \quad & x^T(A + B)y - a - b \\ s.t. \quad & Ay - aR_1^T \le 0 \\ & B^T x - bR_2^T \le 0 \\ & R_1^T x = 1 \\ & R_2^T y = 1 \\ & x \ge 0, \quad y \ge 0 \end{aligned}$$

where a and b are unknown scalars. The optimal values of a and b are the expected payoffs of Players 1 and 2, respectively, and the optimal value of the objective function of the problem (P_1) is zero.

2.4. **Constrained Bi-matrix games.** Sometimes the players have some constraints to choose their strategies (e.g., because of budget limitations). Such game problems are called constrained bi-matrix games. Formally, a constrained bi-matrix game is represented as $CBiG(\bar{X}, \bar{Y}, A, B)$, where the sets of strategies of Players 1 and 2 are

$$\bar{X} = \{x \in \mathbb{R}^m | N_1 x \geq b_1, \quad R_1^T x = 1, \quad x \geq 0\},$$
$$\bar{Y} = \{y \in \mathbb{R}^n | N_2 y \geq b_2, \quad R_2^T y = 1, \quad y \geq 0\}$$

respectively, A, B are their payoff matrices, $N_1 \in \mathbb{R}^{p \times m}$, $N_2 \in \mathbb{R}^{q \times n}$ are the constraint coefficient matrices of players 1 and 2, respectively, and $b_1 = (b_{11}, b_{12}, \ldots, b_{1p})^T$, $b_2 = (b_{21}, b_{22}, \ldots, b_{2q})^T$ are the right-hand side value vectors of the constraints of players 1 and 2, respectively.

Definition 2.3. The pair of strategies $(x^*, y^*) \in \bar{X} \times \bar{Y}$ is a Nash equilibrium solution of the constrained game $CBiG(\bar{X}, \bar{Y}, A, B)$ if

$$\begin{cases} x^T A y^* \leq x^{*T} A y^* & \forall x \in \bar{X}, \\ x^{*T} B y \leq x^{*T} B y^* & \forall y \in \bar{Y}. \end{cases}$$

x^* and y^* are the equilibrium strategies for Players 1 and 2, respectively.

3. Constrained bi-matrix games with intuitionistic fuzzy goals

Sometimes in real-world situations, due to ambiguities and lack of information, the preferences of decision makers (Players) are not precise, and through certain interactions, we find that their preferences have to be modeled in intuitionistic fuzzy environment, instead of fuzzy or some other uncertain environments. For example, consider the problem of competitive investment portfolio management: Two investment firms (players) allocate funds among assets. Their profit goals are intuitionistic fuzzy due to market volatility. Each firm also faces budgetary and regulatory constraints. We want to construct a mathematical model to find the equilibrium investment strategy that maximizes each player's acceptance degree and minimizes rejection of their target profit under these constraints. For another example, consider the problem of advertising budget allocation between rivals: Two competing companies allocate advertising budgets to media channels. The resulting market share payoff is an intuitionistic fuzzy goal due to uncertain consumer behavior, subject to campaign budget and minimum exposure constraints. The model have to determine the equilibrium budget allocation.

In this section, we want to find a Nash equilibrium solution for a constrained bi-matrix game with intuitionistic fuzzy goals. We show this problem by $IFCBiG(\bar{X}, \bar{Y}, A, B)$. The sets of expected payoffs of Players 1 and 2 are as follows:

$$I_1 = \{x^T A y | x \in \bar{X}, y \in \bar{Y}\} \quad and \quad I_2 = \{x^T B y | x \in \bar{X}, y \in \bar{Y}\}.$$

Suppose that the intuitionistic fuzzy aspiration level of expected payoff for Player 1 is $\tilde{z_1}$, which is defined by the following membership and non-membership functions:

$$\mu_{\tilde{z_1}} : I_1 \rightarrow [0,1] \;\; and \;\; \nu_{\tilde{z_1}} : I_1 \rightarrow [0,1].$$

Also, suppose that the intuitionistic fuzzy aspiration level of expected payoff for Player 2 is $\tilde{z_2}$, which is defined by

$$\mu_{\tilde{z_2}} : I_2 \rightarrow [0,1] \;\; and \;\; \nu_{\tilde{z_2}} : I_2 \rightarrow [0,1].$$

Each player seeks to maximize his membership function and minimize his non-membership function.
For Player 1, we define the linear membership and non-membership functions as follows:

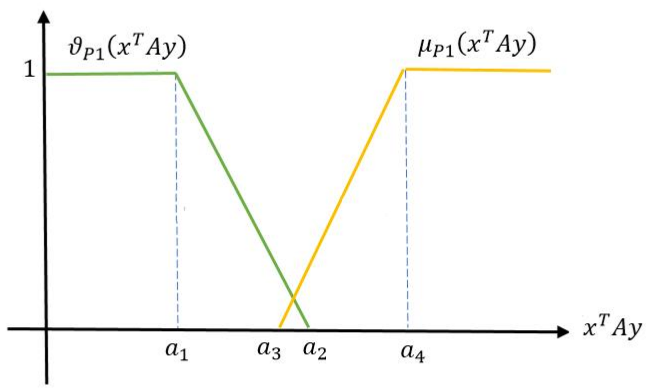

FIGURE 2. Membership and non-membership functions of expected payoff of Player 1

$$\mu_{\tilde{z_1}}(x^T Ay) = \begin{cases} 0 & x^T Ay \leq a_3 \\ \frac{x^T Ay - a_3}{a_4 - a_3} & a_3 \leq x^T Ay \leq a_4 \\ 1 & x^T Ay \geq a_4 \end{cases}$$

and

$$\nu_{\tilde{z_1}}(x^T Ay) = \begin{cases} 1 & x^T Ay \leq a_1 \\ \frac{a_2 - x^T Ay}{a_2 - a_1} & a_1 \leq x^T Ay \leq a_2 \\ 0 & x^T Ay \geq a_2 \end{cases}$$

By setting $\bar{A}^{\mu} = \frac{A}{a_4 - a_3}$, $\bar{A}^{\nu} = \frac{A}{a_2 - a_1}$, $c_{11} = \frac{a_3}{a_4 - a_3}$ *and* $c_{12} = \frac{a_2}{a_2 - a_1}$ and we have

$$\mu_{\tilde{z_1}}(x^T Ay) = \begin{cases} 0 & x^T Ay \leq a_3 \\ -c_{11} + x^T \bar{A}^{\mu} y & a_3 \leq x^T Ay \leq a_4 \\ 1 & x^T Ay \geq a_4 \end{cases}$$

and

$$\nu_{\tilde{z_1}}(x^TAy)=\begin{cases} 1 & x^TAy\le a_1\\ c_{12}-x^T\bar{A}^{\nu}y & a_1\le x^TAy\le a_2\\ 0 & x^TAy\ge a_2\end{cases}$$

. If we assume that $a_4-a_3=a_2-a_1$ then we have $\bar{A}^{\mu}=\bar{A}^{\nu}$. So

$$\mu_{\tilde{z_1}}(x^TAy)=\begin{cases} 0 & x^TAy\le a_3\\ -c_{11}+x^T\bar{A}y & a_3\le x^TAy\le a_4\\ 1 & x^TAy\ge a_4\end{cases}\tag{3.1}$$

and

$$\nu_{\tilde{z_1}}(x^TAy)=\begin{cases} 1 & x^TAy\le a_1\\ c_{12}-x^T\bar{A}y & a_1\le x^TAy\le a_2\\ 0 & x^TAy\ge a_2\end{cases}\tag{3.2}$$

where $\bar{A}=\frac{A}{a_4-a_3}=\frac{A}{a_2-a_1}$.
Similarly, for Player 2 we define

$$\mu_{\tilde{z_2}}(x^TBy)=\begin{cases} 0 & x^TBy\le d_3\\ \frac{x^TBy-d_3}{d_4-d_3} & d_3\le x^TBy\le d_4\\ 1 & x^TBy\ge d_4\end{cases}$$

and

$$\nu_{\tilde{z_2}}(x^TBy)=\begin{cases} 1 & x^TBy\le d_1\\ \frac{d_2-x^TBy}{d_2-d_1} & d_1\le x^TBy\le d_2\\ 0 & x^TBy\ge d_2\end{cases}$$

By setting $\bar{B^{\mu}}=\frac{B}{d_4-d_3},\ \bar{B^{\nu}}=\frac{B}{d_2-d_1},\ c_{21}=\frac{d_3}{d_4-d_3}\ and\ c_{22}=\frac{d_2}{d_2-d_1}$ we have

$$\mu_{\tilde{z_2}}(x^TBy)=\begin{cases} 0 & x^TBy\le d_3\\ -c_{21}+x^T\bar{B^{\mu}}y & d_3\le x^TBy\le d_4\\ 1 & x^TBy\ge d_4\end{cases}$$

and

$$\nu_{\tilde{z_2}}(x^TBy)=\begin{cases} 1 & x^TBy\le d_1\\ c_{22}-x^T\bar{B^{\nu}}y & d_1\le x^TBy\le d_2\\ 0 & x^TBy\ge d_2\end{cases}$$

If we assume that $d_4-d_3=d_2-d_1$ then we have $\bar{B^{\mu}}=\bar{B^{\nu}}$ .So

$$\mu_{\tilde{z_2}}(x^TBy)=\begin{cases} 0 & x^TBy\le d_3\\ -c_{21}+x^T\bar{B}y & d_3\le x^TBy\le d_4\\ 1 & x^TBy\ge d_4\end{cases}\tag{3.3}$$

and

$$\nu_{\tilde{z_2}}(x^TBy)=\begin{cases} 1 & x^TBy\le d_1\\ c_{22}-x^T\bar{B}y & d_1\le x^TBy\le d_2\\ 0 & x^TBy\ge d_2\end{cases}\tag{3.4}$$

where $\bar{B} = \frac{B}{d_4 - d_3} = \frac{B}{d_2 - d_1}$.

Definition 3.1. The strategy pair $(x^*, y^*) \in \bar{X} \times \bar{Y}$ is called a Nash equilibrium solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$, if

$$\begin{cases} \mu_{\tilde{z}_1}(x^T A y^*) \leq \mu_{\tilde{z}_1}(x^{*T} A y^*) \\ \nu_{\tilde{z}_1}(x^T A y^*) \geq \nu_{\tilde{z}_1}(x^{*T} A y^*) \end{cases} \qquad \forall x \in \bar{X},$$

and

$$\begin{cases} \mu_{\tilde{z}_2}(x^{*T} B y) \leq \mu_{\tilde{z}_2}(x^{*T} B y^*) \\ \nu_{\tilde{z}_2}(x^{*T} B y) \geq \nu_{\tilde{z}_2}(x^{*T} B y^*) \end{cases} \qquad \forall y \in \bar{Y}.$$

Theorem 3.2. *A strategy pair* $(x^*, y^*) \in \bar{X} \times \bar{Y}$ *satisfies*

$$\begin{cases} x^T A y^* \leq x^{*T} A y^* & \forall x \in \bar{X} \\ x^{*T} B y \leq x^{*T} B y^* & \forall y \in \bar{Y} \end{cases} \tag{3.5}$$

if and only if it satisfies

$$\begin{cases} x^T \bar{A} y^* \leq x^{*T} \bar{A} y^* & \forall x \in \bar{X} \\ x^{*T} \bar{B} y \leq x^{*T} \bar{B} y^* & \forall y \in \bar{Y} \end{cases} \tag{3.6}$$

Proof. The proof is trivial, becuse $\bar{A}$ $(\bar{B})$ is a positive scaler multipelication of A (B). □

Theorem 3.3. $(x^*, y^*) \in \bar{X} \times \bar{Y}$ *satisfies the conditions* (3.5) *(or equivalently,* (3.6)*), if and only if it satisfies the following conditions functions.*

$$\begin{cases} \mu_{\tilde{z}_1}(x^T A y^*) \leq \mu_{\tilde{z}_1}(x^{*T} A y^*) \\ \nu_{\tilde{z}_1}(x^T A y^*) \geq \nu_{\tilde{z}_1}(x^{*T} A y^*) \end{cases} \qquad \forall x \in \bar{X}, \tag{3.7}$$

and

$$\begin{cases} \mu_{\tilde{z}_2}(x^{*T} B y) \leq \mu_{\tilde{z}_2}(x^{*T} B y^*) \\ \nu_{\tilde{z}_2}(x^{*T} B y) \geq \nu_{\tilde{z}_2}(x^{*T} B y^*) \end{cases} \qquad \forall y \in \bar{Y}. \tag{3.8}$$

Proof. The result can be obtained easily by the fact that be $\mu_{\tilde{z}_1}$ and $\mu_{\tilde{z}_2}$ are increasing and $\nu_{\tilde{z}_1}$ and $\nu_{\tilde{z}_2}$ are decreasing. □

We want to obtain the solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$. Consider the fixed strategy y^0 of Player 2. The best response of Player 1 to y^0 is obtained by maximizing $\mu_{\tilde{z}_1}(x^T \bar{A} y^0)$ and minimizing $\nu_{\tilde{z}_1}(x^T \bar{A} y^0)$. However, according to (3.1) and (3.2) it is sufficient to maximize $x^T \bar{A} y^0$. Therefore, the best response of Player 1 to each fixed strategy y^0 of

Player 2 is obtained by solving the following linear programming problem:

$$
\begin{aligned}
(P_2):\max_{x}\quad & x^T\bar{A}y^0 \\
s.t.\quad & N_1x \geq b_1 \\
& 2x^T\bar{A}y^0 - c_{11} - c_{12} \geq 0 && (3.9)\\
& x^T\bar{A}y^0 \leq c_{12} && (3.10)\\
& R_1^T x = 1 \\
& x \geq 0.
\end{aligned}
$$

Similarly, the best response of Player 2 to the fixed strategy x^0 of Player 1 is obtained by solving the following linear programing problem:

$$
\begin{aligned}
(P_3):\max_{y}\quad & x^{0T}\bar{B}y \\
s.t.\quad & N_2x \geq b_2 \\
& 2x^{0T}\bar{B}y - c_{21} - c_{22} \geq 0 && (3.11)\\
& x^{0T}\bar{B}y \leq c_{22} && (3.12)\\
& R_2^T y = 1 \\
& y \geq 0.
\end{aligned}
$$

Remark 3.4. The conditions $\mu_{\tilde{z}_1}(x^T\bar{A}y) \geq 0$ and $\nu_{\tilde{z}_1}(x^T\bar{A}y) \geq 0$ for the second parts of formulas (3.1) and (3.2) imply that $x^T\bar{A}y \geq c_{11}$ and $x^T\bar{A}y \leq c_{12}$, or equivalently, $x^TAy \geq a_3$ and $x^TAy \leq a_2$. Therefore, for the consistency of the constraint (3.10), we must have $c_{11} \leq c_{12}$ or equivalently, $a_3 \leq a_2$. Otherwise, we must modify a_2 and a_3 by interaction with Player 1. Similarly, we must have $c_{21} \leq c_{22}$, or equivalently, $d_3 \leq d_2$.

Therefore, we can say that $(x^*, y^*) \in \bar{X} \times \bar{Y}$ is a Nash equilibrium solution of the game, when x^* and y^* are the optimal solution of the following pair of linear programming problems simultaneously:

$$
\begin{aligned}
x^{*T}\bar{A}y^* = \max_{x}\ & \{x^T\bar{A}y^* | x^T\bar{A}y^* \leq c_{12},\ 2x^T\bar{A}y^* \geq c_{11} + c_{12},\ N_1x \geq b_1,\\
& R_1^Tx = 1, x \geq 0\}, && (3.13)\\
x^{*T}\bar{B}y^* = \max_{y}\ & \{x^{*T}\bar{B}y | x^{*T}\bar{B}y \leq c_{22},\ 2x^{*T}\bar{B}y \geq c_{21} + c_{22},\ N_2y \geq b_2,\\
& R_2^Ty = 1,\ y \geq 0\}. && (3.14)
\end{aligned}
$$

Theorem 3.6 states that a Nash equilibrium solution of the game $IFCBiG(\bar{X}, \bar{Y}, A, B)$ can be obtained by solving a mathematical programming problem. Before the theorem, we proof the folowing lemma to facilitate the proof of the theorem.

Lemma 3.5. *The strategy pair $(x^*, y^*) \in \bar{X} \times \bar{Y}$ is a Nash equilibrium solution of the game $IFCBiG(\bar{X}, \bar{Y}, A, B)$ if and only if*

$$\begin{cases} x^{*T}\bar{A}y^*R_1 \geq \bar{A}y^* \\ x^{*T}\bar{B}y^*R_2^T \geq x^{*T}\bar{B}. \end{cases} \tag{3.15}$$

Proof. According to Theorems 3.2 and 3.3, (x^*, y^*) is a Nash equilibrium solution of the game $IFCBiG(\bar{X}, \bar{Y}, A, B)$ if and only if

$$\begin{cases} x^T\bar{A}y^* \leq x^{*T}\bar{A}y^* & \forall x \in \bar{X}, \\ x^{*T}\bar{B}y \leq x^{*T}\bar{B}y^* & \forall y \in \bar{Y}. \end{cases} \tag{3.16}$$

In the first inequality of (3.16), choose $x = (0, ..., 1, ..., 0)$ with 1 in each of the m spots, and in the second inequality of (3.16), choose $y = (0, ..., 1, ..., 0)$ with 1 in each of the n spots. Then we have $x^{*T}\bar{A}y^* \geq \bar{A}_{i.}y^*$ for all $i \in \{1, ..., n\}$, and similarly $x^{*T}\bar{B}y^* \geq x^{*T}\bar{B}_{.j}$ for all $j \in \{1, ..., m\}$ in which $\bar{A}_{i.}$ is the i-th row of $\bar{A}$ and $\bar{B}_{.j}$ is the j-th column of $\bar{B}$. In matrix form,

$$\begin{cases} x^{*T}\bar{A}y^*R_1 \geq \bar{A}y^* \\ x^{*T}\bar{B}y^*R_2^T \geq x^{*T}\bar{B}. \end{cases} \tag{3.17}$$

Conversely suppose (3.17) holds. Choose any $x \in \bar{X}$ and $y \in \bar{Y}$ and multiply the first and second inequality of (3.17) respectively, by x^T and y. Using the relations $R_1^T x = 1$ and $R_2^T y = 1$, we have

$$\begin{cases} x^T\bar{A}y^* \leq x^{*T}\bar{A}y^* & \forall x \in \bar{X}, \\ x^{*T}\bar{B}y \leq x^{*T}\bar{B}y^* & \forall y \in \bar{Y}. \end{cases}$$

Therefore, (x^*, y^*) is a Nash equilibrium solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$ according to Theorems 3.2 and 3.3. So, (x^*, y^*) is a Nash equilibrium solution if and only if (3.15) is satisfied. □

Theorem 3.6. *Let $\bar{X}$ and $\bar{Y}$ be non-empty. The following problem gives a Nash equilibrium solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$. In other words, (x^*, y^*) is a Nash equilibrium solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$ if and only if there exist the scalars u^* and v^* such that (x^*, y^*, u^*, v^*) is the* *optimal solution of the following optimization problem.*

$$(P_4): \quad \max_{x,y,u,v} \quad x^T\bar{A}y + x^T\bar{B}y - u - v$$
$$s.t.$$
$$\bar{A}y \leq uR_1 \tag{3.18}$$
$$x^T\bar{B} \leq vR_2^T \tag{3.19}$$
$$x^T\bar{A}y \leq c_{12} \tag{3.20}$$
$$2x^T\bar{A}y \geq c_{11} + c_{12} \tag{3.21}$$
$$N_1x \geq b_1 \tag{3.22}$$
$$x^T\bar{B}y \leq c_{22} \tag{3.23}$$
$$2x^T\bar{B}y \geq c_{21} + c_{22} \tag{3.24}$$
$$N_2y \geq b_2 \tag{3.25}$$
$$R_1^Tx = 1 \tag{3.26}$$
$$R_2^Ty = 1 \tag{3.27}$$
$$x, y \geq 0 \tag{3.28}$$

Furthermore, the optimal value of the objective function of (P_4) *is zero.*

Proof. Let (x^*, y^*) be a Nash equilibrium solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$. We show that $\exists u^*, v^*$ such that (x^*, y^*, u^*, v^*) is the optimal solution of Problem (P_4).
Since (x^*, y^*) is a Nash equilibrium solution of the game, it satisfies the constraints of Problems (3.13) and (3.14). Therefore, it satisfies the constraints (3.20) – (3.28) of (P_4). Setting $u^* = x^{*T}\bar{A}y^*$ and $v^* = x^{*T}\bar{B}y^*$, by lemma 3.5 we have $u^{*T}R_1 \geq \bar{A}y^*$ and $v^{*T}R_2 \geq x^{*T}\bar{B}$. Therefore (x^*, y^*, u^*, v^*) satisfies the constraints of problem (P_4), i.e. , it is a feasible solution for (P_4). Now we show that (x^*, y^*, u^*, v^*) is the optimal solution of (P_4). Let (x, y, u, v) be an arbitary feasible solution of (P_4). Multiplying the inequalities (3.18) and (3.19) respectively by x^T and y and summing the results yields

$$x^T\bar{A}y + x^T\bar{B}y - u - v \leq 0,$$

i.e. the objective function of (P_4) is non-positive. On the other hand, the value of the objective function of (P_4) for the feasible solution (x^*, y^*, u^*, v^*) is

$$x^{*T}\bar{A}y^* + x^{*T}\bar{B}y^* - u^* - v^* = 0.$$

Therefore, (x^*, y^*, u^*, v^*) is the optimal solution of (P_4) and the optimal value of its the objecive function is zero.
Conversely, suppose that there exist u^*, v^* such that (x^*, y^*, u^*, v^*) is

the optimal solution of the problem (P_4). We show that (x^*, y^*) is a Nash equilibrium solution of the game problem. Let $x \in \bar{X}$ and $y \in \bar{Y}$ be arbitrary. Multiplying (3.18) and (3.19) respectively by x^T and y and using (3.26) and (3.27) yields

$$x^T \bar{A} y^* \leq u^*, \tag{3.29}$$

$$x^{*T} \bar{B} y \leq v^*. \tag{3.30}$$

By setting $x = x^*$ and $y = y^*$ in (3.29) and (3.30), we have

$$x^{*T} \bar{A} y^* \leq u^*, \tag{3.31}$$

$$x^{*T} \bar{B} y^* \leq v^*. \tag{3.32}$$

Since the optimal value of the objective function of (P_4) is zero, we have

$$x^{*T} \bar{A} y^* + x^{*T} \bar{B} y^* - u^* - v^* = 0. \tag{3.33}$$

The relations (3.31), (3.32) and (3.33) imply that

$$x^{*T} \bar{A} y^* = u^*, \tag{3.34}$$

$$x^{*T} \bar{B} y^* = v^*. \tag{3.35}$$

Substituting (3.34) and (3.35) in (3.29) and (3.30) yields

$$x^T \bar{A} y^* \leq x^{*T} \bar{A} y^* \qquad \forall x \in \bar{X}, \qquad x^{*T} \bar{B} y \leq x^{*T} \bar{B} y^* \qquad \forall y \in \bar{Y}$$

Finally, using Theorem (3.2) we have

$$x^T A y^* \leq x^{*T} A y^* \qquad \forall x \in \bar{X}, \qquad x^{*T} B y \leq x^{*T} B y^* \qquad \forall y \in \bar{Y}.$$

This proves that the pair of strategies (x^*, y^*) is a Nash equilibrium solution of $IFCBiG(\bar{X}, \bar{Y}, A, B)$. □

4. Numerical Examples

In this section, two numerical examples are solved using Problem (P_4).

Example 4.1. Assume that the following data are received from Players 1 and 2:

$$A = \begin{bmatrix} 20 & 40 \\ 40 & 30 \end{bmatrix}, N_1 = \begin{bmatrix} 12 & 24 \end{bmatrix}, b_1 = 10,\ a_1 = 20,\ a_2 = 40,\ a_3 = 30,\ a_4 = 50,$$

$$B = \begin{bmatrix} 20 & 45 \\ 30 & 15 \end{bmatrix}, N_2 = \begin{bmatrix} 18 & 12 \end{bmatrix}, b_2 = 14,\ d_1 = 15,\ d_2 = 30,\ d_3 = 20,\quad d_4 = 35.$$

For these data, Problem (P_4) is as follows:

$$\begin{aligned}
\max_{x,y,u,v} \quad & \frac{7}{3}x_1y_1 + 5x_1y_2 + 4x_2y_1 + 2.5x_2y_2 - u - v \\
s.t. \quad & y_1 + 2y_2 \leq u \\
& 2y_1 + 1.5y_2 \leq u \\
& 3x_1 + x_2 \leq v \\
& \frac{4}{3}x_1 + 2x_2 \leq v \\
& x_1y_1 + 2x_1y_2 + 2x_2y_1 + 1.5x_2y_2 \leq 2 \\
& 2x_1y_1 + 4x_1y_2 + 4x_2y_1 + 3x_2y_2 \geq 3.5 \\
& \frac{4}{3}x_1y_1 + 3x_1y_2 + 2x_2y_1 + x_2y_2 \leq 2 \\
& \frac{8}{3}x_1y_1 + 6x_1y_2 + 4x_2y_1 + 2x_2y_2 \geq \frac{10}{3} \\
& 12x_1 + 24x_2 \geq 10 \\
& 18y_1 + 12y_2 \geq 14 \\
& x_1 + x_2 = 1 \\
& y_1 + y_2 = 1 \\
& x,\ y \geq 0
\end{aligned}$$

Solving the above problem with LINGO (version 18) yields the optimal solution:

$$x_1 = 0, \quad x_2 = 1, \quad y_1 = 1, \quad y_2 = 0, \quad u = 2, \quad v = 2.$$

Example 4.2. Let's assume that the data received from Players 1 and 2 are as follows:

$$A = \begin{bmatrix} 20 & 30 & 25 \\ 50 & 20 & 20 \\ 60 & 25 & 40 \end{bmatrix}, N_1 = \begin{bmatrix} 75 & 50 & 10 \end{bmatrix},\ b_1 = 40,\ a_1 = 10, a_2 = 35, a_3 = 15,$$
$a_4 = 40,$

and

$$B = \begin{bmatrix} 30 & 50 & 40 \\ 40 & 20 & 60 \\ 70 & 50 & 30 \end{bmatrix}, N_2 = \begin{bmatrix} 30 & 20 & 25 \end{bmatrix},\ b_2 = 15,\ d_1 = 30, d_2 = 50,\ d_3 = 35,,$$
$d_4 = 55.$

For these data, Problem (P_4) is as follows:

$$
\begin{aligned}
\max_{x,y,u,v}\quad & 2.3x_1y_1 + 4x_2y_1 + 5.9x_3y_1 + 3.7x_1y_2 + 1.8x_2y_2 + 3.5x_3y_2 + 3x_1y_3\\
& 3.8x_2y_3 + 3.1x_3y_3 - u - v\\
s.t.\quad & 0.8y_1 + 1.2y_2 + y_3 \le u\\
& 2y_1 + 0.8y_2 + 0.8y_3 \le u\\
& 2.4y_1 + y_2 + 1.6y_3 \le u\\
& 1.5x_1 + 2x_2 + 3.5x_3 \le v\\
& 2.5x_1 + x_2 + 2.5x_3 \le v\\
& 2x_1 + 3x_2 + 1.5x_3 \le v\\
& 0.8x_1y_1 + 2x_2y_1 + 2.4x_3y_1 + 1.2x_1y_2 + 0.8x_2y_2 + x_3y_2 + x_1y_3\\
& + 0.8x_2y_3 + 1.6x_3y_3 \le \frac{7}{5}\\
& 1.6x_1y_1 + 4x_2y_1 + 4.8x_3y_1 + 2.4x_1y_2 + 1.6x_2y_2 + 2x_3y_2 + 2x_1y_3\\
& + 1.6x_2y_3 + 3.2x_3y_3 \ge 2\\
& 1.5x_1y_1 + 2x_2y_1 + 3.5x_3y_1 + 2.5x_1y_2 + x_2y_2 + 2.5x_3y_2 + 2x_1y_3\\
& + 3x_2y_3 + 1.5x_3y_3 \le 2.5\\
& 3x_1y_1 + 4x_2y_1 + 7x_3y_1 + 5x_1y_2 + 2x_2y_2 + 5x_3y_2 + 4x_1y_3 + 6x_2y_3\\
& + 3x_3y_3 \ge \frac{17}{4}\\
& 75x_1 + 50x_2 + 10x_3 \ge 40\\
& 30y_1 + 20y_2 + 25y_3 \ge 15\\
& x_1 + x_2 + x_3 = 1\\
& y_1 + y_2 + y_3 = 1\\
& x,\ y \ge 0
\end{aligned}
$$

Solving the above problem with LINGO (version 18) yields the optimal solution:

$$
x_1 = 0.5,\quad x_2 = 0,\quad x_3 = 0.5,\quad y_1 = 0.1111111,\quad y_2 = 0.8888889,\quad y_3 = 0,\\
u = 1.155556,\quad v = 2.5.
$$

In the two above examples, we observed that the optimal value of the objective function is zero, as stated in Theorem (3.6). The Nash equilibrium solution of the first example was obtained as pure strategies of the players; however, the Nash equilibrium solution of the second example was obtained as the mixed strategies of the players.

5. Concluding remarks

A constrained bi-matrix game with intuitionistic fuzzy goals of players was studied in this research. Based on the degree of achievement and dissatisfaction of the player's intuitionistic fuzzy goals, the concept of Nash equilibrium for the problem was defined, and it was proved that it is equivalent to the definition of Nash equilibrium for the ordinary constrained bi-matrix game problem. Then, a mathematical optimization problem was proposed to solve the problem, and it was proved that its optimal solution is Nash equilibrium solution of the constrained bi-matrix problem with intuitionistic fuzzy goals.

In this paper, a constrained bi-matrix game was considered. This problem can be extended to games with more than two players. In addition, other intuitionistic fuzzy parameters can be included in the problem. Studying the constrained bi-matrix game problem with intuitionistic fuzzy goals and intuitionistic fuzzy constraints, and studying the problem with some ambiguous data, such as type-2 fuzzy payoffs, Z-number payoffs, and type-2 fuzzy goals, can be considered for future researches. Another idea to extend the paper is to consider cooperative games instead of the bi-matrix game.

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