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Stochastic modeling of AAPL stock using classical stochastic differential equation models

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Abstract. This study investigates the ability of classical stochastic differential equations models to replicate the real behavior of AAPL stock prices over the 2020–2024 period. Several models from stochastic calculus—including standard Brownian motion, geometric Brownian motion, Brownian motion with drift, Brownian bridge, and the Ornstein–Uhlenbeck process—are implemented and analyzed, along with their corresponding simulation algorithms. Using actual adjusted price data, each model is simulated and compared with the observed price trajectory, with particular attention to key features such as hitting times, exit times, long-term trends, and mean-reversion behavior. The results indicate how these models can be applied to capture the structural and random components of financial time series. The analysis suggests that while classical models provide useful insights into certain statistical aspects of asset prices, they also exhibit limitations in capturing complex market behavior. To ensure full reproducibility and methodological transparency, all simulations were performed in R, and the appendix includes the corresponding code, visualizations, and explicit algorithms for all models. This comparative framework may support future efforts in selecting or combining stochastic models for more robust financial forecasting.

Keywords: Geometric Brownian motion; Ornstein–Uhlenbeck process; Brownian bridge; hitting time; exit time; financial time series.

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1. Introduction

Stochastic calculus is a branch of mathematics employed to model phenomena exhibiting non-deterministic behavior over time. One of its most fundamental tools is the Brownian motion process, which, due to its well-established mathematical structure and favorable statistical properties, has been widely used in financial modeling—particularly in asset pricing. Assuming that price fluctuations occur randomly over time, models such as the standard Brownian motion or its extended form, the GBM[2], are employed to describe the dynamics of asset prices or returns. These models are capable of simulating market volatility and constitute the foundation of many financial pricing formulas, most notably the well-known Black–Scholes model. Nevertheless, simplifications such as the assumption of normally distributed returns or the absence of directional trends in price movements introduce certain limitations to these classical models.

To address some of these limitations, more advanced concepts have been introduced, including hitting times and exit times. These refer to the first moments at which a stochastic process reaches a predetermined level or exits a predefined interval. For instance, the moment when the price of a stock reaches a given threshold may correspond to the triggering of a buy/sell order, the onset of bankruptcy, or the activation of an option contract. The calculation of the probability and distribution of such first-passage times is of critical importance in risk management and the pricing of path-dependent derivatives.

In addition, other stochastic processes such as the Brownian bridge and the OU[3] process have been developed to model constrained or mean-reverting behaviors. A Brownian bridge is constructed by conditioning a Brownian motion to take fixed values at both the initial and terminal points of a given interval. This constraint causes the path to exhibit bridge-like behavior: it fluctuates randomly in the interior of the interval but is bound to return to the specified terminal value. The Brownian bridge is a Gaussian and Markov process and can be interpreted as the conditional distribution of a Brownian path given its endpoints. It is particularly useful for comparing simulated trajectories with real asset prices and for identifying to what extent observed fluctuations are purely stochastic or influenced by structural trends. The Ornstein–Uhlenbeck process, on the other hand, is especially applicable in modeling mean-reverting time series, such as interest rates. It is characterized by a tendency to revert toward a long-term equilibrium level over time. This

[2]Geometric Brownian Motion
[3]Ornstein–Uhlenbeck

behavior is of great interest in financial contexts where certain quantities (e.g., spread between asset prices, interest rates) exhibit temporary deviations but eventually return toward their mean.

The basic price model can be expressed in discrete time as $S(n+1) = S(n) + B(n)$ where $B(n)$ is a Gaussian random variable, enabling high-frequency modeling at granular time intervals [15]. Separating jump components from Gaussian noise in returns significantly reduces volatility estimation error, highlighting the importance of modeling jumps for accurate risk assessment [15]. In [9], the traditional jump-diffusion model is extended to a Lévy process model with stochastic interest rates for European-style option pricing. In [8], the simultaneous long–short stock-trading result is generalized and the stability of the model is guaranteed w.r.t. stock prices that are controlled by a geometric Brownian motion whose drift and volatility parameters are unknown to the trader. Under idealized market conditions where stock prices follow a non-trivial Geometric Brownian Motion, combining two static linear feedbacks - one long and one short - results in a positive expected trading gain for all $t > 0$, regardless of the model parameters [7]. In an idealized market, it is assumed that the trader can transact continuously, that the market has perfect liquidity, and that the trader is a price taker, operating without transaction costs or resource constraints [7]. In [16], modeling was performed using the ARIMA[4] and logistic regression time series models and VEC[5], LSTM[6], XGBoost[7], and Prophet in predicting the stock price of AAPL[8]. Real AAPL market data shows that if actual market volatility can be estimated accurately on a moment-by-moment basis, it becomes possible both to better control the risk of daily trading and to design short-term strategies that profit from these volatility changes. In [14], the average run length of homogenously weighted moving average control chart is analyzed for Apple Inc. data. The Geometric Brownian motion is used as a suitable model in stock prices (see [11, 13]). The Brownian bridges are also used for modeling financial data (see [3, 4]). In [2], the optimal stopping of a Brownian bridge and its application to American option trading are investigated. Applications of the Ornstein-Uhlenbeck process in finance are mentioned as examples in [2, 6]. In [17], the first hitting time and option pricing problem under the geometric Brownian motion with singular volatility are discussed. In [1], the first exit time of the geometric Brownian motion from stochastic exponential boundaries is also analyzed.

[4] Autoregressive Integrated Moving Average
[5] Vector Error Correction
[6] Long Short-Term Memory
[7] eXtreme Gradient Boosting
[8] Apple Inc.

The present study investigates the performance of classical stochastic calculus models in replicating the actual price behavior of AAPL stock. Specifically, we analyze and simulate the adjusted closing prices over the 2020–2024 period using various models, including the GBM, The Brownian motion with drift, The Brownian bridge, and the Ornstein–Uhlenbeck process. For each model, the simulated path is compared with real price data, and key features such as hitting and exit points relative to defined thresholds are identified and analyzed. The simulation codes and technical implementation details for each model are provided in the appendix.

The structure of this paper is as follows. In Section 2, we will present the theoretical background and essential concepts from stochastic calculus that underpin the models implemented in this study, including standard Brownian motion, geometric Brownian motion, Brownian motion with drift, Brownian bridge, and the Ornstein-Uhlenbeck process, along with their corresponding simulation algorithms. Section 3, presents the main simulations and analyses, where various stochastic models are applied to real stock data to evaluate their ability to replicate observed financial behavior. Section 4, offers concluding remarks and discusses potential directions for future research. Also, all implementation details and simulation codes are provided in the appendix to ensure full reproducibility of the results.

2. Preliminaries and theoretical foundations

To analyze the behavior of stock prices and evaluate the extent to which mathematical models can replicate real market dynamics, it is essential to understand some foundational concepts in stochastic calculus. This section, provides a concise introduction to these key notions.

A stochastic process is, simply put, a sequence of random variables that depend on time. More formally, a stochastic process is a collection of random variables $\{X_t; t \in I\}$. The index t typically represents time, and the set I is considered the index set of the process.

Since financial data such as stock prices or cumulative returns exhibit fluctuations over time, stochastic processes are considered natural and effective tools for their analysis. A set of mathematical and statistical tools has been developed to examine these processes more precisely. Because these tools are intertwined with classical calculus (i.e., concepts like derivatives and integrals), they are collectively referred to as stochastic calculus.

The Brownian motion process, also known as the Wiener process, is one of the most fundamental continuous-time stochastic processes. It is widely used in modeling random phenomena, particularly in financial mathematics. This process provides a framework to represent purely random changes occurring over time.

A stochastic process $\{B(t); t \geq 0\}$ is called a standard Brownian motion (or Wiener process) [5], if it satisfies the following conditions:

- $B(0) = 0$;
- $B(t)$ is a continuous function in t;
- Each increment $B(s+t) - B(s)$ is stationary and distributed as $N(0, \sigma^2 t)$;
- The increments are independent.

Stochastic volatility models extend this framework by allowing the variance to evolve randomly over time; prominent examples include the Heston model and the GARCH[9] family [15]. In real financial markets, variance is not constant, producing heteroskedastic processes; ARCH[10] and GARCH models are widely used to capture such time-varying volatility [15]. The Brownian motion serves as a useful model when considering stock returns as purely random fluctuations with no trend. In this study, the standard Brownian motion is used to reconstruct the cumulative log-return path of AAPL stock. Using this model, we analyze key metrics such as hitting times and exit times, which are discussed below.

Among the core concepts in the study of stochastic processes are hitting times and exit times, which play a significant role in analyzing random paths. The hitting time refers to the first moment when the trajectory of a stochastic process reaches a given threshold α. For Brownian motion $B(t)$, the hitting time is defined as $T_\alpha = \inf\{t > 0; B(t) = \alpha\}$. Also, the exit time from an interval (a, b), denoted τ , is defined as the minimum time the process hits either boundary, i.e., $\tau = \min(T_a, T_b)$.

Theorem 2.1. *([5]). If $a < x < b$, then $P_x(\tau < \infty) = 1$ and $E_x\tau < \infty$.*

The Theorem 2.1 shows that the probability that the exit time τ is finite, given the Brownian motion starts at x is equal to 1, and the expected (mean) exit time from the interval (a, b), given the starting point is x is finite. The Theorem 2.1 guarantees two key properties for a Brownian motion starting inside the interval (a, b),:

1. **Certain Exit:** The process will definitely (with probability 1) eventually exit the interval (a, b). It cannot stay trapped inside forever.
2. **Finite Average Time:** Not only does it exit, but it does so in a finite amount of time on average. The expected or mean exit time, $E_x\tau$, is a finite number.

This result is fundamental in many applications (like finance, physics, or queueing theory), because it assures us that the event of leaving a certain state or region is not just certain, but also happens in a manageable, predictable timeframe on average. We don't have to wait an infinitely long time for it to occur. For the standard Brownian motion (variance

[9]Generalized Autoregressive Conditional Heteroskedasticity

[10]Autoregressive conditional heteroskedasticity

parameter 1) started at $x \in (a, b)$ one has closed-form expressions for the exit probabilities and the mean exit time. In particular the probability of exiting at the right endpoint b before the left endpoint a is $P_x\{T_b < T_a\} = \frac{x-a}{b-a}$ and the expected exit time from (a, b) is $E_x\tau = (x-a)(b-x)$. These formulae quantify both the directional bias of exit and the typical time-scale for leaving the interval in the driftless case. This quadratic form achieves its maximum at $x = \frac{a+b}{2}$ with value $\frac{(b-a)^2}{4}$, confirming finite expected exit time as guaranteed by Theorem 2.1.

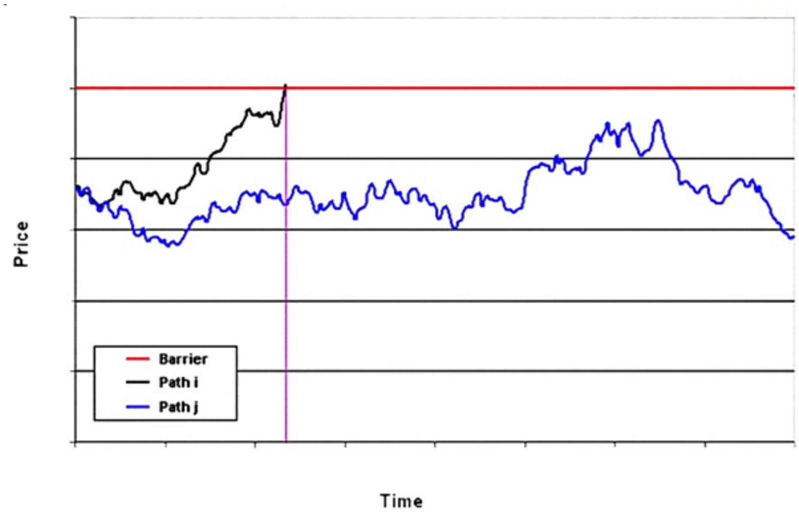


FIGURE 1. Sample trajectories of a stochastic process with a barrier level. Path i reaches the hitting time, while path j remains below it.

Theorem 2.1, implies that if a Brownian motion starts from a point within a bounded interval (a, b), it will almost surely exit the interval in finite time. Moreover, not only is the exit certain, but the expected time to exit is also finite. In other words, on average, the process does not remain within the interval (a, b) indefinitely and will eventually leave it after a finite amount of time. Figure 1 illustrates this behavior.

The standard Brownian motion $B(t)$ is the fundamental building block of continuous-time stochastic processes. It can be simulated by cumulatively summing independent, normally distributed increments. A formal infinitesimal representation of the Brownian increment is $dB(t) = Z_t\sqrt{dt}$ where $Z_t \sim N(0, 1)$. Now Discrete simulation algorithm is introduced. Let the time grid be $0 = t_0 < t_1 < \cdots < t_N = T$ with $\Delta t = t_i - t_{i-1}$ (assumed constant). A simulated path of $B(t)$ on this grid is obtained as follows:

1. Set $B(t_0) = 0$;
2. For $i = 1, \ldots, N$:

- Generate $Z_i \overset{\text{i.i.d.}}{\sim} N(0,1)$;
- Update the path $B(t_i) = B(t_{i-1}) + Z_i\sqrt{\Delta t}$;

3. The sequence $B(t_0), B(t_1), \ldots, B(t_N)$ is a simulated path of the standard Brownian motion.

For completeness, we recall several probabilistic properties used by the simulation. The standard Brownian motion has independent, stationary Gaussian increments $B(t)-B(s) \sim N(0,\ t-s)$, $0 \le s < t$ and increments over disjoint intervals are independent. It is self-similar with index $\frac{1}{2}$, i.e., for any $c > 0$ the process $\{c^{-1/2}B(ct)\}_{t\ge0}$ has the same law as $\{B(t)\}_{t\ge0}$. Moreover, the quadratic variation on a partition $0 = t_0 < \cdots < t_n = t$ satisfies $\sum_{i=1}^{n} \left(B(t_i) - B(t_{i-1})\right)^2 \xrightarrow{P} t$ as $\max_i(t_i - t_{i-1}) \to 0$. Finally, Brownian paths are almost surely continuous but nowhere differentiable, so the formal differential $dB(t)$ must be interpreted in the stochastic (Itô) sense.

A Brownian motion with drift parameter μ and variance parameter σ^2 is defined as $W(t) = \mu t + \sigma B(t)$ where $B(t)$ is a standard Brownian motion. This process exhibits a linear drift component $W(t) = \mu t + \sigma B(t)$ and volatility scaled by σ. A simulated path of $W(t)$ on this grid is obtained as follows. Let $0 = t_0 < t_1 < \cdots < t_N = T$ with $\Delta t = t_i - t_{i-1}$ (assumed constant). For a drifted Brownian motion $W(t) = \mu t + \sigma B(t)$ simulate as follows:

1. Initialize $B(t_0) = 0$, $W(t_0) = 0$;
2. For $i = 1, \ldots, N$:

- Generate $Z_i \overset{\text{i.i.d.}}{\sim} N(0,1)$;
- Update the standard Brownian motion path: $B(t_i) = B(t_{i-1}) + Z_i\sqrt{\Delta t}$;
- Update the drifted Brownian motion path: $W(t_i) = \mu t_i + \sigma B(t_i)$;

3. The sequence $W(t_0), W(t_1), \ldots, W(t_N)$ is a simulated path of the drifted Brownian motion with drift μ and variance parameter σ^2.

Despite its fundamental role in stochastic theory, the Brownian motion alone is insufficient for realistically modeling asset prices, as it can take negative values something that real asset prices, by nature, cannot do. To address this limitation, a modified version known as the geometric Brownian motion has been introduced. Suppose that, $\{B(t); t \ge 0\ \}$ is a standard Brownian motion and $W(t) = \mu t + \sigma B(t)$. Then the geometric Brownian motion with drift parameter μ, volatility σ^2, and initial price Z_0, is defined as $Z(t) = Z_0 \cdot e^{W(t)} = Z_0 \cdot e^{\mu t + \sigma B(t)}$ [5].

One of the drawbacks of the GBM model is that it assumes identically distributed normal returns over fixed time intervals, without considering whether the magnitude of price changes is reasonable relative to the initial price level. For example, it is clear that the probability of a stock price dropping from \$20 to \$15 in a month is not the same as the probability of it falling from \$10 to \$5 over the same period. It is worth mentioning that, the geometric Brownian motion is a continuous-time stochastic process where the logarithm of the asset price follows a Brownian motion with drift. It combines a deterministic growth term, such as an interest rate, with a random component driven by normally distributed noise. This model is widely used in stock price simulation, because it produces lognormally distributed prices and captures both the average growth rate and the volatility of returns [13].

In this approach, the stock price at time t is obtained from its previous value by adding a drift term adjusted for volatility and a random noise term. This makes the GBM suitable for simulating realistic stock price paths over time, especially when combined with historical volatility estimates [13].

The GBM is used to model the stock price $Z(t)$. The simulation involves an exponential transformation of a process with drift. Consider the SDE[11],

$$dZ(t) = \mu Z(t)dt + \sigma Z(t)dB(t).$$

The simulation algorithm is as follows:

1. Set $Z(0)$ to the observed initial AAPL price;
2. Estimate μ (drift) and σ (volatility) from the historical log-returns;
3. For $i = 1, \ldots, N$:

- Generate $Z_i \overset{\text{i.i.d.}}{\sim} N(0,1)$;
- Update $Z(t_i) = Z(t_{i-1}) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma Z_i\sqrt{\Delta t}\right)$.

For GBM one has an exact transition law and simple estimators that are convenient in simulation and inference. In particular

$$\log Z(t) \sim N(\log Z(0) + (\mu - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t)$$

so $Z(t)$ follows a lognormal distribution with expectation

$$E[Z(t)] = Z(0)e^{\mu t}$$

and variance

$$\text{Var}[Z(t)] = Z(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

Given discrete observations on a regular grid with step Δt and log-returns $r_i = \log\left(\frac{Z(t_i)}{Z(t_{i-1})}\right)$, $i = 1, \ldots, n$, the usual estimators are $\hat{\mu} = \frac{\bar{r}}{\Delta t} + \frac{1}{2}\hat{\sigma}^2$ and $\hat{\sigma}^2 = \frac{1}{n\Delta t}\sum_{i=1}^{n}(r_i - \bar{r})^2$ where $\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$. Finally, for pricing or risk-neutral simulations replace μ by the risk-free rate r (i.e. $\mu \mapsto r$) so that the discounted price process is a martingale.

Suppose $\{B(t); t > 0\}$ is a Brownian motion, and we consider its values over the interval $(0, 1)$ under the conditions $B(1) = 0$ and $B(0) = 0$. Then, the process

$$\left\{B^0(t); t \geq 0\right\} = \{B(t); 0 \leq t \leq 1 | B(0) = B(1) = 0\}$$

[11]Stochastic Differential Equation

is called a Brownian Bridge [5] (see Figure 2 for an illustration).

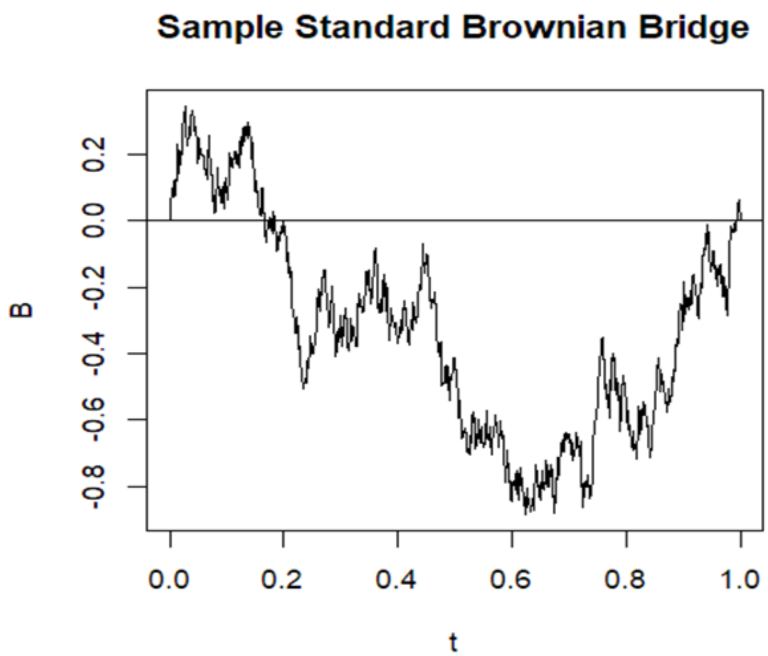


FIGURE 2. An example of a standard Brownian bridge starting at zero, fluctuating over the interval, and returning to zero at the final time.

The Brownian bridge process can be explicitly constructed from standard Brownian motion by $B^0(t) = B(t) - tB(1)$, $0 \le t \le 1$, which satisfies the boundary conditions $B^0(0) = B^0(1) = 0$ almost surely. We have $E[B^0(t)] = 0$, $\mathrm{Cov}(B^0(s), B^0(t)) = \min(s,t) - st$, $\mathrm{Cov}\big(B^0(s), B^0(t)\big) = \min(s,t) - st$, $0 \le s,t \le 1$. For $0 < s < t < 1$, $B^0(t) \mid B^0(s) = y \sim N\left(y \cdot \frac{1-t}{1-s}, \frac{(t-s)(1-t)}{1-s}\right)$ [5].

Let $0 = t_0 < t_1 < \cdots < t_n = 1$. To simulate a Brownian bridge on this grid:

1. Set $B^0(0) = 0$ and $B^0(1) = 0$;
2. For each intermediate point t_i simulate
$$B^0(t_i) \sim N\left(\frac{(1-t_i)B^0(t_{i-1}) + t_i B^0(1)}{1-t_{i-1}}, \ \frac{(t_i - t_{i-1})(1-t_i)}{1-t_{i-1}}\right).$$

The Brownian bridge is widely used in statistics (e.g. goodness-of-fit tests), quantization theory, and as a building block for more complex stochastic processes.

In this study, since the process returns to a fixed value (zero) at both the beginning and end of the interval, it can be compared with the actual price path to assess how much of the observed price variation is due to pure random fluctuations and how much may be attributed to structural trends in the market.

Let $\{B(t); t > 0\}$ be a standard Brownian motion on [0,T]. The Brownian bridge pinned to $B(0) = a$ and $B(T) = b$ can be represented as $B^0(t) = B(t) - \frac{t}{T}B(T)$, $0 \leq t \leq T$and, when pinned to endpoints a and b, one uses the appropriate affine adjustment. The affine form with explicit endpoints is $B^{a\to b}(t) = a + (B(t) - B(0)) - \frac{t}{T}(B(T) - B(0)) + \frac{t}{T}(b - a)$.

On a discrete grid $0 = t_0 < t_1 < \cdots < t_N = T$, given a simulated Brownian path $\{B(t_i)\}_{i=0}^N$ with $B(t_0) = a$, the bridge values are computed by $B^{a\to b}(t_i) = B(t_i) - \frac{t_i}{T}(B(T) - b)$, $i = 0, \ldots, N$, so that $\{B^{a\to b}(t_i)\}_{i=0}^N$ is a Brownian bridge from a to b. Equivalently, one may simulate directly from the conditional Gaussian laws on the grid or use iterative midpoint conditioning for improved accuracy when refining the mesh.

The simulation algorithm is as follows. Let $0 = t_0 < t_1 < \cdots < t_N = T$ and assume a simulated Brownian motion path $\{B(t_i)\}_{i=0}^N$ with $B(0) = a$ is available. A Brownian bridge from a to b on this grid is obtained by:

1. Simulate a standard Brownian motion path $\{B(t_0), \ldots, B(t_N)\}$ with $B(t_0) = a$;
2. For each grid point t_i compute the pinned value $B^0(t_i) = B(t_i) - \frac{t_i}{T}(B(T) - b)$;
3. The sequence $\{B^0(t_0), \ldots, B^0(t_N)\}$ is a Brownian bridge from a to b.

Equivalently, one may simulate the bridge directly using the conditional Gaussian distributions on the grid or by the iterative midpoint conditioning scheme for higher accuracy when refining the grid.

The Ornstein–Uhlenbeck model is one of the most important stochastic calculus processes, widely used in finance, economics, and physics. This model is typically employed to describe variables that exhibit oscillatory behavior around a long-term equilibrium level. The mean-reverting property intrinsic to the Ornstein-Uhlenbeck process renders it exceptionally valuable for modeling a variety of financial and economic variables that naturally fluctuate around a long-term equilibrium level. Its analytical tractability and realistic dynamic have led to its adoption in several cornerstone financial models. A prominent application is the Vasicek model for interest rate modeling, where the OU process captures the tendency of interest rates to revert to a long-run mean, influenced by economic cycles and central bank policies. Similarly, in commodity markets, the process is effectively used to model prices that revert to marginal production costs, as high prices stimulate supply and low prices curb it, creating a natural equilibrium mechanism. Furthermore, the OU process serves as a fundamental building block in stochastic volatility frameworks, such as the Scott model, where it describes the

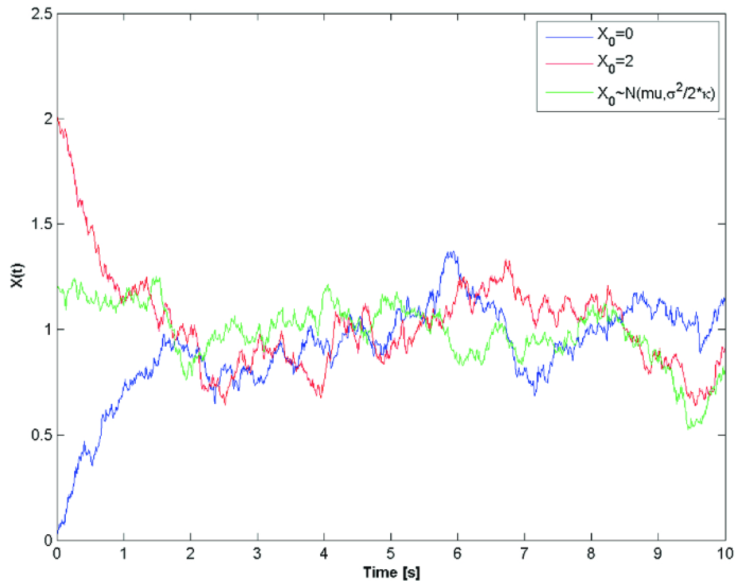


FIGURE 3. Simulated paths of the Ornstein–Uhlenbeck model starting from different initial values. All paths tend to revert toward the common mean over time.

evolution of an unobserved volatility process that meanders around a long-term average, thus capturing the well-documented phenomenon of volatility clustering in asset returns. Beyond these direct modeling applications, the statistical properties of the OU process are harnessed in trading strategies. It forms the theoretical foundation for pairs trading and statistical arbitrage, where traders identify two assets whose price spread is mean-reverting. Deviations from the historical equilibrium of this spread signal trading opportunities, with the expectation that the spread will revert to its mean. Consequently, the Ornstein-Uhlenbeck process provides a powerful and versatile tool for representing the dynamics of key economic variables like interest rates, exchange rates, and commodity prices, whose behavior is characterized by a persistent pull towards a fundamental equilibrium level.

It is defined by the following SDE, $dX(t) = \theta(\mu - X(t))dt + \sigma dB(t)$ [10]. This mean-reverting property is illustrated in Figure 3. In this equation:

- $X(t)$ denotes the value of the process at time t;
- μ is the long-term mean or the expected equilibrium level;
- θ represents the speed of mean reversion;
- σ is the volatility of the process;
- $dB(t)$ denotes a small increment of a Wiener process.

This model consists of two main components:

1. **Deterministic drift term:** The term $\theta(\mu - X(t))dt$ is the drift component of the model. It drives the process toward the equilibrium level μ. If the instantaneous value $X(t)$ is greater than μ, this component tends to pull it downward; and if it's less, it tends to push it upward.
2. **Random diffusion term:** The term $\sigma dB(t)$ is the diffusion component of the model. It introduces random, unpredictable fluctuations into the system, stemming from Brownian motion.

The process has the explicit solution

$$X(t) = X(0)e^{-\theta t} + \mu\big(1 - e^{-\theta t}\big) + \sigma e^{-\theta t} \int_0^t e^{\theta s}\, dB(s).$$

For $t \geq s$, the conditional distribution is Gaussian:

$$X(t) \mid X(s) = x_s \sim N\left(\mu + (x_s - \mu)e^{-\theta(t-s)},\ \tfrac{\sigma^2}{2\theta}(1 - e^{-2\theta(t-s)})\right).$$

As $t \to \infty$, the OU process converges in distribution to the stationary Gaussian $X(\infty) \sim N\left(\mu,\ \frac{\sigma^2}{2\theta}\right)$. The covariance structure for the stationary process is $\mathrm{Cov}(X(s), X(t)) = \frac{\sigma^2}{2\theta} e^{-\theta|t-s|}$.

The Ornstein–Uhlenbeck process is mean-reverting. We use the exact discrete-time solution for simulation. Consider the SDE equation, $dX(t) = \theta\big(\mu - X(t)\big)\, dt + \sigma\, dB(t)$. The simulation algorithm is as follows:

1. Set $X(0)$ to the initial value;
2. Estimate the long-term mean μ, mean-reversion speed θ, and volatility σ;
3. For $i = 1, \ldots, N$:

- Generate $Z_i \overset{\text{i.i.d.}}{\sim} N(0, 1)$;
- Update the process using the exact solution

$$X(t_i) = X(t_{i-1})e^{-\theta\Delta t} + \mu(1 - e^{-\theta\Delta t}) + \sigma\sqrt{\tfrac{1-e^{-2\theta\Delta t}}{2\theta}} Z_i.$$

Equivalently, according to the above algorithm, OU process is simulated as

$$X(t_i) = X(t_{i-1})e^{-\theta\Delta t} + \mu(1 - e^{-\theta\Delta t}) + \sigma\sqrt{\tfrac{1-e^{-2\theta\Delta t}}{2\theta}} + \zeta_i$$

where $\zeta_i \sim N(0, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta\Delta t}))$. To estimate the model parameters, an autoregressive model $AR(1)$ can be used, namely $X_i = a + bX_{i-1} + \varepsilon_i$. So, we have $\hat{\theta} = -\frac{\ln(b)}{\Delta t}$ and $\hat{\mu} = \frac{a}{1-b}$.

3. Modeling and Analysis

In this study, real market data have been used, and a summary of the dataset is presented in Table 1. The data pertain to the stock prices of Apple Inc, traded under the ticker symbol AAPL on the NASDAQ[12] stock exchange. The dataset includes columns such as the opening price,

[12]National Association of Securities Dealers Automated Quotations

daily high and low prices, closing price, trading volume, and adjusted closing price.

For all analyses conducted in this study, only the adjusted closing price was used, as it accounts for dividends and stock splits, thus providing a more accurate representation of the asset's value over time. To better understand the stochastic behavior of stock data, several standard continuous-time stochastic models have been implemented and analyzed. For each model, the simulated trajectory is compared with the actual price path, and the relevant analyses are provided.

TABLE 1. Daily stock prices of AAPL from 2020 to 2024, including key components: opening price, high and low prices, closing price, trading volume, and adjusted closing price.

Date	AAPL.Open	AAPL.High	AAPL.Low	AAPL.Close	AAPL.Volume	AAPL.Adjusted
1/2/2020	74.059998	75.15	73.7975	75.08750153	135480400	72.62083435
1/3/2020	74.287498	75.145	74.125	74.35749817	146322800	71.91481018
1/6/2020	73.447502	74.99	73.1875	74.94999695	118387200	72.48784637
1/7/2020	74.959999	75.225	74.37	74.59750366	108872000	72.14691925
1/8/2020	74.290001	76.11	74.29	75.79750061	132079200	73.30751801
			⋮			
12/23/2024	254.77	255.65	253.45	255.2700043	40858800	254.6557159
12/24/2024	255.49001	258.21	255.29	258.2000122	23234700	257.5786743
12/26/2024	258.19	260.1	257.63	259.019989	27237100	258.3966675
12/27/2024	257.82999	258.7	253.06	255.5899963	42355300	254.9749298

3.1. **The Geometric Brownian Motion Model.** The geometric Brownian motion model is one of the most widely used frameworks for asset pricing. In this model, the asset price grows stochastically, while also following an average trend over time.

The results show that the GBM can partially replicate the general upward trend of the stock price, especially during periods of steady and moderate market growth. However, in intervals such as the year 2023 when the market experienced sustained and strong bullish momentum the GBM model failed to fully capture this growth. In those periods, the simulated path remained below the actual trajectory, indicating the model's limitations in reproducing long-term and robust market trends.

Moreover, in episodes marked by abrupt price surges or sudden crashes such as the drop triggered by the COVID-19[13] crisis in 2020 the model proves insufficient, as real market fluctuations are shaped by structures more complex than those assumed by the GBM. In addition, the model's assumption of constant and homogeneous variance over time is inconsistent with the inherently dynamic and sometimes unstable nature of financial markets.

[13]COronaVIrus Disease 2019

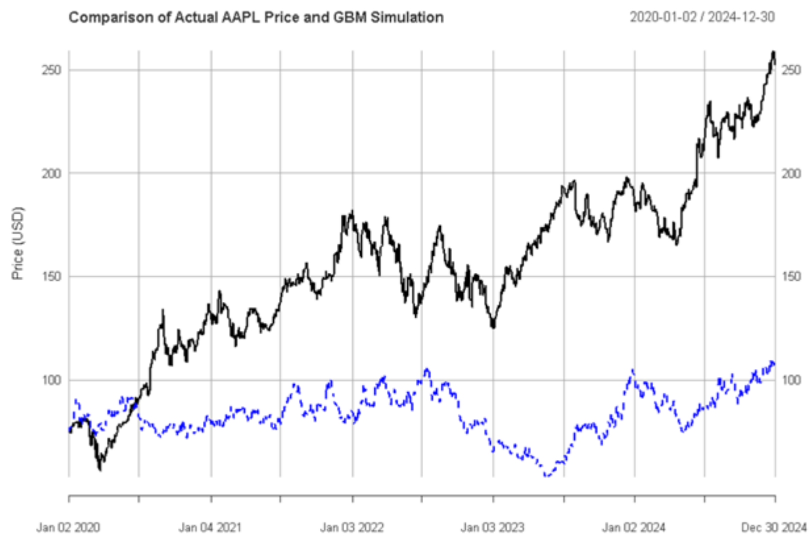


FIGURE 4. Comparison of the actual adjusted closing price path of AAPL (black) and the simulated trajectory from the Geometric Brownian Motion (GBM) model (blue) over the period 2020–2024.

Figure 4 presents a comparison between the actual trajectory of AAPL's adjusted closing price (black line) and the simulated path generated by the GBM model (blue line) over the same time interval. The model parameters growth rate μ and volatility σ have been estimated from real data (Refer to Appendix 1). For further understanding, below we present the R code algorithm given in Appendix 1 in a simplified form. Time series of observed AAPL adjusted closing prices $\mathbf{p} = (p_1, p_2, \ldots, p_{N+1})$, drift parameter μ, volatility σ, number of trading days per year trading_days.

1. Determine the time grid:
- $N \leftarrow \text{length}(\mathbf{p}) - 1$;
- $\Delta t \leftarrow 1/\text{trading\_days}$;
2. Build the time vector t_k:
- Construct time vector: $t_k \leftarrow k \cdot \Delta t$ for $k = 0, 1, \ldots, N$;
3. Simulate the Wiener process $B(t)$ (Brownian motion path):
- Initialize Wiener process: $B(0) \leftarrow 0$;
- For $i = 1$ to N:
  - Generate $Z_i \sim N(0, 1)$;
  - Compute increment: $\Delta B(i) \leftarrow \sqrt{\Delta t} \cdot Z_i$;
  - Update: $B(t_i) \leftarrow B(t_{i-1}) + \Delta B(i)$;
4. Set the initial stock price:
- Set initial price: $S_0 \leftarrow p_1$;

5. Compute the simulated GBM path using the exact solution:
- For $k = 0$ to N:
- Compute simulated price:

$$S_k \leftarrow S_0 \cdot \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) \cdot t_k + \sigma \cdot B(t_k)\right);$$

6. Align simulated path $\mathbf{S} = (S_0, S_1, \ldots, S_N)$ with the dates of $\mathbf{p}$;
7. Visualize the comparison:
- Plot actual prices in solid black line;
- Overlay simulated prices in dashed blue line;

The comparison between the actual and simulated trajectories reveals the following:

- Both paths exhibit an overall upward trend, which reflects a positive μ.
- In periods of low volatility, the GBM model accurately tracks the price path.
- During high-volatility episodes particularly in early 2020, during the COVID-19 crisis the GBM model either underestimates or overestimates the magnitude of fluctuations.

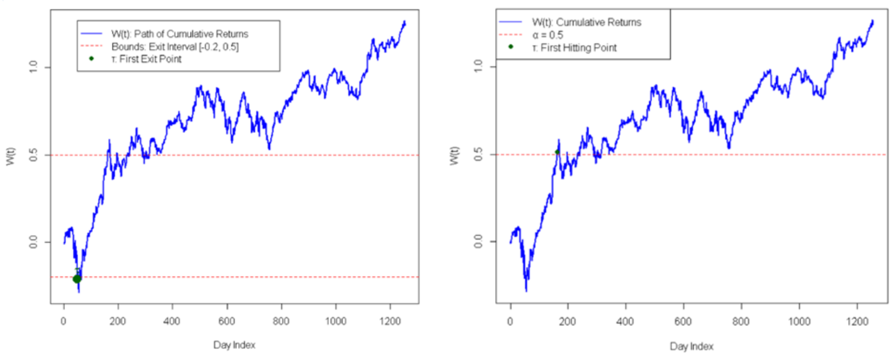
Therefore, while the GBM may be suitable for approximating overall trends, it exhibits limitations in modeling extreme market volatility.

3.2. **The Standard Brownian Motion and The Hitting / Exit Times.** To analyze the oscillatory behavior of the cumulative returns of AAPL stock, an approximate trajectory of a standard Brownian motion was reconstructed using the daily log-returns from the 2020–2024 period. The objective of this analysis is to identify two key events: the first hitting time of a 50% cumulative return threshold, and the exit time from a predefined fluctuation interval.

For this purpose, a symmetric interval was chosen as $(-0.2, 0.5)$, corresponding to a 20% drawdown or a 50% gain in cumulative returns. The bounds of this interval were selected to be wide enough to match the actual volatility range of AAPL during the period under study, while also ensuring a sufficiently high probability of exit within the considered time window so that the statistical analysis remains meaningful.

In the simulation, the Brownian path first hits the upper threshold of $(\alpha = 0.5)$ around day 200. The corresponding point is marked in green in the diagram. It is observed that after reaching this level, the path enters a region of higher volatility (see Figure 5, left). Furthermore, the exit from the interval occurs on March 10, 2020, with a cumulative return drop of approximately 19%. This event coincides with the onset of the global COVID-19 crisis and the market's initial negative reaction (Figure 5, right).

This behavior indicates that even in the absence of any directional component, a standard Brownian motion model can statistically capture critical points and stress events in the market. Although the model



(A) Hitting time (+0.5) (B) Exit time $(-0.2, 0.5)$

FIGURE 5. Comparison of stopping-time scenarios

assumes purely random and independent price changes, reconstructing a Brownian path based on real log-returns of AAPL over a five-year window naturally reproduces significant events such as reaching the 50% gain level or exiting the volatility band.

From a statistical standpoint, the occurrence of such events may inform the design of stop-loss or take-profit strategies especially for short-term investors sensitive to market fluctuations. However, the absence of structural trends in the model reduces its predictive power when facing large-scale external shocks.

It is also observed that, after reaching critical levels, the path tends to become more volatile a behavior visible in both charts. This feature may reflect underlying hidden dynamics in the market.

In conclusion, while the Brownian motion provides a useful framework for analyzing critical thresholds in price fluctuations, it has notable limitations in capturing structural or behavioral patterns in market dynamics.

This type of analysis can be used for early detection of volatility breakouts and for assessing a stock's resilience to sharp price shocks (see Appendix 2 for simulation details). Below we give a simplified version of the R code algorithm that appears in Appendix 2 to aid understanding. Time series of AAPL adjusted closing prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, lower boundary $a < 0$, upper boundary $b > 0$.

1. Compute daily log-returns: $r_t \leftarrow \log(p_{t+1}/p_t)$ for $t = 1, \ldots, n-1$;
2. Construct the discrete Brownian motion path: $B(0) \leftarrow 0$, $B(k) \leftarrow \sum_{i=1}^{k} r_i$, for $k = 1, \ldots, n-1$;
3. Define the time index vector: time $\leftarrow 1, 2, \ldots, (n-1)$;
4. Identify the first exit time index:
$$\tau_{\text{index}} \leftarrow \min\{k \geq 1 \mid B(k) < a \quad \text{or} \quad B(k) > b\};$$
5. If no exit occurs, set $\tau_{\text{index}} \leftarrow n - 1$ (or handle as infinite);

6. Extract the exit value: $B(\tau) \leftarrow B(\tau_{\text{index}})$;
7. Output:

- Plot of $B(k)$ vs. time (solid blue line);
- Horizontal boundary lines at a and b (dashed red);
- Mark the exit point $(\tau_{\text{index}}, B(\tau))$ with a filled green circle;
- Label the point with τ slightly above the point.

3.3. **The Hitting Time under The Drifted Brownian Motion.** To investigate the effect of directional price movements, the cumulative return path was simulated by incorporating a drift component into the Brownian motion model. Unlike the standard Brownian motion, which assumes fluctuations centered around zero, the drifted version allows the trajectory to exhibit a systematic tendency toward growth or decline. This makes the model more suitable for market conditions characterized by a clear directional trend.

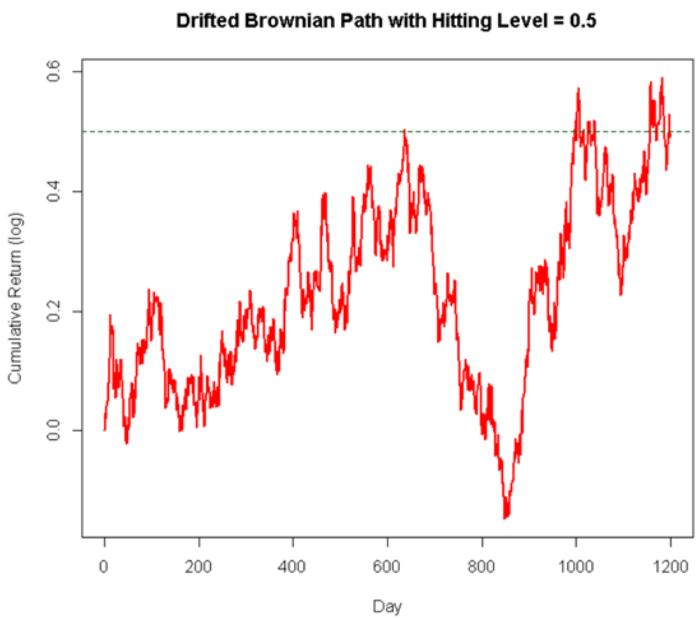


Figure 6. Simulated trajectory of a drifted Brownian motion hitting +0.5 return threshold.

Specifically, the inclusion of a positive drift causes the trajectory to, on average, move upward over time. As a result, the likelihood of reaching a given profit threshold increases, and the time to hit that level tends to decrease.

In this simulation, a return level of +0.5 was set as the threshold, and the time of the first crossing was computed. The results showed that the process reached this level on day 1003 from the start of the dataset, corresponding to September 30, 2022 (see Figure 6 and Appendix 3 for details). To clarify the implementation, we present a condensed R-code algorithm adapted from Appendix 3. Time series of AAPL adjusted closing prices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, Drift parameter $\mu > 0$, Target hitting level $a > 0$, Number of trading days per year trading_days.

1. Compute daily log-returns: $r_t \leftarrow \log(p_{t+1}/p_t)$ for $t = 1, \ldots, n-1$;
2. Set time step: $\Delta t \leftarrow 1/\text{trading\_days}$;
3. Construct the "drifted Brownian motion path" $B(t)$: $B(0) \leftarrow 0$, $B(t_k) \leftarrow \mu \cdot t_k + \sum_{i=1}^{k} r_i$, for $k = 1, \ldots, n-1$ where $t_k = k \cdot \Delta t$;
4. Define the time index vector: day $\leftarrow 1, 2, \ldots, (n-1)$;
5. Initialize: hitting_index $\leftarrow$ NA, hitting_time $\leftarrow$ NA;
6. Search for the "first hitting time" of level a:

$$\text{hitting\_index} \leftarrow \min\{k \geq 1 \mid B(t_k) \geq a\};$$

7. Calculate approximate hitting day:

- If hitting_index exists (i.e., not NA): $B(\text{hit}) \leftarrow B(t_{\text{hitting\_index}})$ and hitting_time $\leftarrow$ hitting_index;
  Else: No hitting occurred within the observation period;

8. Output:

- Plot of $B(t_k)$ vs. day (solid red line);
- Horizontal line at level a (dashed dark green);
- Title: "Drifted Brownian Path with Hitting Level = a";
- Console output: "Hitting index: hitting_index, Approximate day: hitting_time".

This behavior indicates that when a drift component is present, the probability of reaching a specified profit level not only increases, but the trajectory also tends to reach that level more quickly. Such a model can serve as an effective analytical tool in directional markets, where upward or downward trends dominate.

3.4. **Simulation of The Brownian Bridge and Comparison with Real Data.** In this section, the normalized trajectory of AAPL's adjusted closing price from 2020 to 2024 is compared with a simulated Brownian bridge. The aim of this comparison is to evaluate the difference between the behavior of actual prices and a purely random, trendless path.

Unlike the standard Brownian motion, which evolves freely beyond the time interval, the Brownian bridge is constrained to take fixed values at both endpoints (usually zero) and fluctuates only within the interval. To enable a meaningful comparison, both paths were normalized to the interval $[0, 1]$.

In Figure 7, the actual price path (blue line) displays asymmetric and trend-oriented behavior, while the Brownian bridge (red line) exhibits symmetric fluctuations with no directional tendency. This contrast suggests that AAPL's price dynamics are not purely random but also influenced by structural and market-driven factors. For instance, a substantial portion of the company's shares was concentrated in the hands of mutual funds. Regulatory constraints combined with anticipated tax changes led to large-scale selling, which in turn contributed to a sharp decline in the stock price [12].

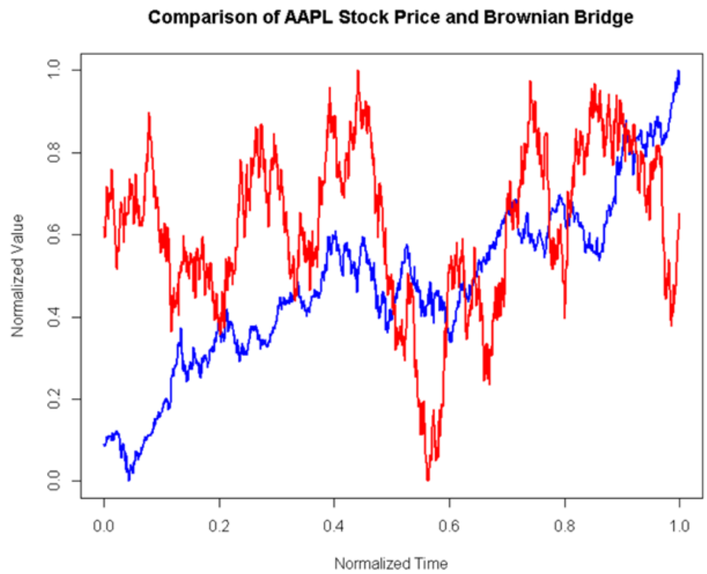


Figure 7. Comparison between the normalized trajectory of AAPL's adjusted closing price (blue) and the normalized Brownian bridge (red).

Furthermore, the Brownian bridge is inherently constructed to return to zero at both the start and end of the interval.

As a result, the simulated Brownian bridge maintains fluctuations around a stable central value. In contrast, the actual price path not only deviates from this center but also shows a clear upward trend in several parts—particularly in the second half of the interval. This asymmetry and directional bias indicate that models based solely on random noise—such as the Brownian bridge—are insufficient for fully capturing real stock market dynamics.

Such discrepancies may reflect the influence of fundamental drivers, market sentiment, or external interventions, all of which push the price trajectory away from purely stochastic patterns.

This comparison highlights the combination of noise and structure in real price behavior and can serve as a foundation for developing more advanced models (see Appendix 4 for simulation details). For convenience, a simplified R implementation of the algorithm from Appendix 4 is shown below. AAPL adjusted closing prices time series: $p_1, p_2, \ldots, p_n$, Number of time points: n (length of price vector).

1. Extract price values into a vector: price_vector $= [p_1, p_2, \ldots, p_n]$;
2. Define normalization function: For any vector x:

$$\text{normalized\_x}=\frac{\text{x- min(x)}}{\text{max(x)- min(x)}};$$

3. Normalize the AAPL price path:

$$\text{price\_norm} = \text{normalize(price\_vector)};$$

4. Set time step: $\Delta t = \frac{1}{n}$;
5. Simulate standard Brownian motion increments:
- For $i = 1$ to n: $dB(i)$random normal, (mean=0, sd= $\sqrt{\Delta t}$);

6. Build the Brownian motion path $B(t)$: $B(0) = 0, \quad B(t_k) = \sum_{i=1}^{k} dB(i) \quad \text{for } k = 1 \text{ to } n$;
7. Create the normalized time vector: time $= \left[0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right]$;
8. Construct the Brownian Bridge: For each $k = 0$ to n: Bridge$(t_k) = B(t_k) - \text{time}_k \times B(\text{final})$ where $B(\text{final}) = B(t_n)$;
9. Normalize the Brownian Bridge path:

$$\text{bridge\_norm} = \text{normalize(Bridge)};$$

10. Plot the comparison:
- Blue solid line: price_norm vs. time;
- Red solid line: bridge_norm vs. time;
- X-axis label: "Normalized Time";
- Y-axis label: "Normalized Value";
- Title: "Comparison of AAPL Stock Price and Brownian Bridge".

3.5. **Simulation of The Ornstein–Uhlenbeck Model.** To examine mean-reverting behavior in the time series of AAPL stock, the Ornstein–Uhlenbeck model was applied to the logarithm of the adjusted closing prices. The simulated OU path was then compared to the actual price trajectory over the period from 2020 to 2024. The main objective of this comparison is to determine whether the OU model can adequately reflect the long-term dynamics of assets like AAPL stock, which often exhibit non-cyclical trends and structural changes.

The results show that the OU model generates fluctuations centered around a long-term mean, such that large deviations tend to diminish over time. In contrast, the actual price path of AAPL exhibits a clear long-term upward trend with significantly higher volatility (see Figure 8). This upward trend intensifies particularly after day 600, while the OU model continues to revert to its mean, thus failing to reproduce the structural shifts observed in the real market.

This comparison demonstrates that while the OU model is useful for analyzing stability components and simulating noise behavior, it falls short in capturing directional trends and structural shifts typical of growing equity prices. (See Appendix 5 for simulation details.)

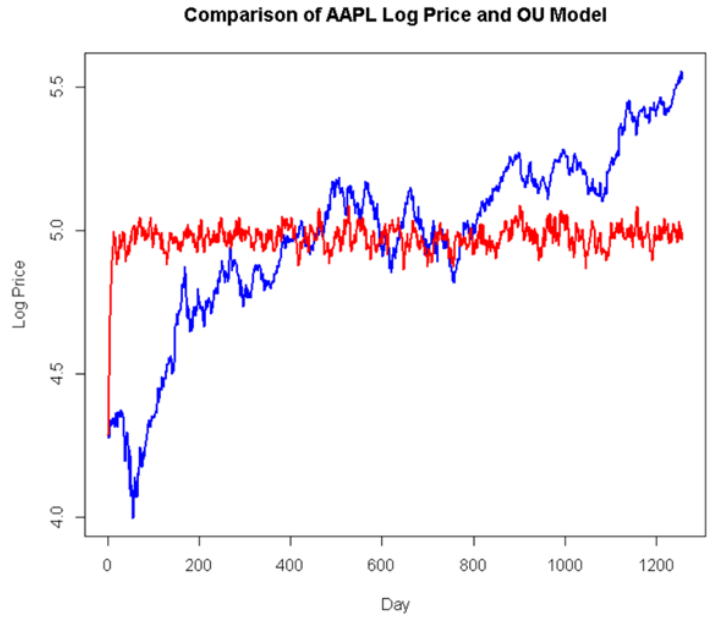


FIGURE 8. Comparison of the logarithmic adjusted price trajectory of AAPL (blue) with the simulated Ornstein–Uhlenbeck (OU) process (red). The real path shows a long-term upward trend and high volatility, whereas the OU model exhibits stable, mean-reverting behavior.

For a clearer view of the steps, we include a pared-down R code algorithm taken from Appendix 5. Time series of AAPL log-adjusted prices: $\log(p_1), \log(p_2), \ldots, \log(p_n)$, Mean reversion speed: $\theta > 0$, Long-term mean: μ, Volatility: $\sigma > 0$, Number of time points: n, Time step: $\Delta t = \frac{1}{252}$ (assuming 252 trading days per year).

1. Define the observed log-price vector:

$$\text{log\_price} = [\log(p_1), \log(p_2), \ldots, \log(p_n)];$$

2. Set time step: $\Delta t = \frac{1}{252}$;
3. Initialize the simulated OU path $X(t)$: $X(0) = \log(p_1)$;
4. Simulate the OU path using Euler–Maruyama discretization:

- For each $i = 2$ to n:

$$X(t_i) = X(t_{i-1}) + \theta(\mu - X(t_{i-1}))\Delta t + \sigma \cdot Z_i \cdot \sqrt{\Delta t}$$

where $Z_i \sim N(0, 1)$ is a standard normal random variable;

5. Create the day index vector: day $= [1, 2, \ldots, n]$;
6. Plot the comparison:

- Blue solid line: log_price vs. day;
- Red solid line: $X(t_i)$ vs. day;
- X-axis label: "Day";

- Y-axis label: "Log Price";
- Title: "Comparison of AAPL Log Price and OU Model".

3.6. **Modeling AAPL Stock with The Diffusion Bridges (GBM and OU).** As discussed in the previous sections, the GBM and the OU process are two fundamental models in financial analysis: the former being growth-oriented, and the latter characterized by mean-reverting behavior. In this section, to extend the analysis, we examine the diffusion bridge versions of these models.

For both processes, a bridge is simulated by conditioning on the actual initial and final prices of AAPL. This framework allows us to evaluate how well each process can reproduce the observed stock trajectory under model-consistent constraints.

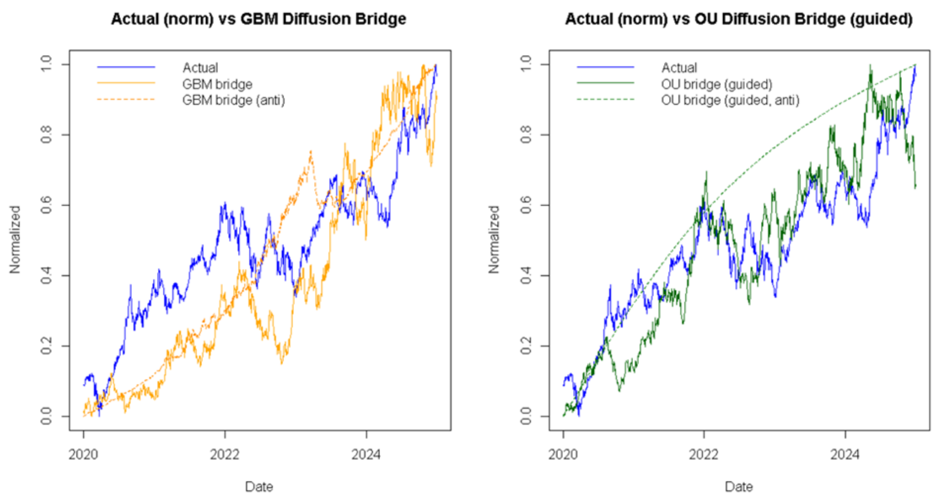


FIGURE 9. The left panel compares the normalized AAPL trajectory (blue) with the GBM diffusion bridges: a standard bridge (orange, solid) and a variance-reduced version (orange, dashed). The right panel presents the OU diffusion bridges: a guided bridge (green, solid) and its variance-reduced counterpart (green, dashed).

Figure 9 presents the results. The left panel compares the normalized actual price path of AAPL (blue) with the two GBM diffusion bridges: a standard bridge (orange, solid) and a variance-reduced bridge (orange, dashed; implemented via antithetic simulation). Both paths replicate the long-term upward trend, but the variance-reduced version smooths out fluctuations, producing a trajectory that remains closer to the empirical data. This highlights the role of variance-reduction techniques (in particular, the antithetic method) in improving simulation stability.

The right panel shows the OU diffusion bridges: a guided bridge (green, solid) and a variance-reduced version (green, dashed; antithetic). As expected, the OU process tends to revert toward a mean trajectory; hence, the OU bridges avoid explosive growth, but in most periods overestimate the actual price level. Nevertheless, the guided OU bridge follows the general trend, and during the years 2023–2024 it overshoots the sharp upward momentum observed in the data.

Overall, the comparison demonstrates that GBM bridges replicate the growth-oriented nature of AAPL prices more faithfully, while the OU bridges emphasize mean reversion. In both cases, the use of variance-reduced bridges (antithetic paths) decreases variability across simulations and improves robustness of the results. These findings suggest that, for forecasting stock prices, GBM-based bridges are more suitable, whereas the OU-based bridges are better aligned with assets that display strong mean-reverting patterns (see Appendix 6 for simulation details). The following presents a simplified R algorithm (derived from Appendix 6) for easier comprehension. Time series of AAPL adjusted closing prices: $p_1, p_2, \ldots, p_n$, Number of trading days per year: 252.

1. Extract and clean price data:
- $\mathbf{p} \leftarrow$ adjusted closing prices,
- dates $\leftarrow$ corresponding dates;
- Remove any missing values;
2. Compute time parameters: $N \leftarrow n-1, \quad T_{\text{yr}} \leftarrow \frac{N}{252}, \quad \Delta t \leftarrow \frac{T_{\text{yr}}}{N}$;
3. Set initial and terminal prices: $S_0 \leftarrow p_1, \quad S_T \leftarrow p_n$, $S_T \leftarrow p_n$;
4. Normalize actual price path for visualization:

$$\text{act} = \frac{p_i - \min(\mathbf{p})}{\max(\mathbf{p}) - \min(\mathbf{p})}, \quad i = 1 \text{ to } n;$$

5. Estimate GBM parameters from log-returns:

$$\text{lr}_t = \log(p_{t+1}/p_t), \quad \sigma = \sqrt{252} \cdot \text{sd(lr)}, \quad \mu = 252 \cdot \text{mean(lr)} + 0.5\sigma^2;$$

6. Simulate GBM Diffusion Bridge (standard and antithetic): For a given random vector $Z \sim N(0,1)^N$:
- Simulate Brownian increments: $dB(i) = \sqrt{\Delta t} \cdot Z_i, \quad B(t_k) = \sum_{i=1}^{k} dB(i)$;
- Construct Brownian bridge: $\tilde{B}(t_k) = B(t_k) - \frac{t_k}{T} \cdot B(T)$;
- Add drift and linear trend to match endpoints:

$$\text{drift}(t_k) = \left(\mu - \tfrac{1}{2}\sigma^2\right) t_k, \quad \text{lin}(t_k) = \frac{(\log S_T - \log S_0) - (\mu - 0.5\sigma^2)T}{T} \cdot t_k;$$

- Form bridge path:

$$\text{GBM\_bridge}(t_k) = \exp\left(\log S_0 + \text{drift}(t_k) + \text{lin}(t_k) + \sigma \tilde{B}(t_k)\right);$$

- For antithetic version: average with path using $-Z$;
7. Estimate OU parameters via linear regression on log prices:
- Let $X_t = \log(p_t)$;

- Fit: $X_{t+1} = a + bX_t + \epsilon_t$;
- Then: $\kappa = -\frac{\log b}{\Delta t}, \quad \theta = \frac{a}{1-b}, \quad \sigma_{\mathrm{OU}} = \sqrt{\frac{2\kappa \cdot \mathrm{Var}(\epsilon)}{1-\exp(-2\kappa\Delta t)}}$;

8. Simulate OU Diffusion Bridge (standard and antithetic): For a given random vector $Z \sim N(0,1)^N$:

- Initialize: $X(0) = \log S_0$;
- Euler–Maruyama step:

$$X(t_i) = X(t_{i-1}) + \kappa(\theta - X(t_{i-1}))\Delta t + \sigma_{\mathrm{OU}}\sqrt{\Delta t} \cdot Z_i;$$

- Correct endpoint: $\tilde{X}(t_k) = X(t_k) + \frac{t_k}{T}(\log S_T - X(T))$;
- Exponentiate: $\mathrm{OU\_bridge}(t_k) = \exp(\tilde{X}(t_k))$;
- For antithetic version: average with path using $-Z$;

9. Output:

- Simulated paths: gbm_std, gbm_vr, ou_std, ou_vr;
- Visual comparison with normalized actual path act.

4. Discussion & Conclusions

This study implemented and evaluated various classical stochastic calculus models on real stock market data from Apple Inc. (AAPL). The models included the geometric Brownian motion, the standard Brownian motion with and without drift, the Brownian bridge, and the OU process. Rather than treating these models as abstract mathematical tools, the analysis focused on how well they capture actual price behavior and volatility dynamics observed in financial markets.

The findings suggest that each model possesses distinct strengths and limitations. The GBM model effectively reflects the medium-term upward trend in stock prices but underrepresents extreme volatility and market shocks. In practice, asset prices may exhibit both upward and downward abrupt changes. These abrupt movements, often referred to as jumps in the finance literature, suggest that extending the basic price model to include jump components can improve its ability to capture real-world price dynamics.

Introducing a drift term improves the model's ability to simulate directional movement and accelerates the path toward predefined thresholds, aligning more closely with observed market behavior during bullish periods.

The Brownian bridge offered a useful benchmark for distinguishing between random and structurally driven price behavior, revealing that AAPL's price fluctuations are not purely stochastic but also shaped by directional trends and asymmetries. The OU model, with its inherent mean-reversion property, was more suitable for capturing stable fluctuations around an equilibrium, yet failed to replicate the persistent growth trend seen in the actual stock price.

Importantly, this comparative modeling approach demonstrates that while classical stochastic processes provide powerful tools for stylized analysis, they are insufficient when applied in isolation to real-world financial time series. Real asset prices are influenced by a mix of noise, structural shifts, external shocks, and behavioral dynamics that lie beyond the scope of basic stochastic assumptions.

Future research could benefit from using models that, beyond gradual changes, also allow for sudden jumps in prices such as jump-diffusion switching models that better reflect market behavior under stress.

Incorporating non-Gaussian statistical frameworks would also help address the heavy tails and extreme deviations observed in real financial data, which classical models tend to underestimate.

Furthermore, the application of machine learning methods in financial modeling may open new paths of analysis, enabling deeper insight into real market dynamics and improving forecasting accuracy. These advancements could contribute to the development of more realistic models and support better-informed decision-making in finance and investment.

5. Compliance with ethical standards

Conflict of interest: The authors declare no conflict of interest.

Data Availability: The data that support the findings of this study are openly available in: `https://finance.yahoo.com/quote/AAPL/history/?period1=1577836800&period2=1735603200`.

Ethical standard: This article does not contain any studies with human participants or animals.

Financial support: Not applicable.

Appendices

In this section, we provide the R codes used throughout the analysis presented in the paper. Each appendix contains a specific simulation relevant to the models and results discussed in the main text.

Appendix 1. Simulation of the Geometric Brownian Motion Model for AAPL Stock. This code simulates a geometric Brownian motion path based on actual AAPL stock data, and compares it with the real price trajectory.

```
N <- length(prices) - 1
dt <- 1 / trading_days
W <- c(0, cumsum(rnorm(N, 0, sqrt(dt))))
S0 <- as.numeric(prices[1])
t <- seq(0, N * dt, by = dt)
S_sim <- S0 * exp((mu - 0.5 * sigma^2) * t + sigma * W)
sim_xts <- xts::xts(S_sim, order.by = index(prices))
plot(prices, col = "black", lwd = 2,
main = "Comparison of Actual AAPL Price and GBM Simulation",
xlab = "Date", ylab = "Price (USD)")
lines(sim_xts, col = "blue", lwd = 2, lty = 2)
```

Appendix 2. Hitting and Exit Analysis in the Standard Brownian Motion (No Drift). In the first part of the code below, we compute the first time the cumulative log-return process exits the interval, estimated from AAPL data. This band reflects typical price fluctuations over 2020–2024.

```
log_returns <- diff(log(adj_prices))
W <- cumsum(na.omit(log_returns))
a <- -0.2
b <- 0.5
tau_index <- which(W < a | W > b)[1]
tau_value <- W[tau_index]
time_num <- 1:length(W)
tau_label <- expression(tau)
plot(time_num, as.numeric(W), type = "l", col = "blue", lwd =
2,
main = "W(t) and First Exit Time",
xlab = "Day Index", ylab = "W(t)")
abline(h = c(a, b), col = "red", lty = 2)
points(time_num[tau_index], tau_value, col = "darkgreen", pch
= 19, cex = 2)
text(time_num[tau_index], tau_value + 0.05, labels =
"tau_label", col = "darkgreen", cex = 1.2)
```

Now, we calculate the first time $W(t)$ reaches a profit level of $\alpha = 0.5$. This helps identify the earliest time point at which the asset achieves a 50% gain based on real AAPL stock data.

```
time <- 1:length(cumulative_returns)
alpha <- 0.5
T_alpha_index <- which(cumulative_returns >= alpha)[1]
T_alpha <- time[T_alpha_index]
plot(time, cumulative_returns, type = "l", col = "blue", lwd
= 2,
main = "W(t) and Hitting Time at \alpha = 0.5",
xlab = "Day Index", ylab = "W(t)")
abline(h = alpha, col = "red", lty = 2)
points(T_alpha, cumulative_returns[T_alpha_index], col =
"darkgreen")
```

Appendix 3. Simulation of Hitting Time under the Drifted Brownian Motion. Based on the previous section, this code adds a drift component to the cumulative log-return process and calculates the first hitting time of the 0.5 level.

```
hitting_index <- NA
hitting_time <- NA
plot(xproc, type = "l", col = "red", lwd = 2,
xlab = "Day", ylab = "Cumulative Return (log)",
main = paste("Drifted Brownian Path with Hitting Level =",
a))
abline(h = a, col = "darkgreen", lty = 2)
cat("Hitting index:", hitting_index, ", Approximate day:",
hitting_time, "\n")
```

Appendix 4. Simulation of the Brownian Bridge. This section simulates a Brownian Bridge and compares it with the normalized adjusted price path of AAPL.

```
price_vector <- as.numeric(Ad(AAPL))
normalize <- function(x) {
(x - min(x)) / (max(x) - min(x))
}
price_norm <- normalize(price_vector)
n <- length(price_vector) - 1
time <- seq(0, 1, length.out = n + 1)
dt <- 1 / n
dW <- rnorm(n, mean = 0, sd = sqrt(dt))
W <- c(0, cumsum(dW))
bridge <- W - time * W[length(W)]
bridge_norm <- normalize(bridge)
plot(time, price_norm, type = "l", col = "blue", lwd = 2,
xlab = "Normalized Time", ylab = "Normalized Value",
main = "Comparison of AAPL Stock Price and Brownian Bridge")
lines(time, bridge_norm, col = "red", lwd = 2)
```

Appendix 5. Simulation of the Ornstein–Uhlenbeck Model for AAPL Log Prices. This code compares the log-adjusted prices of AAPL with a simulated OU process to assess mean-reverting behavior.

```
X[1] <- log_price[1]
for (i in 2:n) {
X[i] <- X[i - 1] + theta * (mu - X[i - 1]) * dt + sigma *
rnorm(1, 0, sqrt(dt))
}
plot(log_price, type = "l", col = "blue", lwd = 2,
main = "Comparison of AAPL Log Price and OU Model",
xlab = "Day", ylab = "Log Price")
lines(X, col = "red", lwd = 2)
```

Appendix 6. Simulation of the Diffusion Bridges (GBM and OU) for AAPL Stock. This appendix provides the cleaned and unified R code for simulating diffusion bridges:

- GBM bridge with drift and endpoint-alignment;
- OU diffusion bridge using affine correction.

```
p <- na.omit(Ad(AAPL)); dates <- index(p)
N <- NROW(p)-1; Tyr <- N/252; dt <- Tyr/N
S0 <- as.numeric(first(p)); ST <- as.numeric(last(p))
norm <- function(x){r<-range(x);(x-r[1])/(r[2]-r[1])}
act <- norm(as.numeric(p))
lr <- diff(log(p)); sig <- sd(lr)*sqrt(252)
mu <- mean(lr)*252+0.5*sig^2
gbm_bridge <- function(S0,ST,mu,sig,T,N,antithetic=FALSE){
dt<-T/N; t<-seq(0,T,length.out=N+1)
z0<-log(S0); zT<-log(ST)
make<-function(Z){B<-c(0,cumsum(sqrt(dt)*Z))
B<-B-(t/T)*B[length(B)]
drift<-(mu-0.5*sig^2)*t
lin<-((zT-z0)-(mu-0.5*sig^2)*T)*(t/T)
exp(z0+drift+lin+sig*B)}
Z<-rnorm(N); if(!antithetic) make(Z) else
0.5*(make(Z)+make(-Z))
}
gbm_std<-gbm_bridge(S0,ST,mu,sig,Tyr,N);
gbm_vr<-gbm_bridge(S0,ST,mu,sig,Tyr,N,TRUE)
X<-as.numeric(log(p)); fit<-lm(X[-1]~X[-length(X)])
b<-coef(fit)[2]; a<-coef(fit)[1]; kap<--log(b)/dt;
th<-a/(1-b)
resv<-var(resid(fit)); sig_o<-sqrt(2*kap*resv/
(1-exp(-2*kap*dt)))
ou_bridge<-function(x0,xT,kap,th,sig,T,N,antithetic=FALSE){
dt<-T/N;t<-seq(0,T,length.out=N+1)
step<-function(Z){X<-numeric(N+1);X[1]<-x0
for(i in 1:N) X[i+1]<-X[i]+kap*(th-X[i])*dt+sig*sqrt(dt)*Z[i]
X+(t/T)*(xT-X[N+1])}
Z<-rnorm(N);out<-if(!antithetic) step(Z) else
0.5*(step(Z)+step(-Z))
exp(out)}
ou_std<-ou_bridge(log(S0),log(ST),kap,th,sig_o,Tyr,N)
ou_vr <-ou_bridge(log(S0),log(ST),kap,th,sig_o,Tyr,N,TRUE)
```

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