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On the Ring of Continuous Functions with Countable Values and Compact Support

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Abstract. In this paper, we investigate the structure of $C_{cK}(X)$, the set of all functions $f \in C_c(X)$ whose support, defined as $\mathrm{cl}_X(X \setminus Z(f))$, is compact. We study $C_{cK}(X)$ as an ideal of $C_c(X)$ and characterize its closure in the topological ring $C_{c_m}(X)$ as the intersection of all maximal ideals containing it. Additionally, we introduce the space X_{cL} and examine its relationship with $C_{cK}(X)$, particularly in connection with the purity and projectivity of the ideal. We establish necessary and sufficient conditions for $C_{cK}(X)$ to be a pure or projective $C_c(X)$-module. Moreover, we show that $C_c(X)$ is a pp-ring if and only if the space X is c-basically disconnected. Finally, we prove that $C_{cK}(X)$ is a pure ideal and that X_{cL} is c-basically disconnected if and only if every principal ideal (f), with $f \in C_{cK}(X)$, is a projective $C_c(X)$-module.

Keywords:Projective, pure, pp-ring, c-basically disconnected.

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1. Introduction

A central theme in the study of rings of continuous functions has been regarded as the examination of how topological properties of a space X are reflected in the algebraic structure of $C(X)$. In this context, attention has been directed toward the subring $C_c(X)$, consisting of all continuous functions on X with countable image, as well as other related subrings, see [4], [5]. It has been established that $C_c(X)$ retains many features of $C(X)$, while exhibiting distinct characteristics of its own. Specifically, the role of z-ideals in $C(X)$ has been shown to have a precise analogue in $C_c(X)$. Moreover, it has been demonstrated that each $C_c(X)$ is isomorphic to $C_c(Y)$ for some zero-dimensional space Y, thereby permitting the reduction of problems to the zero-dimensional setting without loss of generality. This parallels the classical result that $C(X)$ is isomorphic to $C(Y)$ for some completely regular space Y. Furthermore, several properties absent in the ring $C^*(X)$ of bounded continuous functions have been verified to hold within $C_c(X)$. Although $C_c(X)$ is not algebraically defined, it has been observed, analogously to $C^*(X)$, to be preserved under isomorphism: if $C(X) \cong C(Y)$, then both $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$ follow, where $C^F(X)$ denotes the subring of $C(X)$ consisting of functions with finite image. This preservation arises from the fact that for any ring homomorphism $\varphi : C(X) \to C(Y)$, one has $\mathrm{Im}(\varphi(f)) \subseteq \mathrm{Im}(f)$ (see the discussion following [4, Corollary 3.5]). It should also be recalled that both $C_c(X)$ and $C^F(X)$ are algebraically closed in $C(X)$ ([4, Proposition 3.1]). In addition, $C_c(X)$ has been recognized as an algebraic subring of $C(X)$, since it contains all constant functions and satisfies the property that $f^2 \in C_c(X)$ implies $f \in C_c(X)$ for each $f \in C(X)$. It is also noteworthy that $C^F(X)$ forms a regular ring and constitutes the smallest algebraic subring of $C(X)$ ([6, 16.29]; [2, Proposition 2.1]). In summary, $C_c(X)$ has been acknowledged as more than a mere replica of $C(X)$. From the results presented in [5], [4], [7], [8], [15], [10], and [2], the fundamental properties of $C_c(X)$ may be systematically learned. Furthermore, it has been perceived that $C_c(X)$ and certain locally related constructions, such as $L_c(X)$ ([8]) and $L_{cc}(X)$ ([10]), serve purposes comparable to those of $C(X)$ in many contexts of study. A significant observation is that the Stone–Čech compactification βX, commonly identified with $\mathrm{Max}(C(X))$, represents the space of maximal ideals of $C(X)$ equipped with the Zariski topology. In a similar vein, the Banaschewski compactification $\beta_0 X$—as described in [14, Sec. 4.7]—corresponds to $\mathrm{Max}(C_c(X))$, the space of maximal ideals of $C_c(X)$ under the Zariski topology, as discussed in [2, Remarks 3.6, 3.7]. The support of a function $f \in C(X)$ is defined as the closure of the set $X \setminus Z(f)$. The subring

$C_K(X)$ is given by $C_K(X) = \{f \in C(X) : \text{the support of } f \text{ is compact}\}$. The equivalence between $C_K(X)$ and the intersection of all free maximal ideals in $C(X)$—where an ideal I is said to be *free* if $\bigcap Z[I] = \emptyset$, and otherwise *fixed*—was first established by Kaplansky for discrete spaces. Kaplansky also raised the question of whether this equivalence holds more generally. Subsequently, Kohls extended the result to P-spaces, and further generalizations were obtained using the notion of the *socle* in [9]. Recent work in [16] has characterized the topological spaces for which $C_K(X)$ equals the intersection of free maximal ideals. In particular, for *pseudo-finite* spaces—those in which every compact subspace is finite—it was shown in [9] that the socle of $C(X)$ coincides with $C_K(X)$. In analogy with Kaplansky's original question, the general equivalence between $C_K(X)$ and the socle of $C(X)$ was also posed in [9]. Let $C_\infty(X)$ denote the ideal of $C^*(X)$ consisting of functions f that vanish at infinity, i.e., for every $n > 0$, the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact [6, 7F]. Azarpanah introduced the notation $C_F(X)$ for the socle of $C(X)$, and showed that $C_K(X) = C_F(X) \quad$ (respectively, $C_\infty(X) = C_F(X)$) if and only if X is *pseudo-discrete*, meaning that every compact subspace of X has finite interior (respectively, X is pseudo-discrete with only finitely many isolated points). It is clear that $C_K(X) \subseteq C_\infty(X)$. The ideal $C_K(X)$ represents the intersection of all free ideals in both $C(X)$ and $C^*(X)$, while $C_\infty(X)$ corresponds to the intersection of free maximal ideals in $C^*(X)$ [6, 7F]. Moreover, both $C_K(X)$ and $C_\infty(X)$ can be expressed as intersections of essential ideals. However, it is known that the intersection of essential ideals in $C(X)$ may be trivial, particularly when X contains no isolated points. This leads to a natural question: under what conditions do intersections such as $C_K(X)$ or $C_\infty(X)$ remain essential? This problem is investigated further in [1]. We define $C_{cK}(X)$ as the set of all functions $f \in C_c(X)$ such that the closure $\text{cl}_X(X \setminus Z(f))$ is compact. In this paper, we study $C_{cK}(X)$ as an ideal of $C_c(X)$, focusing on both its algebraic and topological characteristics. In particular, we examine its behavior in the topological ring $C_{c_m}(X)$, where $C_c(X)$ is endowed with the m_c-topology. We prove that the closure of $C_{cK}(X)$ in $C_{c_m}(X)$ coincides with the intersection of all maximal ideals of $C_{c_m}(X)$ containing $C_{cK}(X)$. To gain deeper insight into the structure of $C_{cK}(X)$, we introduce the subspace X_{cL} of X and investigate its connection with $C_{cK}(X)$. We provide necessary and sufficient conditions for $C_{cK}(X)$ to be a pure ideal in $C_c(X)$, especially in the cases where X is compact or C-pseudocompact. Assuming that $C_{cK}(X)$ is pure, we show that for every ideal $I \subseteq C_{cK}(X)$, the set $X \setminus Z_c(I)$ is contained in X_{cL}. We also characterize the conditions under which $C_{cK}(X)$ becomes a projective $C_c(X)$-module. Moreover, we demonstrate that $C_c(X)$ is a pp-ring if and only if the space X is c-basically disconnected. Finally, we establish that X_{cL} is c-basically disconnected and that $C_{cK}(X)$ is pure if and only if, for every $f \in C_c(X)$, the principal ideal (f) is a projective $C_c(X)$-module.

2. $C_{cK}(X)$

We denote by $C_{cK}(X)$ the set of all functions in $C_c(X)$ with compact support, that is,

$$C_{cK}(X) = \{f \in C_c(X) : Supp(f) = \mathrm{cl}_X(X \setminus Z(f)) \text{ is compact}\}.$$

In what follows, we study $C_{cK}(X)$ as an ideal of $C_c(X)$.

Recall that a topological space X is called locally compact if every point $x \in X$ has a neighborhood whose closure is compact. Similarly, X is said to be nowhere locally compact if no point of X has a compact neighborhood. Equivalently, X is nowhere locally compact if and only if for each $x \in X$ and any neighborhoods U of x, $\overline{U}$ is not compact.

The following lemma corresponds to [18, Lemma 4.4], and its proof follows exactly the same reasoning as in that result. We include it here since it will be used in the proof of the Theorem 2.2.

Lemma 2.1. *An ideal I of $C_c(X)$ (or $C_c^*(X)$) is a free ideal if and only if for every compact subset $A \subseteq X$ there exists $f \in I$ such that $f(x) \neq 0$ for all $x \in A$.*

Theorem 2.2. *Let X be a Hausdorff, zero-dimensional, and countable completely regular space. Then the following hold:*

1. *The set $C_{cK}(X)$ is an ideal of $C_c^*(X) = C_c(X) \cap C^*(X)$.*
2. *If X is compact, then $C_{cK}(X) = C_c(X)$.*
3. *The set $C_{cK}(X)$ is a free ideal if and only if X is locally compact and non-compact.*
4. *The set $C_{cK}(X)$ is contained in every free ideal of $C_c(X)$ (or $C_c^*(X)$).*
5. *The space X is nowhere locally compact if and only if $C_{cK}(X) = \{0\}$; equivalently, this occurs precisely when the remainder $\beta_0 X \setminus X$ is dense in $\beta_0 X$.*

Proof. (1) Let $f \in C_{cK}(X)$. Then

$$f(X) \setminus \{0\} = f(X \setminus Z(f)) \subseteq f(\mathrm{cl}(X \setminus Z(f))).$$

Since $\mathrm{cl}(X \setminus Z(f))$ is compact and f is continuous, it follows that $f(\mathrm{cl}(X \setminus Z(f)))$ is compact and hence closed in $\mathbb{R}$. Thus $f(X) \setminus \{0\}$ is compact, and therefore $f(X)$ is closed in $\mathbb{R}$. Hence $f \in C_c(X)$, and so $C_{cK}(X) \subseteq C_c(X)$.

Now, let $g \in C_c^*(X)$ and $f \in C_{cK}(X)$. Then

$$\mathrm{cl}(X \setminus Z(gf)) = \mathrm{cl}\left((X \setminus Z(g)) \cap (X \setminus Z(f))\right).$$

Since $\mathrm{cl}(X \setminus Z(f))$ is compact and the right-hand side is a closed subset of it, we conclude that $\mathrm{cl}(X \setminus Z(gf))$ is compact. Hence $gf \in C_{cK}(X)$.

(2) Suppose X is compact. For every $f \in C_c(X)$, the set $\mathrm{cl}(X \setminus Z(f))$ is closed in X, hence compact. Thus $C_{cK}(X) = C_c(X)$.

(3) Assume $C_{cK}(X)$ is a free ideal. Then by Lemma 2.1, there exists $f \in C_{cK}(X)$ such that $f(x) \neq 0$ for all $x \in X$. This implies that $X \setminus Z(f)$ is dense in X and contained in the compact set $\mathrm{cl}(X \setminus Z(f))$, showing that X is locally compact. Moreover, X cannot be compact, since otherwise $C_{cK}(X) = C_c(X)$, contradicting the assumption that $C_{cK}(X)$ is a proper free ideal.

Conversely, assume X is locally compact but not compact. By part (1), $C_{cK}(X)$ is an ideal. To show it is free, take any compact set $A \subseteq X$. Since X is locally compact, each $x \in A$ has a compact neighborhood. Using the complete regularity and countability assumptions, there exists $f \in C_c(X)$ with $A \subseteq X \setminus Z(f) \subseteq N$, where N is compact. Hence $\mathrm{cl}(X \setminus Z(f)) \subseteq N$ is compact, so $f \in C_{cK}(X)$ and $f(x) \neq 0$ for all $x \in A$. Thus $C_{cK}(X)$ is free.

(4) Let $f \in C_{cK}(X)$ and I be a free ideal of $C_c(X)$. By Lemma 2.1, there exists $g \in I$ that does not vanish on $\mathrm{cl}(X \setminus Z(f))$, i.e.,

$$\mathrm{cl}(X \setminus Z(f)) \subseteq X \setminus Z(g).$$

Thus $Z(g) \subseteq \operatorname{int} Z(f)$. By [6, Problem 1D.1], this implies that f is a multiple of g, so $f \in I$. A similar argument works in $C_c^*(X)$.

(5) If X is nowhere locally compact, then for every $f \in C_c(X)$ the set $\mathrm{cl}(X \setminus Z(f))$ is non-compact. Hence no nonzero f can belong to $C_{cK}(X)$, so $C_{cK}(X) = \{0\}$. Conversely, if $C_{cK}(X) = \{0\}$, then no nonzero function has compact support, which means X is nowhere locally compact. The equivalent characterization follows from the fact that $\beta_0 X \setminus X$ is dense in $\beta_0 X$ exactly in this case. □

Theorem 2.3. *Suppose that $f \in C_c(X)$ is such that $cl_{\beta_0 X} Z(f)$ is a neighbourhood of $\beta_0 X \setminus X$, then $f \in C_{cK}(X)$.*

Proof. It suffices to show that $cl_X(X \setminus Z(f))$ is closed in $\beta_0 X$ and hence compact. Since $Z(f)$ is closed in X, we infer that $cl_{\beta_0 X} Z(f) \cap (X \setminus Z(f)) \neq \emptyset$. By hypothesis, there exists an open set U in $\beta_0 X$ such that $\beta_0 X \setminus \subset U \subset cl_{\beta_0 X} Z(f)$. Hence $U \cap (X \setminus Z(f)) = \emptyset$, which further implies because U is open in $\beta_0 X$ that $U \cap cl_{\beta_0 X}(X \setminus Z(f)) = \emptyset$. Consequently $U \cap cl(X \setminus Z(f)) = \emptyset$. Since $\beta_0 X \setminus X \subset U$, we infer that no point of $\beta_0 X \setminus X$ is a limit point of $cl_X(X \setminus Z(f))$ in the space $\beta_0 X$. Thus there does not exist any limiting point of $cl_X(X \setminus Z(f))$ out side it in the entire space $\beta_0 X$. Hence $cl_X(X \setminus Z(f))$ is closed in $\beta_0 X$. $\square$

We recall that for every zero-dimensional space X, the maximal ideals of $C_c(X)$ are precisely of the following form:

$$M_c^p = \{f \in C_c(X) : p \in \mathrm{cl}_{\beta_0 X} Z(f)\}, \qquad (p \in \beta_0 X).$$

Moreover, for each $p \in \beta_0 X$, we recall that

$$O_c^p = \{f \in C_c(X) : p \in \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)\}.$$

For additional properties and related results, see [2].

Remark 2.4. $C_{cK}(X) \subseteq \bigcap\{O_c^p \ : \ p \in \beta_0 X \setminus X\}$. This follows from Theorem 2.2, which states that $C_{cK}(X)$ is contained in every free ideal of $C_c(X)$ ($C_c^*(X)$). Moreover, since for each $p \in \beta_0 X$, the ideal O_c^p is free, the claim follows.

Theorem 2.5. *Let X be zero-dimensional and Hausdorff. Then*

$$C_{cK}(X) = \bigcap\{O_c^p \ : \ p \in \beta_0 X \setminus X\}.$$

Proof. Let $f \in O_c^p$ for each $p \in \beta_0 X \setminus X$. Then $cl_{\beta_0 X} Z(f)$ is a neighbourhood of each point of $\beta_0 X \setminus X$ in the space $\beta_0 X$. It follows from Theorem 2.3 that $f \in C_{cK}(X)$. Thus $\bigcap\{O_c^p \ : \ p \in \beta_0 X \setminus X\} \subset C_{cK}(X)$. The reversed implication relation is already realized in Remark 2.4. Hence $C_{cK}(X) = \bigcap\{O_c^p : p \in \beta_0 X \setminus X\}$. $\square$

Corollary 2.6. *The ideal $C_{cK}(X)$ is the intersection of all free ideals in $C_c(X)$, that is, in $C_c^*(X)$.*

Proof. By Theorem 2.2(4), if E denotes the family of all free ideals in $C_c(X)$ (i.e., $C_c^*(X)$), then

$$C_{cK}(X) \subseteq \bigcap E.$$

Moreover, if $p \in \beta_0 X \setminus X$, then O_c^p is a free ideal. Hence,

$$\bigcap E \subseteq \bigcap\{O_c^p : p \in \beta_0 X \setminus X\}.$$

Now, by Theorem 2.5, it follows that

$$C_{cK}(X) = \bigcap E.$$

$\square$

Let us recall that $\mathcal{U}_c^+(X) = \{u \in \mathcal{U}^+(X) : u \in C_c(X)\}$. The m_c-topology on $C_c(X)$ is defined by taking the subset of the form

$$\mathcal{B}_c(f, u) = \{g \in C_c(X) : |f(x) - g(x)| < u(x), \ \ \forall x \in X\},$$

as a base for a neighborhood system at f, for each $f \in C_c(X)$ and $u \in \mathcal{U}_c^+(X)$. The set $C_c(X)$ endowed with the m_c-topology is denoted by $C_{cm}(X)$, see [11], [17].

Similar to [6] and [19], it can be shown that the closure of an ideal $I \subseteq C_{cm}(X)$ is precisely the intersection of all maximal ideals containing I. Consequently, an ideal of $C_{cm}(X)$ is closed if and only if it is the intersection of maximal ideals. Therefore, every maximal ideal in $C_c(X)$ is closed with respect to the m_c-topology.

We characterize the closure of $C_{cK}(X)$ in $C_{cm}(X)$ as the intersection of all maximal ideals containing it.

Theorem 2.7. $cl_m(C_{cK}(X)) = \bigcap_{p\in\beta_0 X\setminus X} M_c^p$.

Proof. $C_{cK}(X) = \bigcap_{p\in\beta_0 X\setminus X} O_c^p$, so $C_{cK}(X) \subseteq \bigcap_{p\in\beta_0 X\setminus X} M_c^p$. Since every maximal ideal of $C_{cm}(X)$ is closed, we infer that the intersection of maximal ideals is closed. Therefore $cl_m(C_{cK}(X)) \subseteq \bigcap_{p\in\beta_0 X\setminus X} M_c^p$. Now, we suppose that $f \in \bigcap_{p\in\beta_0 X\setminus X} M_c^p$, then $\beta_0 X \setminus X \subseteq cl_{\beta_0 X} Z(f)$. We must prove that $B_c(f,u) \cap C_{cK}(X) \neq \emptyset$ for $u \in \mathcal{U}_c^+(X)$. For this purpose, we define the following function:

$$g(x) = \begin{cases} f(x) + \frac{u(x)}{2} & , \quad f(x) \leqslant \frac{-u(x)}{2} \\ 0 & , \quad |f(x)| \leqslant \frac{u(x)}{2} \\ f(x) - \frac{u(x)}{2} & , \quad f(x) \geqslant \frac{u(x)}{2}, \end{cases}$$

It is evident that $g \in C_c(X)$. we set $H := \{x \in X : |f(x)| \geqslant \frac{u(x)}{2}\}$, then H is a zero-set in X. Suppose that $h \in C_c(X)$ such that $H = Z(h)$ and we show that $Z(f) \subseteq X \setminus Z(h) \subseteq Z(g)$. For this main, we suppose that $f(x) = 0$, so $x \in Z(f)$. Hence $f(x) = 0 < \frac{u(x)}{2}$, and therefore $x \in X \setminus Z(h)$. Also, if $x \in X \setminus Z(h)$, then $|f(x)| < \frac{u(x)}{2}$, so $x \in Z(g)$. Hence, $cl_{\beta_0 X} Z(g)$ is a zero-set and $\beta_0 X \setminus X \subseteq cl_{\beta_0 X} Z(f) \subseteq int_{\beta_0 X} cl_{\beta_0 X} Z(g)$. thus, from the other side, for any $g \in \bigcap_{p\in\beta_0 X\setminus X} O_c^p = C_{cK}(X)$. That is, $|f(X) - g(X)| < u(X)$ for all $x \in X$. Which means $g \in B_c(f,u)$. So $|f - g| < u$, and therefore $g \in B_c(f,u) \cap C_{cK}(X)$. □

3. Purity of the ideal $C_{cK}(X)$

In this section, we introduce the space X_{cL}, examine its relationship with $C_{cK}(X)$, and conclude by stating the conditions under which $C_{cK}(X)$ is pure.

Definition 3.1. If I is an ideal of $C_c(X)$, then

$$X \setminus Z_c(I) = \bigcup_{f\in I} X \setminus Z(f).$$

Definition 3.2. Let X be zero-dimension. X_{cL} is the set of all point in X having compact neighborhood.

Remark 3.3. X is locally compact if and only if $X = X_{cL}$. Also if $X_{cL} = \emptyset$, then X is nowhere locally compact.

Lemma 3.4. *Let X be countable completely regular, then*

$$X_{cL} = X \setminus Z_c(C_{cK}(X))$$

.

Proof. Suppose $x \in X \setminus Z_c(C_{cK}(X))$. There exists an element f in $C_{cK}(X)$ exists such that $f(x) \neq 0$, which imply that $\overline{X \setminus Z(f)}$ is compact and $X \setminus Z(f)$, and so $X \setminus \overline{Z(f)}$ is a compact neighborhood of x. But we know $x \in X \setminus Z(f) \subseteq \overline{X \setminus Z(f)}$, given that $X \setminus Z(f)$ is an open set containing x, so $X \setminus Z(f)$ is a compact neighborhood of x. Therefore $x \in X_{cL}$. Contrariwise, we prove $X_{cL} \subseteq X \setminus Z_c(C_{cK}(X))$. Suppose $x \in X_{cL}$. By definition space X_{cL}, x has a compact neighborhood like U which $x \in int\,U$ and since X is countable completely regular, there exists $f \in C_c(X)$ such that $x \in X \setminus Z(f) \subseteq U$. Since U is compact, we infer that U is close. So $x \in X \setminus Z(f) \subseteq \overline{X \setminus Z(f)} \subseteq \overline{U} = U$. Therefore $\overline{X \setminus Z(f)}$ is a subset compact of U. So $\overline{X \setminus Z(f)}$ is compact and $f \in C_{cK}(X)$. Therefore $X_{cL} = X \setminus Z_c(C_{cK}(X))$. □

Remark 3.5. Throughout this paper for every $f \in C(X)$ we set $f^* := -1 \vee (f \wedge 1)$. Also, the continuous extension of f^* over $\beta_0 X$ is called closure of f and we write $\bar{f} = (f^*)^{\beta_0}$.

Theorem 3.6. *The following statements are true for any zero-dimension space X.*

1. $X_{cL} = \text{int}_{\beta_0} X$, *which is an open subset of X and of $\beta_0 X$.*
2. X_{cL} *is locally compact subset of X.*
3. *For each $f \in C_{cK}(X)$, $X \setminus Z(f)$ is an open subset of $\beta_0 X$.*

Proof. (1) Let $x \in int_{\beta_0 X} X$, then there exists an open set U of $\beta_0 X$ such that $x \in X \subseteq U$. Regularity of $\beta_0 X$ implies that there exists an open set V of $\beta_0 X$ such that $x \in V \subseteq cl_{\beta_0 X} \subseteq U$. Hence, $cl_{\beta_0 X} V$ is compact neighborhood of X and so $x \in X_{cL}$
Part(2) from the first part, it easily follows.
(3) $X \setminus Z(f)$ is open in X and $(X \setminus Z(f)) \cap X_{cL} = X \setminus Z(f)$. so $X \setminus Z(f)$ is open in X_{cL} and by part (2), X_{cL} is open in $\beta_0 X$. therefore $X \setminus Z(f)$ is open in X and $\beta_0 X$.

□

Corollary 3.7. $X_{cL} = \emptyset$, *if and only if $\beta_0 X \setminus X$ be dense in $\beta_0 X$, if and only if $C_{cK}(X) = \{0\}$.*

Proof. By Proposition 3.6, $X_{cL} = int_{\beta_0 X} X$ and $\beta_0 X \setminus X_{cL} = \beta_0 X \setminus int_{\beta_0 X} X$. Therefore, $\beta_0 X \setminus X_{cL} = cl_{\beta_0 X}(\beta_0 X \setminus X)$. So $X_{cL} = \emptyset$ if and only if $\beta_0 X = cl_{\beta_0 X}(\beta_0 X \setminus X)$ if and only if $\beta_0 X \setminus X$ be dense in $\beta_0 X$. Now, we show $\beta_0 X \setminus X$ is dense in $\beta_0 X$ if and only if $C_{cK}(X) = \{0\}$. Suppose $X_{cL} = \emptyset$ and By Lemma 3.4, $X \setminus Z_c(C_{cK}(X)) = \emptyset$. Therefore $\bigcup_{f \in C_{cK}(X)} X \setminus Z(f) = \emptyset$ and we have $X \setminus Z(f) = \emptyset$, for each $f \in C_{cK}(X)$. So $X \setminus Z(f) = \emptyset$; that is $Z(f) = X$. So $f = 0$ and $C_{cK}(X) = \{0\}$. For the converse, if $C_{cK}(X) = \{0\}$, then $X_{cL} = X \setminus Z_c(C_{cK}(X)) = X \setminus Z_c(\{0\}) = \emptyset$. Hence $X_{cL} = \emptyset$. □

4. Purity of the ideal $C_{cK}(X)$ in compact space and C-pesudocompact space

We recall from [6] that an ideal I of a commutative ring R is called pure if for each $a \in I$, there exists $b \in I$ such that $a = ab$.

Lemma 4.1. *If I be a pure ideal in $C_c(X)$, then*

$$X \setminus Z_c(I) = \bigcup_{f \in I} Supp(f).$$

Proof. It is clear that $X \setminus Z_c(I) = \bigcup_{f \in I} X \setminus Z(f) \subseteq \bigcup_{f \in I} Supp(f)$. Conversely, suppose $f \in I$. Since I is pure, we infer that there exists $g \in I$ such that $f = fg$ and $g\big|_{Supp(f)} = 1$. So for each $f \in I$, $Supp(f) \subseteq X \setminus Z(g)$. Hence

$$\bigcup_{f \in I} Supp(f) \subseteq \bigcup_{g \in I} X \setminus Z(g) = X \setminus Z_c(I).$$

Therefore, $X \setminus Z_c(I) = \bigcup_{f \in I} Supp(f)$. □

The following theorem is the counterpart of [13, Theorem 2.3], and its proof follows essentially the same line of reasoning. For the sake of clarity and completeness, we provide the full argument here.

Theorem 4.2. *Let I be a Z_c-ideal including $C_{cK}(X)$. Then the following statements are equivalent:*

(1) *I is pure.*
(2) *$I = O_c^{\beta_0 X \setminus (X \setminus Z_c(I))}$.*
(3) *$X \setminus Z_c(I) = \bigcup_{f \in I} Supp(f)$.*

Proof. (1) ⇒ (2). Suppose I be a pure, in this case $I = O_c^A$ such that $A = \bigcap Z(\bar{f})$, see [12]. So $\beta_0 X \setminus A = \bigcup_{f \in I} \beta_0 X \setminus Z(\bar{f})$. Since $X \subseteq \beta_0 X$, we infer that $(\beta_0 X \setminus Z(f)) \cap X = \beta_0 X \cap (X \setminus Z(f)) \cap X = X \setminus Z(f)$. So

$$\beta_0 X \setminus A = \bigcup_{f \in I} \beta_0 X \setminus [Z(f) \cup (\beta_0 X \setminus X)] = \bigcup_{f \in I} (\beta_0 X \setminus Z(f)) \cap X = \bigcup_{f \in I} X \setminus Z(f).$$

(2) $\Rightarrow$ (3). By Corollary 3.6, $X \setminus Z(f)$ is an open subset of $\beta_0 X$. So $\beta_0 X \setminus (X \setminus Z_c(I))$ is a closed subset of $\beta_0 X$. So I is a pure. Then by Lemma 4.1 we have $X \setminus Z_c(I) = \bigcup_{f \in I} Supp(f)$.
(3) $\Rightarrow$ (1). Let $g \in I$. Then

$$Supp(g) \subseteq \bigcup_{f \in I} Supp(f) = X \setminus Z_c(I) = \bigcup_{f \in I} X \setminus Z(f).$$

Since $Supp(g)$ is compact, we infer that $Supp(g) \subseteq \bigcup_{i=1}^{n} X \setminus Z(f_i)$ for each $f_1, f_2, \ldots, f_n \in I$. Suppose $h = \sum_{i=1}^{n} f_i^2$. So $h \in I$ and $X \setminus Z(h) = \bigcup_{i=1}^{n} X \setminus Z(f_i)$. Let $k \in C_c(X)$ such that $k(Supp(g)) = 1$ and $k(Z(h)) = 0$. Then $g = gk$ and $Z(h) \subset Z(k)$. Since I is Z_c-ideal, we infer that $k \in I$ and I is a pure ideal. □

Theorem 4.3. *Let $C_{cK}(X)$ be a pure ideal. Then $X \setminus Z_c(I)$ is proper subset X_{cL} for each proper ideal I of $C_{cK}(X)$.*

Proof. We prove the case with the help of reverse proof. Suppose I be an ideal of $C_{cK}(X)$ such that $X \setminus Z_c(I) = X_{cL}$. Also let $f \in C_{cK}(X)$. Hence by purity ideal I and Lemma 4.1, $X \setminus Z_c(I) = \bigcup_{f \in I} Supp(f)$. So $Supp(f) \subseteq X_{cL} = X \setminus Z_c(I)$ and therefore $Supp(f) \subseteq \bigcup_{i=1}^{n} X \setminus Z(f_i)$ for each $f_i \in I$. Now suppose $\sum_{i=1}^{n} f_i^2 = g$. Since $f_i \in I$ for each i, we infer that $g \in I$ and

$$X \setminus Z(g) = \bigcup_{i=1}^{n} X \setminus Z(f_i).$$

Now, we define function $h(x)$;

$$h(x) = \begin{cases} \dfrac{f}{g} & , x \in X \setminus Z(g) \\ 0 & , \text{in other points.} \end{cases}$$

Since $Supp(f) \subseteq X \setminus Z(g)$, $h \in C_c(X)$ and $f = gh$, we infer that $I = C_{cK}(X)$. That it contradicts the assumption. □

Theorem 4.4. *Let $C_{cK}(X)$ and $C_{cK}(Y)$ be pure ideals. If X_{cL} is homeomorphic to Y_{cL}, then $C_{cK}(X)$ is isomorphic to $C_{cK}(Y)$.*

Proof. Let $\varphi : X_{cL} \to Y_{cL}$ be a homeomorphism. For $f \in C_{cK}(Y)$, define $f_1 = f|_{Y_{cL}}$. Since $f : Y \to \mathbb{R}$, the composition

$$f_1 \circ \varphi : X_{cL} \longrightarrow Y_{cL} \longrightarrow \mathbb{R}$$

belongs to $C_c(X_{cL})$. By Lemma 3.4, we know

$$X_{cL} = X \setminus Z_c(C_{cK}), \qquad Y_{cL} = Y \setminus Z_c(C_{cK}).$$

If $f \in C_{cK}(Y)$, then $Y \setminus Z(f) \subseteq Y_{cL}$. Since $f_1 = f|_{Y_{cL}}$, it follows that

$$Y \setminus Z(f) = Y \setminus Z(f_1).$$

We claim that

$$Y \setminus Z(f) = \varphi\big[X \setminus Z(f_1 \circ \varphi)\big].$$

Indeed, let $x \in X \setminus Z(f_1 \circ \varphi)$. Then $(f_1 \circ \varphi)(x) \neq 0$, hence $(f \circ \varphi)(x) \neq 0$ and so $\varphi(x) \in Y \setminus Z(f)$. Thus

$$\varphi[X \setminus Z(f_1 \circ \varphi)] \subseteq Y \setminus Z(f).$$

Conversely, if $y \in Y \setminus Z(f)$, then $f(y) \neq 0$. Since $y \in Y_{cL}$ and φ is a homeomorphism, there exists $x \in X_{cL}$ with $y = \varphi(x)$. As $f_1(y) \neq 0$, we get $(f_1 \circ \varphi)(x) \neq 0$, hence $x \in X \setminus Z(f_1 \circ \varphi)$. Therefore

$$Y \setminus Z(f) \subseteq \varphi[X \setminus Z(f_1 \circ \varphi)].$$

This proves the claim, and consequently

$$\varphi^{-1}[Y \setminus Z(f)] = X \setminus Z(f_1 \circ \varphi).$$

Taking closures in X_{cL} yields

$$\mathrm{cl}_{X_{cL}}(X \setminus Z(f_1 \circ \varphi)) = \varphi^{-1}\big(\,\mathrm{Supp}(f)\big),$$

since $\mathrm{Supp}(f) \subseteq Y_{cL}$ by the purity of $C_{cK}(Y)$.

Now, define $g_f : X \to \mathbb{R}$ by

$$g_f(x) = \begin{cases} f_1 \circ \varphi(x) & , x \in X_{cL}, \\ 0 & , x \in X \setminus \varphi^{-1}(\mathrm{Supp}(f)). \end{cases}$$

Since $f_1 \circ \varphi(x)$ is continuous by Theorem 7.6 in [20], we infer that g_f is continuous. Also, since $f_1 \in C_{cK}(Y)$, i.e., $|f_1(Y)| \leq \aleph_0$ and $\mathrm{cl}_{Y_{cL}}(Y \setminus Z(f_1))$ is compact, it follows that

$$|(f_1 \circ \varphi)(X_{cL})| \leq |f_1(Y)| \leq \aleph_0,$$

and

$$X \setminus Z(f_1 \circ \varphi) = \varphi^{-1}(Y \setminus Z(f_1)) \subseteq X \setminus Z(f_1).$$

Hence, $g_f \in C_{cK}(X)$, because every closed subset of a compact space is compact. Therefore, the support of f,

$$\mathrm{Supp}(f) = \mathrm{cl}_{X_{cL}}(X \setminus Z(f_1 \circ \varphi)),$$

is compact. Define

$$\bar{\varphi} : C_{cK}(Y) \to C_{cK}(X), \qquad \bar{\varphi}(f) = g_f.$$

Since $\varphi : X_{cL} \to Y_{cL}$ is a homeomorphism, we infer that $\bar{\varphi}$ is a ring homomorphism. *Injectivity.* Suppose $\bar{\varphi}(f) = 0$. Then $f_1 \circ \varphi(x) = 0$ for every $x \in X_{cL}$, which means $X \setminus Z(f_1 \circ \varphi) = \varphi^{-1}(Y \setminus Z(f)) = \emptyset$. Hence $f = 0$. Thus $\bar{\varphi}$ is injective.

Surjectivity. Let $h \in C_{cK}(X)$. Define $g : Y \to \mathbb{R}$ by

$$g(y) = \begin{cases} h \circ \varphi^{-1}(y) & , y \in Y_{cL}, \\ 0 & , y \in Y \setminus \varphi(\mathrm{Supp}(h)). \end{cases}$$

Then $g \in C_{cK}(Y)$, since $\varphi(\mathrm{Supp}(h))$ is compact and $\mathrm{Supp}(h) \subseteq X_{cL}$ by purity. Moreover,

$$\bar{\varphi}(g)(x) = \begin{cases} (g \circ \varphi)(x) & , x \in X_{cL}, \\ 0 & , x \in X \setminus \varphi^{-1}(\mathrm{Supp}(g)), \end{cases} = h(x).$$

Thus $\bar{\varphi}(g) = h$, so $\bar{\varphi}$ is surjective.

Therefore $\bar{\varphi}$ is a ring isomorphism, and hence $C_{cK}(X) \cong C_{cK}(Y)$. □

We emphasize that the proof of the following result proceeds in exactly the same manner as the proof of its analogue in $C(X)$; see [3, Theorem 3.4].

Theorem 4.5. *The principal ideal (f) is a projective $C_c(X)$-module if and only if, Supp(f) is open.*

We recall that a topological space X is said to be c-basically disconnected if for every function $f \in C_c(X)$, the support *Supp*(f) is an open subset of X.

A commutative ring is called a pp-ring if every principal ideal is a projective module.

Theorem 4.6. *$C_c(X)$ is a pp-ring if and only if, X is c-basically disconnected.*

Proof. Suppose that $C_c(X)$ is a pp-ring. Then every principal ideal of $C_c(X)$ is projective. Therefore, by Theorem 4.5, the support Supp(f) is open for each $f \in C_c(X)$, which is equivalent to X being c-basically disconnected. This completes the proof, which is also recursive in nature. □

The following facts are the counterparts of [13, Theorem 4.5, Corollary 4.6, and Corollary 4.7], and their proofs follow exactly the same arguments as in those results. Nevertheless, we provide a proof here for the sake of completeness.

Theorem 4.7. *Let I be a pure ideal containing $C_{cK}(X)$. Then $X \setminus Z_c(I)$ is c-basically disconnected if and only if every principal ideal of I is a projective $C_c(X)$-module.*

Proof. Suppose $Y = X \setminus Z_c(I)$ and Y be c-basically disconnected and $f \in I$. Since I is pure, we infer that by Lemma 4.1, $X \setminus Z_c(I) = \bigcup_{f \in I} Supp(f)$. So $Supp(f) \subseteq Y$. We consider $f_1 = f\big|_Y$. Hence $cl_Y(Y \setminus$

$Z(f_1)) = Supp(f)$. Which according to the assumption Y is c-basically disconnected. So $Supp(f)$ is open in Y and X. Therefore according to Theorem 4.5, ideal (f) is a projective $C_c(X)$-module.
For the converse, Suppose each principal ideal of I is $C_c(X)$-projective module. First we show for every $f \in C_{cK}(Y)$, $Supp(f)$ is clopen then using it for every $f \in C_c(Y)$, $Supp(f)$ is clopen point. Let $f_1 \in C_{cK}(Y)$ and define

$$f(x) = \begin{cases} f_1(x) & , x \in cl_Y(Y \setminus Z(f_1)) \\ 0 & , x \in X \setminus (Y \setminus Z(f_1)). \end{cases}$$

Since f is Z_c-ideal and $Supp(f)$ is compact set containing Y, we infer that $f \in I$. So (f) is a principal ideal of I. Hence

$$cl_Y(Y \setminus Z(f_1)) = Supp(f)$$

is clopen. Now, suppose $a \in cl_Y(Y \setminus Z(k)) \subseteq Y$ and $k \in C_c(Y)$. The open set U exists such that $\overline{U}$ is compact and $a \in U \subseteq \overline{U} \subseteq Y$. Since X is quite regular, we infer that $f \in C_c(X)$ exists such that $f(a) = 1$ and $f(X \setminus U) = 0$. Since $Supp(f)$ is compact and containing Y, we infer that $f \in I$. We know $(Y \setminus Z(f_1)) \subseteq cl_Y(Y \setminus Z(f_1)))$. So

$$\begin{aligned} a \in cl_Y(Y \setminus Z(f_1)) \cap cl_Y(Y \setminus Z(k)) &\subseteq cl_Y(Y \setminus Z(f_1)) \cap cl_Y(Y \setminus Z(k)) \\ &= cl_Y(Y \setminus Z(h)) \cap (Y \setminus Z(k)) \\ &= cl_Y(Y \setminus Z(f_1 k)) \subseteq cl_Y(Y \setminus Z(k)). \end{aligned}$$

But $cl_Y(Y \setminus Z(f_1 k))$ is compact. $f_1 k \in C_{cK}(X)$, so it is clopen. Hence $cl_Y(Y \setminus Z(k))$ is clopen in Y. So $Y = X \setminus Z_c(I)$ is c-basically disconnected. □

Corollary 4.8. *The space X_{cL} is c-basically disconnected, and $C_{cK}(X)$ is pure if and only if, for every $f \in C_{cK}(X)$, the principal ideal (f) is a projective $C_c(X)$-module.*

Proof. Assume that Supp(f) is clopen. Let g denote the characteristic function of Supp(f). Then $g \in C_{cK}(X)$ and clearly $f = fg$. □

Corollary 4.9. *Let X be a locally compact space. Then the following conditions are equivalent:*

(1) *$C_c(X)$ is a pp-ring.*
(2) *X is c-basically disconnected.*
(3) *Every principal ideal of $C_{cK}(X)$ is a projective $C_c(X)$-module.*

Proof. (1) $\Rightarrow$ (2). If $C_c(X)$ is a pp-ring, then every principal ideal is projective. By definition, this implies that X is c-basically disconnected, i.e., Supp(f) is open for each $f \in C_{cK}(X)$. Hence, the argument follows as in Theorem 4.5.

(2) ⇒ (3) and (3) ⇒ (1). These implications follow directly from the definitions. □

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