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Applications of F_h-convex functions to Integral Inequalities and Economics on time scales

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Abstract. Some new properties for products of F_h-convex functions and $\diamond_{(F_{h(\lambda)})^s}$ dynamics are applied to integral inequalities of Hermite-Hadamard type on time scales. Economic applications to dynamic Optimization problem of household utility on time scales are also discussed.

Keywords: F_h-convex, diamond-$(F_{h(\lambda)})^s$, Time scales, Hermite-Hadamard, Dynamic model.

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1. Introduction

Convexity theory plays cogent roles in various fields of pure and applied sciences ([4], [8], [11], [15], [16], [18]).

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A function $f : K \to \mathbb{R}$ is said to be convex in the classical sense if $\forall\ x, y \in K,\ \lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Consequently, this classical concept, due to its close relationship with the theory of inequalities, has been widely used and generalized by several authors to derive many useful inequalities in different directions, see [1], [4].

Among these inequalities, one of the most celebrated, important and useful results, that has attracted, and continues to attract attention from researchers in inequalities study in the last few decades, is the Hermite-Hadamard(H-H)'s inequality, which gives us an estimate, from below and from above, of the mean value of a convex function. The inequality

$$(b-a) f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}, \quad a < b, \tag{1.1}$$

holds for any convex functions f defined on $\mathbb{R}$, see, [16].

The double inequality (1.1) is known to be a fundamental result for convex functions. It has a natural geometrical interpretation, various interesting extensions, generalizations, improvements, variations and applications for uni- and multi-variate convex functions, as well as other classes of convex functions on classical intervals with recent extensions to time scales theory. See for example, [8]-[13], [19], [21]-[24].

Recently, new developments of the theory of time scales were introduced [17], in order to unify and extend the theory of difference and differential calculus with accuracy and also introduce the delta (Δ) and nabla (∇) time scales calculi (see [5]-[6]). Consequently, the diamond-alpha $(\diamond_\alpha)$ dynamic calculus on time scales, which is essentially a linear combination of the Δ and ∇ calculi, was developed by Sheng et al. [22]. From literature, the classical H-H inequality (1.1) has been further extended and improved to time scales via the Δ, ∇ and $\diamond_\alpha$ calculi for convex functions (see [9],[24] and the references therein).

More recently, different concept of a more generalized class of convex functions, called F_h-convex functions on time scales, including a more general, combined dynamic calculus, referred to as the diamond-F_h $(\diamond_{(F_{h(\lambda)})^s})$ dynamic calculus, which is a generalization and a unification of the Δ, ∇ and $\diamond_\alpha$ calculi, were introduced on time scales [11].

The interested reader is referred to [11]-[14] for applications of these interesting concepts to H-H inequalities on time scales.

In this article, the reader will find some preliminary results on time scales in this first section. Next section contains a review of some necessary concepts recently introduced by the authors [11] and proofs of some further interesting properties on time scales needed for our purpose. Representative applications of the new concepts in the previous section are shown in section three, by establishing some new integral inequalities of Hermite-Hadamard type for products of two F_h-convex functions on time scales. Some direct applications in the field of Economics are further provided to illustrate our results in section four, followed by conclusion.

2. On F_h-convexity and $\diamond_{(F_{h(\lambda)})^s}$ dynamics on time scales

We recall the concept of F_h-convexity recently introduced by the authors [11].

Definition 2.1. [11] A mapping $f : I_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ is said to be F_h-convex on time scales if

$$f(\lambda x + (1-\lambda)y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y), \tag{2.1}$$

for all $s \in [0,1]$, $0 \leq \lambda \leq 1$, $x, y \in I_{\mathbb{T}}$ and $\lambda x + (1-\lambda)y \in I_{\mathbb{T}}$.

Remark 2.2. Let $SX(F_h, I_{\mathbb{T}})$ denote the class of F_h-convex functions on time scales. Then (2.1) is h-convex$(SX(h, I_{\mathbb{T}}))$ on time scales if $h(\lambda) = \lambda^{\frac{s}{s+1}}$. For $s = 0$, f is said to be P-convex $(SX(P, I_{\mathbb{T}}))$ on time scales. Definition 1.1 reduces to MT-convexity$(SX(MT, I_{\mathbb{T}}))$ on time scales when $s = 1$, $h(\lambda) = 2\sqrt{\lambda(1-\lambda)}$ while it represents mid-point convexity$(SX(J, I_{\mathbb{T}}))$ on time scales if $s = 1$, $h(\lambda) = 1$, and $\lambda = \frac{1}{2}$. It is convex $(SX(I_{\mathbb{T}}))$ on time scales if $s = 1$ and $h(\lambda) = 1$. Thus, if $\mathbb{T} = \mathbb{R}$, f is h-convex, MT-convex, P-convex, mid-point convex and convex on classical intervals respectively(see [11], [8], [1], [4], [18]).

Note that $SX(P, I_{\mathbb{T}}) \subseteq SX(h, I_{\mathbb{T}}) \subseteq SX(F_h, I_{\mathbb{T}})$ for $0 \leq s \leq 1$. If inequality (1.2) is reversed, then f is F_h-concave, that is, $f \in SV(F_h, I_{\mathbb{T}})$.

Now, we introduce some properties of addition and scalar multiplication which show that $SX(F_h, I_{\mathbb{T}})$ is a linear space for $\beta \geq 0$.

Throughout this paper, let $h_1, h_2 : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be nonzero non negative functions, where $\mathbb{J}_{\mathbb{T}}$ is an F_h-convex subset of the real $\mathbb{T}$. h_1, h_2 have the property that $h(t) > 0$ for all $t \geq 0$.

Proposition 2.3. *Let* $f,\ g \in SX(F_h, I_{\mathbb{T}}), I_{\mathbb{T}}$ *and* $\beta \geq 0$, $x, y \in I_{\mathbb{T}}$, $\beta \in \mathbb{R}$, *then* $f + g$ *and* βf *are both* F_h*-convex on time scales.*

Proof. Since $f,\ g \in SX(F_h, I_{\mathbb{T}})$, then

$$(f+g)(\lambda x+(1-\lambda)y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s [(f+g)(x)] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [(f+g)(y)]$$

and

$$(\beta f)(\lambda x + (1-\lambda)y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s \beta f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \beta f(y),$$

$\forall\ \lambda \in [0,1]$ and $s \in [0,1]$.
If $f,\ g \in SV(F_h, I_{\mathbb{T}})$ and $\beta \leq 0$, then $f+g$ and βf are both F_h-concave on time scales. □

The following example was given in [11].

Example 2.4. [11] Consider the function f to be a non-negative convex function on $I_{\mathbb{T}}$ and h, a non zero non negative function on $I_{\mathbb{T}}$ with $h(t) > 0$ for all $t \geq 0$ satisfying

$$h(\lambda) \leq \lambda^{1-\frac{1}{m}}, \qquad m \in (0,1],\ 0 \leq \lambda \leq 1.$$

Then, we have that

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y),$$

showing that $f \in SX(F_h, I_{\mathbb{T}})$.

Remark 2.5. Example 2.4 implies that all convex functions on time scales are examples of the generalized class of F_h-convex function (2.1) on $I_{\mathbb{T}}$ provided the condition $h(\lambda) \leq \lambda^{1-\frac{1}{m}}$ is satisfied. In particular, an example of such $h(\lambda)$ is $h(\lambda) = \lambda^k$ for $k > 1 - \frac{1}{m}$, $m \in (0,\ 1],\ 0 \leq \lambda \leq 1$.

Any non-negative concave function f belongs to the class $SV(F_h, I_{\mathbb{T}})$ i.e. f is F_h-concave provided h satisfies $h(\lambda) \geq \lambda^{1-\frac{1}{m}}$ for any $\lambda \in [0,1]$ and $m \in (0,1]$.

The following new concepts establish properties of two F_h-convex functions on time scales.

Proposition 2.6. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ for all $t \in [a,b]_{\mathbb{T}}$. Then*

$$\left(\frac{\lambda}{h_1(\lambda)}\right)^s \leq \left(\frac{\lambda}{h_2(\lambda)}\right)^s, \qquad \forall\ \lambda \in [0,1] \text{ and } s \in [0,1].$$

If $f \in SX(F_{h_1}, I_{\mathbb{T}})$, then $f \in SX(F_{h_2}, I_{\mathbb{T}})$; $f \in SV(F_{h_1}, I_{\mathbb{T}})$, then $f \in SV(F_{h_2}, I_{\mathbb{T}})$.

Proof. Suppose $f \in SX(F_{h_1}, I_{\mathbb{T}})$, then for any $x, y \in J_{\mathbb{T}}$,

$$f(\lambda x + (1-\lambda)y) \leq \left(\frac{\lambda}{h_1(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s f(y)$$

$$\leq \left(\frac{\lambda}{h_2(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(y),$$

that is, $f \in SX(F_{h_2}, I_{\mathbb{T}})$. □

Proposition 2.7. *Let f and g be similarly ordered on $\mathbb{I}_{\mathbb{T}}$, i.e $(f(x) - f(y))(g(x) - g(y)) \geq 0$, for all $x, y \in I_{\mathbb{T}}$. If $f \in SX(F_{h_1}, I_{\mathbb{T}})$ and $g \in SX(F_{h_2}, I_{\mathbb{T}})$ such that $\left(\frac{\lambda}{h_2(\lambda)}\right)^s + \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \leq \beta^s$ for all $s \in [0,1]$, $0 \leq \lambda \leq 1$ with $h(\lambda) = max_{\lambda\in[0,1]}\{h_1(\lambda), h_2(\lambda)\}$ and $\beta > 0$, $\beta \in \mathbb{R}$. Then the product $fg \in SX(F_{\beta h}, I_{\mathbb{T}})$.*

Proof. Let f and g be similarly ordered, then

$$\begin{aligned}
& f(\lambda x + (1-\lambda)y)g(\lambda x + (1-\lambda)y) \\
\leq\ & \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s f(y)\right] \\
\times\ & \left[\left(\frac{\lambda}{h_2(\lambda)}\right)^s g(x) + \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s g(y)\right] \\
\leq\ & \left(\frac{\lambda}{h(\lambda)}\right)^{2s} (fg)(x) + \left(\frac{\lambda}{h(\lambda)}\frac{1-\lambda}{h(1-\lambda)}\right)^s f(x)g(y) \\
+\ & \left(\frac{1-\lambda}{h(1-\lambda)}\frac{\lambda}{h(\lambda)}\right)^s f(y)g(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^{2s} (fg)(y) \\
\leq\ & \left(\frac{\lambda}{h(\lambda)}\right)^{2s} (fg)(x) + \left(\frac{\lambda}{h(\lambda)}\frac{1-\lambda}{h(1-\lambda)}\right)^s [f(x)g(x) + f(y)g(y)] \\
+\ & \left(\frac{1-\lambda}{h(1-\lambda)}\right)^{2s} (fg)(y) \\
\leq\ & \left(\frac{\lambda}{\beta h(\lambda)}\right)^s (fg)(x) + \left(\frac{1-\lambda}{\beta h(1-\lambda)}\right)^s (fg)(y),
\end{aligned}$$

that is, $(fg) \in SX(F_{\beta h}, I_{\mathbb{T}})$.
Hence, the proof of the assertion of the proposition is complete. □

Remark 2.8. If f and g are oppositely ordered on $\mathbb{I}_{\mathbb{T}}$, i.e., $(f(x) - f(y))(g(x) - g(y)) \leq 0$, for all $x, y \in I_{\mathbb{T}}$. If $f \in SV(F_{h_1}, I_{\mathbb{T}})$, $g \in SV(F_{h_2}, I_{\mathbb{T}})$ and $\left(\frac{\lambda}{h_2(\lambda)}\right)^s + \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \geq \beta^s$ for all $s \in [0,1]$, $0 \leq \lambda \leq 1$ with $h(\lambda) = min_{\lambda\in[0,1]}\{h_1(\lambda), h_2(\lambda)\}$ and $\beta > 0$, $\beta \in \mathbb{R}$. Then the product $fg \in SX(F_{\beta h}, I_{\mathbb{T}})$.

In the paper [11] appears the following generalized diamond-F_h dynamic calculus on time scales.

Definition 2.9. [11] A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *diamond-F_h differentiable* on $\mathbb{T}_k^k$ in the sense of Δ and ∇, for all $t \in \mathbb{T}_k^k$, we write

$$f^{\diamond(F_{h(\lambda)})^s}(t) = \left(\frac{\lambda}{h(\lambda)}\right)^s f^{\Delta}(t) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f^{\nabla}(t),$$

where $\lambda \in [0,1]$ *and* $s \in [0,1]$.

If f is defined in $t \in \mathbb{T}$ such that for any $\epsilon > 0$, there is a neighbourhood U of m and $n \in U$, with $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$, we have,

$$\left| \left(\frac{\lambda}{h(\lambda)}\right)^s [f(\sigma(m)) - f(n)]\nu_{mn} + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [f(\rho(m)) - f(n)]\mu_{mn} \right.$$
$$\left. - f^{\diamond(F_{h(\lambda)})^s}(t)\mu_{mn}\nu_{mn} \right| < \epsilon|\mu_{mn}\nu_{mn}|, \qquad \lambda \in [0,1] \text{ and } s \in [0,1].$$

Remark 2.10. $f^{\diamond(F_{h(\lambda)})^s}(t)$ reduces to the $\diamond_\alpha$ derivative for $F_h = \alpha, s = 1$ and $h(\lambda) = 1$; thus every diamond-α differentiable function on $\mathbb{T}$ is diamond-F_h differentiable but the converse is not true. For $F_h = 1$, $s = 1$ and $h(\lambda) = 1$, it reduces to the standard Δ derivative or the standard ∇ derivative for $F_h = 0$, $s = 1$ and $h(\lambda) = 1$, while it represents a "weighted dynamic derivative" for $F_h \in (0,1)$, $s = 1$ and $h(\lambda) = 1$. When $F_h = \frac{1}{2}, s = 1$ and $h(\lambda) = 1$, the combined dynamic derivative gives a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$. If f is diamond-F_h differentiable for $0 \le s \le 1$, and $0 \le \lambda \le 1$, then f is both Δ and ∇ differentiable. When $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$ and $f^{\diamond(F_{h(\lambda)})^s}(t)$ becomes the ordinary differential derivative(see [22], [9], [11] and [20]).

Definition 2.11. [11] The *diamond-F_h integral* of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by;

$$\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(t)\nabla t, \tag{2.2}$$

$\forall\ \lambda \in [0,1]$ *and* $s \in [0,1]$, such that f has a Δ and ∇ integral on $[a,b]_{\mathbb{T}}$.

The permanence properties of diamond-F_h derivative and convexity operations were presented in the following Proposition in [11].

Proposition 2.12. [11] *Let $f, g \in SX(F_h, I_{\mathbb{T}})$, i.e, these are F_h-convex, $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-F_h differentiable at $t \in I_{\mathbb{T}}$, and c be any constant. Then $f + g, cf, fg, \frac{1}{g}(g \neq 0), \frac{f}{g}(g \neq 0)$ are all diamond-F_h differentiable at $t \in I_{\mathbb{T}}$.*

Some new basic properties of the diamond-F_h integral, which are similar to Theorem 2.2 of [7] and its analogue for the nabla integral, including Theorem 3.7 of [22] for diamond-α integral on time scales are established as follows;

Proposition 2.13. *Let $a, b, c \in \mathbb{T}, \beta \in \mathbb{R}$ and f, g be continuous, non negative functions on $I_{\mathbb{T}}$, then*

(i) $\int_a^b (f(t) \pm g(t)) \diamond_{(F_{h(\lambda)})^s} t = \int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t \pm \int_a^b g(t) \diamond_{(F_{h(\lambda)})^s} t.$

(ii) $\int_a^b (\beta f) t \diamond_{(F_{h(\lambda)})^s} t = \beta \int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t.$

(iii) $\int_a^b (fg)^{\diamond_{(F_{h(\lambda)})^s}}(t) = \int_a^b f^{\diamond_{(F_{h(\lambda)})^s}}(t) g(t) \diamond_{(F_{h(\lambda)})^s} t$
$+ \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f^{\sigma}(t) g^{\Delta}(t) \Delta t + \int_a^b \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f^{\rho}(t) g^{\nabla}(t) \nabla t.$

(iv) $\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t = -\int_b^a f(t) \diamond_{(F_{h(\lambda)})^s} t$

(v) $\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t = \int_a^c f(t) \diamond_{(F_{h(\lambda)})^s} t + \int_c^b f(t) \diamond_{(F_{h(\lambda)})^s} t$

(vi) $\int_a^a f(t) \diamond_{(F_{h(\lambda)})^s} t = 0.$

(vii) *If $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$, then $\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t \geq 0.$*

(viii) *If $f(t) \leq g(t)$ for all t, then $\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t \leq \int_a^b g(t) \diamond_{(F_{h(\lambda)})^s} t.$*

(ix) *If $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$, then $f = 0$ if and only if $\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t = 0.$*

(x) *If $|f(t)| \leq g(t)$ on $[a, b)$, then $|\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t| \leq \int_a^b g(t) \diamond_{(F_{h(\lambda)})^s} t.$*

(xi) *If in (x), we choose $g(t) = |f(t)|$ on $[a, b]$, we have*
$|\int_a^b f(t) \diamond_{(F_{h(\lambda)})^s} t| \leq \int_a^b |f(t)| \diamond_{(F_{h(\lambda)})^s} t.$

Proof. The proofs are straightforward, so we omit. □

3. Applications to integral inequalities

Here, we show representative applications of F_h-convexity and diamond-F_h dynamics to establish some new integral inequalities of Hermite-Hadamard type on time scales.

Theorem 3.1. *Let $f \in SX(F_{h_1}, I_{\mathbb{T}})$ and $g \in SX(F_{h_2}, I_{\mathbb{T}})$ be continuous F_h-convex functions, with fg integrable and non negative on $[a, b]_{\mathbb{T}}$,* *where* $a, b, t \in I_{\mathbb{T}}$*,* $a < b$*. Then*

$$\begin{aligned} & \frac{1}{b-a}[\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x g(x) \\ + & \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b g(x) \Delta x f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b g(x) \nabla x f(x)] \\ \leq & M(a,b) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda \\ + & N(a,b) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda, \end{aligned} \tag{3.1}$$

$\forall\ \lambda \in [0,1]$ *and* $s \in [0,1]$*, where* $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$*.*

Proof. Since f and g are non negative functions and $f \in SX(F_{h_1}, I_{\mathbb{T}})$, $g \in SX(F_{h_2}, I_{\mathbb{T}})$, then, by proposition 2.6, we have for all $s \in [0,1]$, $0 \leq \lambda \leq 1$,

$$\begin{aligned} & [f(\lambda a + (1-\lambda)b) g(\lambda a + (1-\lambda)b)] \\ \leq & \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s f(a)g(a) + \left(\frac{\lambda}{h_2(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(a)g(b) \\ + & \left(\frac{\lambda}{h_2(\lambda)}\right)^s \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s f(b)g(a) \\ + & \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(b)g(b). \end{aligned} \tag{3.2}$$

Taking the $\diamond_{(F_{h(\lambda)})^s}$ integral of (3.2) with respect to λ over [0, 1], we have

$$\begin{aligned} & \int_0^1 [f(\lambda a + (1-\lambda)b) g(\lambda a + (1-\lambda)b)] \diamond_{(F_{h(\lambda)})^s} x \\ \leq & f(a)g(a) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} x \\ + & f(a)g(b) \int_0^1 \left(\frac{\lambda}{h_2(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} x \\ + & f(b)g(a) \int_0^1 \left(\frac{\lambda}{h_2(\lambda)}\right)^s \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} x \\ + & f(b)g(b) \int_0^1 \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} x. \end{aligned} \tag{3.3}$$

By (2.2) and proposition 2.7, (3.3) becomes

$$\begin{aligned}
& \left(\frac{\lambda}{h(\lambda)}\right)^s \int_0^1 [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)]\Delta\lambda \\
+ & \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_0^1 [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)]\nabla\lambda \\
\leq & \ [f(a)g(a)+f(b)g(b)] \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda \\
+ & \ [f(a)g(b)+f(b)g(a)] \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda.
\end{aligned} \tag{3.4}$$

Substituting $x=\lambda a+(1-\lambda)b, \Delta x=(a-b)\Delta\lambda; \nabla x=(a-b)\nabla\lambda$ into (3.4) gives

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)g(x) \diamond_{(F_{h(\lambda)})^s} x \\
\leq & \ M(a,b) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda \\
+ & \ N(a,b) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda,
\end{aligned}$$

which by proposition 2.13(iii) becomes

$$\begin{aligned}
& \frac{1}{b-a}[\int_a^b f(x) \diamond_{(F_{h(\lambda)})^s} x g(x) \\
+ & \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b g(x)\Delta x f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b g(x)\nabla x f(x)] \\
\leq & \ M(a,b) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda \\
+ & \ N(a,b) \int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda,
\end{aligned}$$

that is, the required inequality (3.1). □

Theorem 3.2. *Let $f \in SX(F_{h_1}, I_{\mathbb{T}})$ and $g \in SX(F_{h_2}, I_{\mathbb{T}})$ be continuous non negative F_h-convex functions, $a, b, t \in I_{\mathbb{T}}$, with $a < b$, $s \in [0,1]$ and*

$\lambda \in [0,1]$, *then*

$$\begin{aligned} & 4^s\left[h_1\left(\frac{1}{2}\right)\right]^s\left[h_2\left(\frac{1}{2}\right)\right]^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ \leq & \frac{2}{b-a}\int_a^b f(x)g(x) \diamond_{(F_{h(\lambda)})^s} \lambda \\ + & M(a,b)\int_0^1\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda \\ + & N(a,b)\int_0^1\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda, \end{aligned}$$

where $M(a,b) = f(a)g(a)+f(b)g(b)$ *and* $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof. Since f and g are F_h-convex by (2.1), then making a change of variables $x = \lambda a+(1-\lambda)b$, $y = (1-\lambda)a+\lambda b$ and $\lambda = \frac{1}{2}$, we have that

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ = & f\left(\frac{\lambda a+(1-\lambda)b}{2}+\frac{(1-\lambda)a+\lambda b}{2}\right) \\ & g\left(\frac{\lambda a+(1-\lambda)b}{2}+\frac{(1-\lambda)a+\lambda b}{2}\right) \\ \leq & \left[\left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s\left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s\right] \\ & [f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)] \\ & [g(\lambda a+(1-\lambda)b)+g((1-\lambda)a+\lambda b)] \\ = & [\left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s\left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s \\ & [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)g((1-\lambda)a+\lambda b) \\ + & f(\lambda a+(1-\lambda)b)g((1-\lambda)a+\lambda b)+f((1-\lambda)a+\lambda b)g((1-\lambda)a+\lambda b)]. \end{aligned} \quad (3.5)$$

By a simple computation, (3.5) becomes

$$\begin{aligned}
& f\left(\frac{\lambda a+(1-\lambda)b}{2}+\frac{(1-\lambda)a+\lambda b}{2}\right)g\left(\frac{\lambda a+(1-\lambda)b}{2}+\frac{(1-\lambda)a+\lambda b}{2}\right)\\
\leq\ & \left[\left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s\left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s\right]\\
& [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)g((1-\lambda)a+\lambda b)]\\
+\ & \left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s\left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s\times\\
& \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(a)g(a)+\left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\left(\frac{\lambda}{h_2(\lambda)}\right)^s f(b)g(b)\right]\\
+\ & \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{\lambda}{h_2(\lambda)}\right)^s f(a)g(b)+\left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(b)g(a)\right]\\
+\ & \left[\left(\frac{\lambda}{h_2(\lambda)}\right)^s\left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s f(a)g(a)+\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(b)g(b)\right]\\
+\ & \left[\left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s f(a)g(b)+\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{\lambda}{h_2(\lambda)}\right)^s f(b)g(a)\right].
\end{aligned}\tag{3.6}$$

After a simple rearrangement, and using proposition 2.7, one can transform the inequality (3.6) to

$$\begin{aligned}
& f\left(\frac{\lambda a+(1-\lambda)b}{2}+\frac{(1-\lambda)a+\lambda b}{2}\right)\\
& g\left(\frac{\lambda a+(1-\lambda)b}{2}+\frac{(1-\lambda)a+\lambda b}{2}\right)\\
\leq\ & \left[\left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s\left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s\right]\\
& [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)\\
+\ & f((1-\lambda)a+\lambda b)g((1-\lambda)a+\lambda b)]\\
+\ & \left[\left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s\left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s\right][f(a)g(a)+f(b)g(b)]\\
& \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s+\left(\frac{\lambda}{h_2(\lambda)}\right)^s\left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\right]\\
+\ & [f(a)g(b)+f(b)g(a)]\\
& \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{\lambda}{h_2(\lambda)}\right)^s+\left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s\right].
\end{aligned}$$

Then we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
\leq\ & \left[\left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s \left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s\right] \\
& [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b) \\
& +f((1-\lambda)a+\lambda b)g((1-\lambda)a+\lambda b)] \\
+\ & \left(\frac{\frac{1}{2}}{h_1(\frac{1}{2})}\right)^s \left(\frac{\frac{1}{2}}{h_2(\frac{1}{2})}\right)^s \\
& M(a,b)\left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s + \left(\frac{\lambda}{h_2(\lambda)}\right)^s \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\right] \\
+\ & N(a,b)\left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s + \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s\right].
\end{aligned} \tag{3.7}$$

Taking the $\diamond_{(F_{h(\lambda)})^s}$ integral of (3.7) over [0, 1], we obtain

$$\begin{aligned}
& 4^s\left[h_1\left(\frac{1}{2}\right)\right]^s \left[h_2\left(\frac{1}{2}\right)\right]^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
\leq\ & \int_0^1 [f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b) \\
+\ & f((1-\lambda)a+\lambda b)g((1-\lambda)a+\lambda b)]\diamond_{(F_{h(\lambda)})^s}\lambda \\
+\ & M(a,b) \\
& \int_0^1 \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s + \left(\frac{\lambda}{h_2(\lambda)}\right)^s \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s\right]\diamond_{(F_{h(\lambda)})^s}\lambda \\
+\ & N(a,b) \\
& \int_0^1 \left[\left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s + \left(\frac{1-\lambda}{h_1(1-\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s\right]\diamond_{(F_{h(\lambda)})^s}\lambda.
\end{aligned}$$

Using property (iii) of proposition 2.13 and taking into accounts $x = \lambda a+(1-\lambda)b, \Delta x = (a-b)\Delta\lambda, \nabla x = (a-b)\nabla\lambda;\ \ y = (1-\lambda)a+\lambda b, \Delta y =$

$(b-a)\Delta\lambda, \nabla y = (b-a)\nabla\lambda$ in the above inequality, we have

$$\begin{aligned} & 4^s \left[h_1\left(\frac{1}{2}\right)\right]^s \left[h_2\left(\frac{1}{2}\right)\right]^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ \leq \; & \frac{2}{b-a}\int_0^1 f(x)g(x) \diamond_{(F_{h(\lambda)})^s} \\ + \; & M(a,b)\int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda \\ + \; & N(a,b)\int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s \left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s} \lambda. \end{aligned} \tag{3.8}$$

Hence, Theorem 3.2 is proved. □

4. Economic Applications

The theory of time scales is directly applicable in many fields such as Engineering, Optimization and Economics, in which dynamic processes can be described by discrete or continuous time systems, variables or models ([2], [3], [15]).

In Economics, most dynamic optimization problems are developed thus: a representative consumer seeks to maximize his/her lifetime utility U subject to certain budgetary constraints A. There is the (constant) discount factor δ, which satisfies $0 \leq \delta \leq 1$, C_s is consumption during period s, $u(C_s)$ is the utility the consumer derives from consuming C_s units of consumption in periods $s = 0, 1, 2, ..., T$. Utility is assumed to be concave: $u(C_s)$ has $u(C_s)' > 0$ and $u(C_s)'' < 0$. The consumer receives some income Y in a time period s and decides how much to consume and save during that same period. If the consumer consumes more today, the utility or satisfaction he derives from consumption, is forgone tomorrow as the detterrence. Normally, the consumer is insatiable. However, each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period (Law of diminishing marginal utility, LDMU). This means that the first unit of consumption of a good or service yields more utility than the second or subsequent units, with a continuing reduction for greater amounts.

The individual is constrained by the fact that the value function of his consumption, $u(C)$ must be equal to the value function of his income Y_s, plus the assets/debts, A_s that he might accrue in a period s. Hence, A_{s+1} is the amount of assets held at the beginning of period $s+1$. Also, A could be positive or negative; the consumer might save for the future or borrow against the future at interest rate r in any given period s but the value of A_T, which is the debt accrued with limit or the last period asset holding, has to be nonnegative (the optimal level is naturally zero).

In order to state the necessary and sufficient condition for optimization in the formulation of a dynamic optimization problem as that presented above, it is important to present the simplest form of optimal control problem in terms of (2.2) as;

$$\begin{aligned}\max\ J_{\diamond_{(F_{h(\lambda)})^s}}[x, u_1(t)u_2(t)] &= \int_a^b L(t, x, u_1(t)u_2(t))\diamond_{(F_{h(\lambda)})^s} t \\ &= \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b L\left(t, x^\sigma, (u_1(t)u_2(t))^\sigma\right) \Delta t \\ &+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b L\left(t, x^\rho, (u_1(t)u_2(t))^\sigma\right) \nabla t, \qquad (4.1)\end{aligned}$$

$\forall\ \lambda \in [0,1]$ and $s \in [0,1]$, among all pairs $(x, u_1(t)u_2(t))$ such that $x^\Delta = f(t, x^\sigma, (u_1(t)u_2(t))^\sigma)$ and $x^\Delta = f(t, x^\rho, (u_1(t)u_2(t))^\sigma)$, together with appropriate endpoint conditions
$u^{\diamond_{(F_{h(\lambda)})^s}{}'}(t) = L(t, u_1(t)u_2(t), p)$, $\quad x(0) = u_0$, $\quad u_1(t)u_2(t)(T)$ free for all $t \in [0,T]$.

Hence, a simple utility maximization model of household consumption in Economics for product of two functions can be set up and solved in time scales settings, using the same intuition as that of the dynamic optimization problem presented above, by employing our developed concepts in sections 2 and 3 as follows. The model assumes a perfect foresight.

Theorem 4.1. *The value function of the lifetime utility $U_{\diamond_{F_h}}$ as a product of two continuous F_h-concave functions to be maximized subject to certain constraints is;*

$$\begin{aligned}& Maximize\ \ U_{\diamond_{F_h}} \\ = \ & \frac{2}{b-a}\int_a^b u_1(C(t))e_{-\delta}(t,\,0)u_2(C(t))e_{-\delta}(t,\,0))\diamond_{F_h} t \\ \geq \ & 4^s\left[h_1\left(\frac{1}{2}\right)\right]^s\left[h_2\left(\frac{1}{2}\right)\right]^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ - \ & M(a,b)\int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{1-\lambda}{h_2(1-\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s}\lambda \\ + \ & N(a,b)\int_0^1 \left(\frac{\lambda}{h_1(\lambda)}\right)^s\left(\frac{\lambda}{h_2(\lambda)}\right)^s \diamond_{(F_{h(\lambda)})^s}\lambda, \qquad (4.2)\end{aligned}$$

subject to the budget constraints

$$\begin{aligned} A^{\nabla}(t) &= (rA+Y-C)(\rho(t)), \\ A^{\Delta}(t) &= \tfrac{r}{1+r\mu(t)}A^{\sigma}(t)+\tfrac{1}{1+r\mu(t)}y^{\sigma}(t)-\tfrac{1}{1+r\mu(t)}c^{\sigma}(t), \\ a(0) &= a_0, \qquad a(T)=a_T, \end{aligned} \tag{4.3}$$

where $u_1,\ u_2$ *are* F_h*-concave* $u_1^{'}(C(t_1))u_2^{'}(C(t_2))>0$ *and* $u_1^{''}(C(t_1))u_2^{''}(C(t_2))<0$*),* $0\le\lambda\le 1,\ s\in[0,1]$*,* A^{Δ} *and* A^{∇} *are the partial delta and nabla derivatives of the budget constraints,* e *is the exponential function,* $r,\delta,A,$ *and* Y *are as defined above.*

Proof. Let $f(t)g(t)$ be functions satisfied by the consumption function path that would maximize lifetime utility $u_1(C(t))e_{-\delta}(t,0)u_2(C(t))e_{-\delta}(t,0))$ in (4.2), then the condition for a functional of the form

$$\begin{aligned} &\int_a^b L(t,x,u_1(t)u_2(t))\diamond_{(F_{h(\lambda)})^s}t \\ = &\left(\frac{\lambda}{h(\lambda)}\right)^s\int_a^b L\left(t,x^{\sigma},(u_1(t)u_2(t))^{\sigma}\right)\Delta t \\ + &\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\int_a^b L\left(t,x^{\rho},(u_1(t)u_2(t))^{\sigma}\right)\nabla t, \end{aligned} \tag{4.4}$$

for all $s\in[0,1]$ and $0\le\lambda\le 1$, to have a local extremum for functions $u_1(t)u_2(t)$ and the sufficient condition for an absolute maximum(minimum) of (4.4) hold.

Since both local and absolute extrema hold, then (4.4) satisfies the sufficient conditions for optimization, which in turn satisfies Theorem 3.2. To analyze the model (4.2)-(4.3), (4.2) is written in terms of (2.2), stating the maximum principle and giving the Hamiltonian function for the model. □

5. CONCLUSION

More interesting properties on the notion of F_h-convexity and $\diamond_{(F_{h(\lambda)})^s}$ dynamics have been established on time scales. Also, representative applications of these concepts were shown by establishing some new integral inequalities of H-H type for products of two F_h-convex functions on time scales. Further, using F_h-convexity and $\diamond_{(F_{h(\lambda)})^s}$ dynamics, some direct applications to Economics were provided to illustrate our results. It is expected that the ideas and techniques of this paper would further stimulate research in various fields.

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