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On a new integral type contraction and coupled fixed point theorems in metric spaces with applications to system of integral equations

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Abstract. This paper introduces a new type of integral contraction in metric spaces. We define the notion of an "(ϕ, ψ)-integral-type coupled contraction" and establish coupled fixed point theorems under this new contraction condition. The presented results generalize and extend several existing theorems in the literature, particularly those involving integral-type contractions. We demonstrate the applicability of our results by providing an example and outlining potential applications in solving systems of integral equations.

Keywords: Integral type contraction; Coupled fixed point; Systems of integral equations.

2000 Mathematics subject classification: 45G15, 32H50.

1. Introduction

Fixed point theory is a rich, interesting and exciting branch of mathematics. It is relatively young but fully developed area of research. Study of the existence of fixed points falls within several domains such as functional analysis, operator theory, general topology. Fixed points and

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fixed point theorems have always been a major theoretical tool in fields as widely apart as topology, mathematical economics, game theory, approximation theory and initial and boundary value problems in ordinary and partial differential equations. Moreover, recently, the usefulness of this concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major tool in the arsenal of mathematics.

The Banach fixed point theorem which was first presented by Banach in 1922 is a significant result in fixed point theory. Because of its importance in proving the existence of solutions for functional equations, nonlinear Volterra integral equations and nonlinear integro-differential equations, this result has been extended in many different directions (see, e.g., [1, 2, 4, 5, 7, 11, 12, 13, 16, 17, 18, 20]).

In recent years, different contractive circumstances have been investigated using fixed point theory. Indeed, integral type contraction is among them. In 2002, Branciari [7] analyzed the existence of fixed points for mapping defined on a complete metric space satisfying a general contractive condition of integral type. Following Branciari's finding, other studies have been conducted on generalizing integral type contractive conditions for various contractive mappings that meet a variety of known features (see [3, 9, 14, 15].

On the other hand, Bhaskar and Lakshmikantham [6] introduced the concept of a coupled fixed point in partially ordered metric spaces. They noted that their theorem can be used to investigate a large class of problems and discussed the existence and uniqueness of solution for a periodic boundary value problem. It was after the appearance of a coupled contraction mapping theorem, the coupled fixed point results appeared in a large number of works like [8, 10, 19].

Motivated by the above considerations, this paper introduces a new integral-type contraction condition tailored for coupled mapping and establishes coupled fixed point theorems in complete metric spaces. We then demonstrate the applicability of our results by providing an example and discussing potential applications in the context of integral equations.

2. Preliminaries

Let (X, d) be a metric space.

Definition 2.1 ([6]). A coupled fixed point of a mapping $F: X \times X \to X$ is a pair $(x, y) \in X \times X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Example 2.2. Let X be the set of real numbers and define a mapping $F: X \times X \to X$ as $F(x, y) = x^2 + y$. A coupled fixed point for this mapping would be a pair (x, y) such that $x^2 + y = x$ and $y^2 + x = y$. One solution to this system is $(x, y) = (0, 0)$.

Definition 2.3. A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Notation 2.4. Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

Notation 2.5. Let Φ be the family of Lebesgue integrable mapping $\phi : [0, \infty) \to [0, \infty)$ which is summable, nonnegative, and such that $\int_0^{\epsilon} \phi(t)dt > 0$ for all $\epsilon > 0$.

3. Main results

In this section, we present our main results. At first, we introduce our new integral type contraction condition.

Definition 3.1. Let (X, d) be a metric space and $F : X \times X \to X$ be a mapping. We say that F is an (ϕ, ψ)-integral-type coupled contraction if there exist $(\phi, \psi) \in \Phi \times \Psi$ such that for all $x, y, u, v \in X$:

$$\int_0^{d(F(x,y),F(u,v))} \phi(t)dt \leq \psi \left(\int_0^{\max\{d(x,u),d(y,v)\}} \phi(t)dt \right).$$

Theorem 3.2. Let (X, d) be a complete metric space and $F : X \times X \to X$ be an (ϕ, ψ)-integral-type coupled contraction satisfying the condition in Definition 3.1. Then there exists a unique pair $(x^*, y^*) \in X \times X$ such that $F(x, y^*) = x^*$ and $F(y, x^*) = y^*$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Define the sequences $\{x_n\}$ and $\{y_n\}$ by:

$$x_{n+1} = F(x_n, y_n), \; y_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0.$$

Then, we have

$$\begin{aligned}\int_0^{d(x_{n+1},x_n)} \phi(t)dt &= \int_0^{d(F(x_n,y_n),F(x_{n-1},y_{n-1}))} \phi(t)dt \\ &\leq \psi\left(\int_0^{\max\{d(x_n,x_{n-1}),d(y_n,y_{n-1})\}} \phi(t)dt\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^{d(y_{n+1},y_n)} \phi(t)dt &= \int_0^{d(F(y_n,x_n),F(y_{n-1},x_{n-1}))} \phi(t)dt \\ &\leq \psi\left(\int_0^{\max\{d(y_n,y_{n-1}),d(x_n,x_{n-1})\}} \phi(t)dt\right).\end{aligned}$$

Let $a_n = \max\{d(x_n,x_{n-1}),d(y_n,y_{n-1})\}$. Then,

$$\int_0^{a_{n+1}} \phi(t)dt \leq \psi\left(\int_0^{a_n} \phi(t)dt\right). \tag{3.1}$$

Since ψ is nondecreasing, it follows that $\int_0^{a_{n+1}} \phi(t)dt \leq \int_0^{a_n} \phi(t)dt$. Thus, the sequence $\{\int_0^{a_n} \phi(t)dt\}$ is a decreasing sequence of nonnegative real numbers, and hence it converges to some real number $L \geq 0$.

Suppose $L > 0$. Then, taking the limit as $n \to \infty$ in the inequality (1) we obtain $L \leq \psi(L)$. However, since $\sum_{n=1}^{\infty} \psi^n(L) < \infty$, it follows that $\psi(L) < L$ for all $L > 0$. This is a contradiction. Therefore, $L = 0$. This implies that $\lim_{n\to\infty} \int_0^{a_n} \phi(t)dt = 0$. Consequently, $\lim_{n\to\infty} a_n = 0$, which means $\lim_{n\to\infty} d(x_n,x_{n-1}) = 0$ and $\lim_{n\to\infty} d(y_n,y_{n-1}) = 0$.

Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose not. Then there exists an $\epsilon > 0$ and sequences m_k and n_k of positive integers such that $n_k > m_k > k$ and

$$d(x_{n_k},x_{m_k}) \geq \epsilon.$$

Assume that n_k is the smallest such integer. Then $d(x_{n_k-1},x_{m_k}) < \epsilon$. Therefore,

$$\epsilon \leq d(x_{n_k},x_{m_k}) \leq d(x_{n_k},x_{n_k-1}) + d(x_{n_k-1},x_{m_k}) < d(x_{n_k},x_{n_k-1}) + \epsilon.$$

Taking the limit as $k \to \infty$, we get $\lim_{k\to\infty} d(x_{n_k},x_{m_k}) = \epsilon$. Similarly, we can show that $\lim_{k\to\infty} d(y_{n_k},y_{m_k}) = \epsilon$. Now, consider

$$\begin{aligned}\int_0^{d(x_{n_k},x_{m_k})} \phi(t)dt &= \int_0^{d(F(x_{n_k-1},y_{n_k-1}),F(x_{m_k-1},y_{m_k-1}))} \phi(t)dt \\ &\leq \psi\left(\int_0^{\max\{d(x_{n_k-1},x_{m_k-1}),d(y_{n_k-1},y_{m_k-1})\}} \phi(t)dt\right).\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\int_0^{\epsilon} \phi(t)dt \le \psi\left(\int_0^{\epsilon} \phi(t)dt\right) < \int_0^{\epsilon} \phi(t)dt,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Similarly, we can show that $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Now, we show that (x, y) is a coupled fixed point of F. Consider

$$\begin{aligned}
\int_0^{d(x,F(x,y))} \phi(t)dt &\le \int_0^{d(x,x_{n+1})} \phi(t)dt + \int_0^{d(x_{n+1},F(x,y))} \phi(t)dt \\
&= \int_0^{d(x,x_{n+1})} \phi(t)dt + \int_0^{d(F(x_n,y_n),F(x,y))} \phi(t)dt \\
&\le \int_0^{d(x,x_{n+1})} \phi(t)dt + \psi\left(\int_0^{\max\{d(x_n,x),d(y_n,y)\}} \phi(t)dt\right).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\int_0^{d(x,F(x,y))} \phi(t)dt \le 0 + \psi(0) = 0.$$

Since $\int_0^{\epsilon} \phi(t)dt > 0$ for all $\epsilon > 0$, it follows that $d(x, F(x, y)) = 0$. Therefore, $x = F(x, y)$. Similarly, we can show that $y = F(y, x)$. Thus, (x, y) is a coupled fixed point of F.

To prove uniqueness, suppose (u, v) is another coupled fixed point of F. Then $u = F(u, v)$ and $v = F(v, u)$. Consider

$$\int_0^{d(x,u)} \phi(t)dt = \int_0^{d(F(x,y),F(u,v))} \phi(t)dt \le \psi\left(\int_0^{\max\{d(x,u),d(y,v)\}} \phi(t)dt\right).$$

Similarly,

$$\int_0^{d(y,v)} \phi(t)dt = \int_0^{d(F(y,x),F(v,u))} \phi(t)dt \le \psi\left(\int_0^{\max\{d(y,v),d(x,u)\}} \phi(t)dt\right).$$

Let $M = \max\{d(x, u), d(y, v)\}$. Then

$$\int_0^{M} \phi(t)dt \le \psi\left(\int_0^{M} \phi(t)dt\right).$$

If $M > 0$, then $\int_0^M \phi(t)dt > 0$. But since $\psi(t) < t$ for all $t > 0$, we have a contradiction. Therefore, $M = 0$, which implies $d(x, u) = 0$ and $d(y, v) = 0$. Thus, $x = u$ and $y = v$. Hence, the coupled fixed point is unique. □

Here, we provide examples to illustrate the applicability of our theoretical results.

Example 3.3. Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$. Define $F : X \times X \to X$ by $F(x,y) = \frac{x+y}{8}$. Let $\psi(t) = \frac{t}{2}$ and $\phi(t) = 1$. Then, $\int_0^{\epsilon} \phi(t)dt = \epsilon > 0$ for all $\epsilon > 0$, and $\sum_{n=1}^{\infty} \psi^n(t) = \sum_{n=1}^{\infty} \frac{t}{2^n} = t < \infty$ for all $t > 0$. Now, for any $x, y, u, v \in X$,

$$\begin{aligned}
\int_0^{d(F(x,y),F(u,v))} \phi(t)dt &= \left|\frac{x+y}{8} - \frac{u+v}{8}\right| \\
&\le \frac{1}{4}\max\{|x-u|,|y-v|\} \\
&= \frac{1}{4}\int_0^{\max\{|x-u|,|y-v|\}} \phi(t)dt \\
&= \frac{1}{2}\left(\frac{1}{2}\int_0^{\max\{|x-u|,|y-v|\}} \phi(t)dt\right) \\
&= \psi\left(\int_0^{\max\{|x-u|,|y-v|\}} \phi(t)dt\right).
\end{aligned}$$

Thus, F satisfies the contraction condition defined in Definition 3.1. Therefore, by Theorem 3.2, F has a unique coupled fixed point in X. □

4. Application to system of integral equations

Consider the following system of integral equations:

$$x(t) = \int_0^T K_1(t,s,x(s),y(s))ds, \quad y(t) = \int_0^T K_2(t,s,y(s),x(s))ds,$$

for $t \in [0,T]$, where $K_1, K_2 : [0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $X = C([0,T],\mathbb{R})$ be the space of continuous real-valued functions on $[0,T]$, equipped with the metric

$$d(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|.$$

Then (X,d) is a complete metric space. Define a mapping $F : X \times X \to X$ by

$$F(x,y)(t) = \int_0^T K_1(t,s,x(s),y(s))ds,$$

Similarly, define

$$G(y,x)(t)=\int_0^T K_2(t,s,y(s),x(s))ds$$

A solution to these integral equations is a coupled fixed point of the operator $F(x,y)$.

Theorem 4.1. Assume that there exist $(\psi,\phi)\in(\Psi,\Phi)$ such that for all $t,s\in[0,T]$ and $x,y,u,v\in\mathbb{R}$

$$|K_1(t,s,x,y)-K_1(t,s,u,v)|\le\frac{1}{T}\phi(\max\{|x-u|,|y-v|\}),$$

$$|K_2(t,s,y,x)-K_2(t,s,v,u)|\le\frac{1}{T}\phi(\max\{|x-u|,|y-v|\}).$$

Then the system of integral equations has a unique solution in $C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$.

Proof. For any $x,y,u,v\in X$ and $t\in[0,T]$,

$$\begin{aligned}|F(x,y)(t)-F(u,v)(t)| &= \left|\int_0^T K_1(t,s,x(s),y(s))ds-\int_0^T K_1(t,s,u(s),v(s))ds\right|\\ &\le \int_0^T |K_1(t,s,x(s),y(s))-K_1(t,s,u(s),v(s))|ds\\ &\le \int_0^T \frac{1}{T}\psi(\max\{|x(s)-u(s)|,|y(s)-v(s)|\})ds.\end{aligned}$$

Since ψ,x,y,u,v are bounded, thus

$$\begin{aligned}|F(x,y)(t)-F(u,v)(t)| &\le \psi(\sup_{s\in[0,T]}\max\{|x(s)-u(s)|,|y(s)-v(s)|\})\\ &= \psi(\max\{d(x,u),d(y,v)\}).\end{aligned}$$

Thus,

$$d(F(x,y),F(u,v))=\sup_{t\in[0,T]}|F(x,y)(t)-F(u,v)(t)|\le\psi(\max\{d(x,u),d(y,v)\}).$$

Then, for any $x,y,u,v\in X$

$$\int_0^{d(F(x,y),F(u,v))}\phi(t)dt\le\int_0^{\psi(\max d(x,u),d(y,v))}\phi(t)dt\le\psi\left(\int_0^{\max\{d(x,u),d(y,v)\}}\phi(t)dt\right).$$

Then all conditions of Theorem 3.2 are satisfied, thus there is a unique solution to the system of integral equations.

Declarations

Ethical Approval. Hereby, I consciously assure that the following is fulfilled:

1) This material is the authors' own original work, which has not been previously published elsewhere.

2) The paper is not currently being considered for publication elsewhere.

3) The paper reflects the authors' own research and analysis in a truthful and complete manner.

Competing interests. The author declares no conflicts of interest.

Availability of data and materials. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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