Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2025.28216.1734
Caspian J Math Sci. **14**(2)(2025), 375-393 (Research Article)

Characterizing regular posemirings by ordered high-(quasi,bi)-ideals

M. M. Shamivand [1] and Ahmad Shahsavaran
Department of Mathematics, Bo.C., Islamic Azad University, Borujerd, Iran.

Abstract. In this paper, we study the notions of an ordered high-quasi-ideal and ordered high-bi-ideal of a posemiring and show that ordered high-quasi-ideals and ordered high-bi-ideals coincide in regular posemirings. Then we give characterizations of regular posemirings, regular duo-posemirings and left(right) regular posemirings by their high-quasi-ideals and high-bi-ideals.

Keywords: Posemiring, Regular, High-quasi-ideal, High-bi-ideal.

2000 Mathematics subject classification: 05C20, 05C30.

1. Introduction

The concept of quasi-ideal for rings was introduced in 1953 by Steinfeld [13, 14, 15]. Iseki [5] introduced this concept for semirings without zero and studied some properties. Donges [3] studied quasi-ideals of a semiring with zero and connections between left(right) ideals, bi-ideals and quasi-ideals. Later, Shabir et al. [12] have studied some properties of quasi-ideals, using quasi-ideals to characterize regular and intra-regular semirings and regular duo-semirings. As a generalization of quasi-ideals of semirings the quasi-ideals of Γ-semirings were investigated by many authors: see [1, 2, 6]. In 2011, the notion of an ordered semiring (posemiring) was introduced by Gan and Jiang [4] as a semiring with a partially ordered relation on the semiring such that the relation is compatible

[1]Corresponding author: shamivand53@iau.ir
Received: 16 December 2024
Revised: 17 Feruary 2025
Accepted: 04 June 2025
How to Cite: Shamivand, Mohammadmahdi; Shahsavaran, Ahmad. Characterizing regular posemirings by ordered high-(quasi,bi)-ideals, Casp.J. Math. Sci.,**14**(2)(2025), 375-393.

to the operations of the semirings and the concept of a left(right) ordered ideal, a minimal ordered ideal, and a maximal ordered ideal was defined. The Mandal [9] studied fuzzy ideals in a posemiring with the least element zero and gave a characterization of regular posemirings by their fuzzy ideals. In this paper, the concept of *high-quasi-ideal*, *high-bi-ideal* in posemirings and some new characterizations for regular posemirings are presented. A *posemiring* is a system $(S, +, \cdot, \leq)$ consisting of a non-empty set S, two binary operations $+, \cdot$ on S and a partial order ralation on S such that $(S, +, \cdot)$ is a semiring (i.e. an algebraic structure similar to a ring but without the requirement that each element must have an additive inverse), and for every $a, b, x \in S$ the following conditions are satisfied:

(i) If $a \leq b$, then $a + x \leq b + x$ and $x + a \leq x + b$,

(ii) If $a \leq b$, then $a \cdot x \leq b \cdot x$ and $x \cdot a \leq x \cdot b$.

As usual, we omit the operation "$\cdot$" between every two elements $a, b \in S$ and write ab provided that no confusion arises. A posemiring S is said to be *additively commutative* if $a + b = b + a$ for all $a, b \in S$. An element $0 \in S$ is said to be an *absorbing zero* if $0a = 0 = a0$ and $a+0 = a = 0+a$ for all $a \in S$. Throughout this paper, the word posemiring shall mean an additively commutative posemiring with an absorbing zero 0 unless otherwise stated. For subposets A and B of a posemiring S, let $AB := \{ab \mid a \in A, b \in B\}$, $(A] := \{s \in S \mid s \leq a$ for some $a \in A\}$ and $(a] := \{s \in S \mid s \leq a\}$ for every $a \in S$. Also, for subposets A and B of a posemiring S, we denote:

$A + B = \{a + b \mid a \in A, b \in B\}$,
$\Sigma A = \{\Sigma_{i \in I} a_i \in S \mid a_i \in A$ and I is a finite subset of $\mathbb{N}\}$,
$\Sigma AB = \{\Sigma_{i \in I} a_i b_i \mid a_i \in A, b_i \in B$ and I is a finite subset of $\mathbb{N}\}$,
$\mathbb{N}a = \Sigma\{a\}$.

If $A = \{a\}$, then we denote $\Sigma A = \Sigma a$. By a *subposemiring of a posemiring S we mean a subsemiring of S which is also a subposet under the inherited order relation of S.*

Remark 1.1. For a posemiring S and non-empty subsets A, B of S, we have the following:
(i) $\Sigma(A] \subseteq (\Sigma A]$, (ii) $\Sigma(\Sigma A) = \Sigma A$, (iii) $A(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq \Sigma AB$,

(iv) $\Sigma(A\Sigma B) \subseteq \Sigma AB$ and $\Sigma(\Sigma A)B \subseteq \Sigma AB$, (v) $\Sigma(A + B) = \Sigma A + \Sigma B$.

We note that, for every $A \subseteq S$, $\Sigma A = A$ if and only if $A + A \subseteq A$(($A, +$) is a subsemigroup of $(S, +)$).(See [7, 8]*)*

Recall from [10, 11, 12] *that If S is a posemiring, then a non-empty subset Q of S is called a quasi-ideal of S if $Q + Q \subseteq Q$, $(\Sigma SQ] \cap (\Sigma QS] \subseteq Q$ and $(Q] \subseteq Q$. Also, a non-empty subset B of a posemiring S is called a bi-ideal of S if $B + B \subseteq B$, $\Sigma BSB \subseteq B$ and $(B] \subseteq B$.*

By a left ordered ideal(right ordered ideal) of a posemiring S, we mean a non-empty subset A of S if the following conditions are satisfied(see[4]*):*

(1) *A is a left ideal(right ideal) of S,*

(2) *If $x \leq a$ for some $a \in A$ then $x \in A$(i.e., $A = (A]$).*

We call A an ordered ideal if it is both left ordered ideal and right ordered ideal of S.

The right ordered ideal, left ordered ideal, ideal, quasi-ideal and bi-ideal generated by a subset A of a posemiring S will be denoted by $R(A), L(A), I(A), Q(A)$ and $B(A)$, respectively. If $A = \{a\}$, then we denote $R(A), L(A), I(A), Q(A)$ and $B(A)$ by $R(a), L(a), I(a), Q(a)$ and $B(a)$, respectively.

If S is a posemiring, then $R(a) = (\Sigma a + \Sigma aS]$, $L(a) = (\Sigma a + \Sigma Sa]$, $I(a) = (\Sigma a + \Sigma Sa + \Sigma aS + \Sigma SaS]$, $Q(a) = (\Sigma a + ((\Sigma aS] \cap (\Sigma Sa])]$ and $B(a) = (\Sigma a + \Sigma a^2 + \Sigma aSa]$.

Here we generalize the notions of bi-ideal and quasi-ideal as high-bi-ideal and high-quasi-ideal, respectively. Firstly, for each subset A of a posemiring S we denote:

$$(A]^* := \{s \in S \mid s \leq a^n \text{ for some } a \in A \text{ and } n \in \mathbb{N}\}.$$

Let A and B be non-empty subsets of posemiring S. Clearly, $A \subseteq (A]^*$, $((A]^*]^* = (A]^*$, $(A \cap B]^* \subseteq (A]^* \cap (B]^*$ and $(A \cup B]^* = (A]^* \cup (B]^*$ and $A \subseteq B$ implies that $(A]^* \subseteq (B]^*$ as well.

If S has an identity then we have the following Lemmas:

Lemma 1.2. *For a posemiring S and non-empty subsets A, B and C of S, we have:*

(i) $\Sigma(A]^* \subseteq (\Sigma A]^*$,

(ii) $\Sigma(\Sigma A) = \Sigma A$,

(iii) $A(\Sigma B) \subseteq \Sigma AB$ *and* $(\Sigma A)B \subseteq \Sigma AB$,

(iv) $\Sigma(A\Sigma B) \subseteq \Sigma AB$ *and* $\Sigma(\Sigma A)B \subseteq \Sigma AB$,

(v) $\Sigma(A + B) = \Sigma A + \Sigma B$,

(vi) $A(B]^* \subseteq (A]^*(B]^* \subseteq (AB]^*$ *and* $(A]^*B \subseteq (A]^*(B]^* \subseteq (AB]^*$,

(vii) $A + (B]^* \subseteq (A]^* + (B]^* \subseteq (A + B]^*$ *and* $(A]^* + B \subseteq (A]^* + (B]^* \subseteq (A + B]^*$,

(viii) $A(B + C]^* \subseteq (AB + AC]^*$ *and* $(A + B]^*C \subseteq (AC + BC]^*$.

Lemma 1.3. *Let S ba a posemiring and A be a non-empty subposet of S. If $A \subseteq (\Sigma A^2 + \Sigma ASA]^*$ then, $\Sigma A^2 \subseteq (\Sigma ASA]^*$.*

Proof. Assume that $A \subseteq (\Sigma A^2 + \Sigma ASA]^*$ then,

$$\begin{aligned}
\Sigma A^2 &\subseteq \Sigma(\Sigma A^2 + \Sigma ASA]^* A \\
&\subseteq \Sigma((\Sigma A^2)A + (\Sigma ASA)A]^* \\
&\subseteq \Sigma(\Sigma A^3 + \Sigma ASA]^* \\
&\subseteq (\Sigma(\Sigma A^3 + \Sigma ASA)]^* \\
&= (\Sigma(\Sigma A^3) + \Sigma(\Sigma ASA)]^* \\
&= (\Sigma A^3 + \Sigma ASA]^* \\
&= (\Sigma AAA + \Sigma ASA]^* \\
&\subseteq (\Sigma ASA + \Sigma ASA]^* \\
&= (\Sigma ASA]^*.
\end{aligned}$$

□

Lemma 1.4. *Let A be a nonempty subset of a posemiring S then,*

(i) $(\Sigma SA]^*$ *is a left ordered ideal of S,*

(ii) $(\Sigma AS]^*$ *is a right ordered ideal of S,*

(iii) $(\Sigma SAS]^*$ *is an ordered ideal of S.*

Proof. (i) Let $x, y \in (\Sigma SA]^*$ then $x \leq a^m$ and $y \leq b^n$ for some $a, b \in \Sigma SA$ and $m, n \in \mathbb{N}$ and so $x + y \leq a^m + b^n \leq (a + b)^{m+n}$. By considering $a + b = c \in \Sigma SA$ and $m + n = k \in \mathbb{N}$ we have, $x + y \in (\Sigma SA]^*$. Also, $S(\Sigma SA]^* \subseteq (S\Sigma SA]^* \subseteq (\Sigma SSA]^* \subseteq (\Sigma SA]^*$ and $((\Sigma SA]^*]^* = (\Sigma SA]^*$. Hence, $(\Sigma SA]^*$ is a left ordered ideal of S.

(ii) and (iii) can be proved similar to (i).

□

Corollary 1.5. *Let S be a posemiring then for every $a \in S$,*

(i) $(Sa]^*$ *is a left ordered ideal of S,*

(ii) $(aS]^*$ *is a right ordered ideal of S,*

(iii) $(\Sigma SaS]^*$ *is an ordered ideal of S.*

Lemma 1.6. *Let A be a nonempty subset of a posemiring S then,*

(i) $L(A) = (\Sigma A + \Sigma SA]^*$,

(ii) $R(A) = (\Sigma A + \Sigma AS]^*$,

(iii) $I(A) = (\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]^*$.

Proof. (i) Since S has an absorbing zero then for every $a \in A$, $a = a + 0 \in \Sigma A + \Sigma SA \subseteq (\Sigma A + \Sigma SA]^*$. Hence, $A \subseteq (\Sigma A + \Sigma SA]^*$. Let $x, y \in (\Sigma A + \Sigma SA]^*$ then $x \leq a^m$ and $y \leq b^n$ for some $a, b \in \Sigma A + \Sigma SA$ and $m, n \in \mathbb{N}$. Thus $a = a_1 + b_1$ and $b = a_2 + b_2$ for some $a_1, a_2 \in \Sigma A$ and $b_1, b_2 \in \Sigma SA$. It is easy to show that $a_1 + a_2 \in \Sigma A$ and $b_1 + b_2 \in \Sigma SA$. Clearly, $x + y \leq a^m + b^n \leq (a + b)^{m+n}$. By considering $a + b = c \in \Sigma A + \Sigma SA$ and $m + n = k \in \mathbb{N}$ we have, $x + y \in (\Sigma A + \Sigma SA]^*$. Also, $S(\Sigma A + \Sigma SA]^* \subseteq (S(\Sigma A + \Sigma SA)]^* \subseteq (S\Sigma A + S\Sigma SA]^* \subseteq (\Sigma SA + \Sigma SSA]^* \subseteq (\Sigma SA + \Sigma SA]^* = (\Sigma SA]^* \subseteq (\Sigma A + \Sigma SA]^*$. Since $((\Sigma A + \Sigma SA]^*]^* = (\Sigma A + \Sigma SA]^*$ so L is a left ordered ideal of S. Let K be a left ordered ideal of S containing A so $\Sigma A \subseteq K$ and $\Sigma SA \subseteq K$. Hence $\Sigma A + \Sigma SA \subseteq K$ which implies that $(\Sigma A + \Sigma SA]^* \subseteq (K]^* = K$. Therefore, $(\Sigma A + \Sigma SA]^*$ is the smallest left ordered ideal of S containing A.

(ii) and (iii) can be proved similar to (i).

□

As a special case of Lemma 1.6, if $A = \{a\}$ then we have the following corollary: Let S be a posemiring then for every $a \in S$,

(i) $L(a) = (\mathbb{N}a + Sa]^*$,

(ii) $R(a) = (\mathbb{N}a + aS]^*$,

(iii) $I(a) = (\mathbb{N}a + Sa + aS + \Sigma SaS]^*$.

An element e of a posemiring S is said to be an identity if $ea = a = ae$ for all $a \in S$. If S has an identity, then we denote 1 as the identity of S. It is not difficult to show that if S has an identity, then $L(A) = (\Sigma SA]^*$, $R(A) = (\Sigma AS]^*$ and $I(A) = (\Sigma SAS]^*$ for every $A \subseteq S$. In particular, we have $L(a) = (Sa]^*$, $R(a) = (aS]^*$ and $I(a) = (\Sigma SaS]^*$ for every $a \in S$.

2. Ordered high- quasi- ideal in posemirings

We present a notion of an ordered high- quasi- ideal and an ordered high- bi- ideal of a posemiring. Then, in a posemiring with an identity, we show that every ordered high- quasi- ideal can be expressed as an intersection of an ordered left ideal and an ordered right ideal.

Definition 2.1. Suppose that $(S, +, ., \leq)$ be a posemiring and let $(Q, +)$, $(B, +)$ and $(I, +)$ are subposemirings of $(S, +)$ then Q is said to be an *ordered high- quasi- ideal* of S if the following conditions are satisfied:

(i) $(\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq Q$,
(ii) If $x \leq q^n$ for some $q \in Q$ and $n \in \mathbb{N}$ then $x \in Q$(i.e., $Q = (Q]^*$). Also,

B is said to be an *ordered high- bi- ideal* of S if the following conditions are satisfied:

(i) $B + B \subseteq B$ and $\Sigma BSB \subseteq B$,
(ii) If $x \leq b^n$ for some $b \in B$ and $n \in \mathbb{N}$ then $x \in B$(i.e., $B = (B]^*$). Also,

I is said to be an *ordered high-interior-ideal* of S if the following conditions are satisfied:

(i) $I + I \subseteq I$ and $\Sigma SIS \subseteq I$,
(ii) If $x \leq i^n$ for some $i \in B$ and $n \in \mathbb{N}$ then $x \in I$(i.e., $I = (I]^*$).

We give some characterizations of regular posemirings using their ordered high-bi-ideals and ordered high-quasi-ideals. Clearly, every one-sided ordered ideal of a posemiring S is an ordered high-quasi-ideal of S. For a subset A of a posemiring S, we denote the ordered high-quasi-ideal and ordered high-bi-ideal generated by A by $Q^*(A)$ and $B^*(A)$, respectively. If $A = \{a\}$, we denote $Q^*(A)$ and $B^*(A)$ by $Q^*(a)$ and $B^*(a)$, respectively.

Lemma 2.2. *Let S be a posemiring and A be a nonempty subset of S then,*

$Q^*(A) = (\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)]^*$.

Proof. Let $Q = (\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)]^*$. Since S has an absorbing zero, we have $a = a + 0 \in \Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*) \subseteq Q$ for every $a \in A$ hence $A \subseteq Q$. Let $x, y \in Q$ then, $x \leq a^m$ and $y \leq b^n$ for some $a, b \in \Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)$ and $m, n \in \mathbb{N}$. Thus $a = a_1 + b_1$ and $b = a_2 + b_2$ for some $a_1, a_2 \in \Sigma A$ and $b_1, b_2 \in (\Sigma SA]^* \cap (\Sigma AS]^*$. It is clear that $x + y \leq a^m + b^n \leq (a + b)^{m+n}$. By considering $a + b = c \in \Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*$ and $m + n = k \in \mathbb{N}$ we have, $x + y \in (\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)]^*$ and so, $x + y \in Q$. We have,

$$\begin{aligned}
&(\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq (\Sigma SQ]^* \\
&= (\Sigma S(\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)]^*]^* \\
&\subseteq (\Sigma S(\Sigma A + (\Sigma SA]^*]^*]^* \\
&\subseteq (\Sigma(S\Sigma A + S(\Sigma SA]^*]^*]^* \\
&\subseteq (\Sigma(\Sigma SA + (\Sigma SSA]^*]^*]^* \\
&\subseteq (\Sigma(\Sigma SA + \Sigma SSA]^*]^* \\
&\subseteq (\Sigma(\Sigma SA + \Sigma SA]^*]^* \\
&\subseteq (\Sigma(\Sigma SA]^*]^* \\
&\subseteq ((\Sigma SA]^*]^* \\
&= (\Sigma SA]^*.
\end{aligned}$$

Similarly, we can show that $(\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq (\Sigma AS]^*$. Thus, $(\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq (\Sigma SA]^* \cap (\Sigma AS]^* \subseteq \Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*) \subseteq Q$. Since $(Q]^* = Q$, we obtain that Q is an ordered high- quasi- ideal of S contaning A. Let Q' be an ordered high- quasi- ideal of S contaning A. We have, $(\Sigma SA]^* \cap (\Sigma AS]^* \subseteq (\Sigma SQ']^* \cap (\Sigma Q'S]^* \subseteq Q'$. So, $\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*) \subseteq Q'$. Hence, $Q = (\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*]^* \subseteq (Q']^* = Q'$. Therefore, Q is the smallest ordered high- quasi- ideal of S contaning A. □

Let S be a posemiring and $A = \{a\}$ be a subset of S then,
$Q^*(a) = (\ \mathbb{N}a + ((\Sigma Sa]^* \cap (\Sigma aS]^*)]^*$.

It is clear that every left ordered ideal(right ordered ideal and ordered ideal) of a posemiring S is an ordered high- quasi- ideal of S. Moreover, each ordered high- quasi- ideal of S is a subposemiring of S; indeed, $QQ \subseteq (QQ]^* \subseteq (SQ]^* \cap (QS]^* \subseteq (\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq Q$.

Example 2.3. Let $S = \{a, b, c, d\}$. We define binary operations $+$ and . on S by the following equations:

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	d
d	d	b	d	d

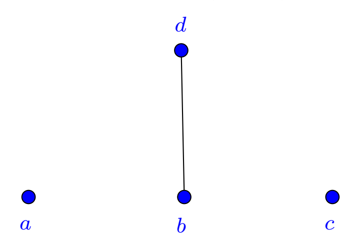
$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	b	b	b

then $(S, +, .)$ is an additively commutative semiring with an absorbing zero a. We define a binary ordering relation on S by:

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (b, d)\}.$$

We give the covering relation and the figure of S by:

$$\prec := \{(b, d)\}$$



now $(S, +, ., \leq)$ is a posemiring. Let $Q = \{a, b\}$ we have $(\Sigma SQ]^* \cap (\Sigma QS]^* = \{a, b, c\} \cap \{a, b\} = Q$ and $(Q]^* = Q$. Hence, Q is an ordered high- quasi- ideal of S but is not a left ordered ideal of S, since $SQ = \{a, b, c\} \nsubseteq Q$.

Lemma 2.4. *Let S be a posemiring and $\{Q_i \mid i \in I\}$ be a family of ordered high- quasi- ideals of S. Then $\bigcap_{i\in I} Q_i$ is an ordered high- quasi- ideal of S.*

Theorem 2.5. *The intersection of a left ordered ideal L and a right ordered ideal R of a posemiring S is an ordered high- quasi- ideal of S.*

Proof. $L \cap R$ is a subsemigroup of $(S, +)$ so by Lemma 1.2 we have

$$\begin{aligned}&(\Sigma S(L\cap R)]^* \cap (\Sigma(L\cap R)S]^* \subseteq (\Sigma S(L\cap R)]^*\\&= (\Sigma(SL\cap SR)]^*\\&\subseteq (\Sigma SL]^*\\&\subseteq L,\end{aligned}$$

$$\begin{aligned}&(\Sigma S(L\cap R)]^* \cap (\Sigma(L\cap R)S]^* \subseteq (\Sigma(L\cap R)S]^*\\&= (\Sigma(LS\cap RS)]^*\\&\subseteq (\Sigma RS]^*\\&\subseteq R.\end{aligned}$$

Hence, $(\Sigma S(L\cap R)]^* \cap (\Sigma(L\cap R)S]^* \subseteq L\cap R$. Let $s \in S$ such that $s \leq x^n$ for some $x \in L\cap R$ and $n \in \mathbb{N}$. Then, $s \in (L\cap R]^* \subseteq (L]^* \cap (R]^* = L\cap R$. □

Let S be a posemiring then, the following statements hold:

(i) $(\Sigma SA]^* \cap (\Sigma AS]^*$ *is an ordered high- quasi- ideal of S, for every $A \subseteq S$,*

(ii) $(\Sigma Sa]^* \cap (\Sigma aS]^*$ *is an ordered high- quasi- ideal of S, for every $a \in S$.*

Proof. (i) By Lemma 1.3 $(\Sigma SA]^*$ and $(\Sigma AS]^*$ are a left and a right ordered ideal of S, respectively. Then by Theorem 2.5 $(\Sigma SA]^* \cap (\Sigma AS]^*$ is an ordered high-quasi- ideal of S.

(ii) It is a particular case of (i).

□

Now, we will show that the converse of Theorem 2.5 is true if S contain an identity by the following theorem:

Theorem 2.6. *Let S be a posemiring with identity then, every ordered high-quasi- ideal Q of S can be written in the form $Q = R\cap L$ for some right ordered ideal R and left ordered ideal L of S.*

Proof. Assume S has an identity and Q be an ordered high- quasi- ideal of S then, $R(Q) = (\Sigma QS]^*$ and $L(Q) = (\Sigma SQ]^*$. Clearly, $Q \subseteq R(Q)\cap L(Q)$ and $R(Q)\cap L(Q) = (\Sigma QS]^* \cap (\Sigma SQ]^* \subseteq Q$. Hence, $Q = R(Q)\cap L(Q)$.

□

3. Regularity of posemirings

In this section, we show that in regular posemirings the converse of Theorem 2.5 is true and ordered high- quasi- ideals concide with ordered high- bi- ideals. Then we give characterizations of regular posemirings, regular duo-posemirings, left regular and right regular posemirings by their ordered high- quasi- ideals.

Definition 3.1. An element a of a posemiring S is said to be *regular* if $a \leq axa$ for some $x \in S$. A posemiring S is said to be *regular* if every element $a \in S$ is regular. (see[9])

The following lemma is characterizations of regular posemiring which directly follows from Definition 3.1.

Lemma 3.2. *Let S be a posemiring then the following statements are equivalent:*

(i) *S is regular,*

(ii) *$A \subseteq (\Sigma ASA]^*$ for every $A \subseteq S$,*

(iii) *$a \in (aSa]^*$ for every $a \in S$.*

Now, we will show that the converse of Theorem 2.5 is true in regular posemirings.

Theorem 3.3. *Every ordered high- quasi- ideal of a regular posemiring S can be written in the form $Q = R \cap L$ for some right ordered ideal R and left ordered ideal L of S.*

Proof. Let Q be an ordered high- quasi- ideal of S. By Lemma 1.6, we have $R(Q) = (\Sigma Q + \Sigma QS]^*$ and $L(Q) = (\Sigma Q + \Sigma SQ]^*$. Obviously, $Q \subseteq R(Q) \cap L(Q)$. Let, $q \in Q$ since S is regular so $q \in (qSq]^* \subseteq (QSQ]^* \subseteq (QS]^* \subseteq (\Sigma QS]^*$. So, $Q \subseteq (\Sigma QS]^*$. Since $Q + Q \subseteq Q$ then $\Sigma Q = Q$. Obviously $(\Sigma QS]^* \subseteq (\Sigma Q + \Sigma QS]^* = (Q + \Sigma QS]^* \subseteq ((\Sigma QS]^* + \Sigma QS]^* \subseteq (\Sigma QS]^*$ which implise that $R(Q) = (\Sigma QS]^*$. Similarly, $L(Q) = (\Sigma SQ]^*$. Hence, $R(Q) \cap L(Q) = (\Sigma QS]^* \cap (\Sigma SQ]^* \subseteq Q$. Therefore, $Q = R(Q) \cap L(Q)$.

□

Theorem 3.4. *Every ordered high- quasi- ideal of a posemiring S is an ordered high- bi- ideal of S.*

Proof. Let Q be an ordered high- quasi- ideal of S then $\Sigma QSQ \subseteq \Sigma QS \subseteq (\Sigma QS]^*$ and $\Sigma QSQ \subseteq \Sigma SQ \subseteq (\Sigma SQ]^*$. So $\Sigma QSQ \subseteq (\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq Q$. Hence, Q is an ordered high- bi- ideal of S.

□

By following example we show that converse of the Theorem 3.4 is not generally true.

Example 3.5. Let $S = \{a, b, c, d, e\}$. We define binary operations $+$ and . on S by the following equations

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	d
c	c	d	d	d	d
d	d	d	d	d	d
e	e	d	d	d	e

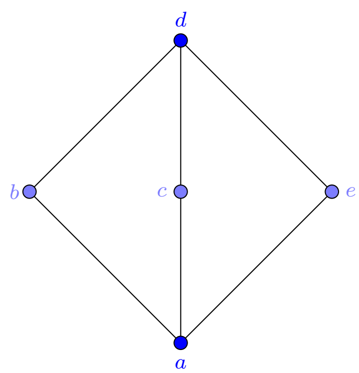
$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	b	b	b
d	a	a	b	b	b
e	a	a	b	b	b

then $(S, +, .)$ is an additively commutative semiring with an absorbing zero a. We define a binary ordering relation on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, e), (a, d), (b, d), (c, d), (e, d)\}.$$

We give the covering relation and the figure of S by:

$$\prec := \{(a, b), (a, c), (a, e), (b, d), (c, d), (e, d)\}$$



now, $(S, +, ., \leq)$ is a posemiring but is not regular, since $d \nleq dxd$ for every $x \in S$. Let $B = \{a, e\}$ then B is an ordered high- bi- ideal but not an ordered high- quasi- ideal of S, since, $(\Sigma SB]^* \cap (\Sigma BS]^* = \{a, b\} \nsubseteq B$.

We show that in regular posemirings, ordered high- bi- ideals and ordered high- quasi- ideals are coincide by the following theorem:

Theorem 3.6. *Let S be a regular posemiring then, ordered high- bi- ideals and ordered high- quasi- ideals coincide in S.*

Proof. By the Theorem 3.4, every ordered high- quasi- ideal of S is an ordered high- bi- ideal of S. We show that every ordered high- bi- ideal of S is an ordered high- quasi- ideal of S. Let B be an ordered high- bi- ideal of S and $a \in (\Sigma SB]^* \cap (\Sigma BS]^*$. By Lemma 3.2 and Lemma 1.2, we obtain $a \in (aSa]^* \subseteq ((\Sigma BS]^* S(\Sigma SB]^*]^* \subseteq ((\Sigma BSS]^*(\Sigma SB]^*]^* \subseteq ((\Sigma BS)(\Sigma SB)]^* \subseteq (\Sigma(BS(\Sigma SB))]^* \subseteq (\Sigma(\Sigma BSSB)]^* \subseteq (\Sigma BSB]^* \subseteq B$. Hence, B is an ordered high- quasi- ideal of S.

□

Theorem 3.7. *Let S be a regular posemiring then, the following statements are equivalent:*

(i) *S is regular,*

(ii) $(\Sigma RL]^* = R \cap L$ *for every right ordered ideal R and left ordered ideal L of S,*

(iii) $B = (\Sigma BSB]^*$ *for every ordered high- bi- ideal B of S,*

(iv) $Q = (\Sigma QSQ]^*$ *for every ordered high- quasi- ideal Q of S.*

Proof. (i) $\Longrightarrow$ (ii) suppose S be a posemiring. Let R and L be a right ordered ideal and a left ordered ideal of S, respectively. So $(\Sigma RL]^* \subseteq (\Sigma R]^* = R$ and $(\Sigma RL]^* \subseteq (\Sigma L]^* = L$. Hence, $(\Sigma RL]^* \subseteq R \cap L$. Let $a \in R \cap L$, since S is regular, $a \leq axa$ for some $x \in S$. Obviously, $a \in (aSa]^* = ((a)Sa]^* \subseteq (RL]^* \subseteq (\Sigma RL]^*$. Therefore, $(\Sigma RL]^* = R \cap L$.

(ii) $\Longrightarrow$ (iii) Let B be an ordered high- bi- ideal of S. It is clear that $(\Sigma BSB]^* \subseteq B$. By assumption (ii) we have $B \subseteq R(B) \cap L(B) = (\Sigma R(B)L(B)]^*$. By Lemma 1.2 and Lemma 1.3, we have,

$$
\begin{aligned}
B &\subseteq (\Sigma B + \Sigma BS]^* \cap (\Sigma B + \Sigma SB]^* \\
&= (\Sigma((\Sigma B + \Sigma BS]^*(\Sigma B + \Sigma SB]^*)]^* \\
&\subseteq (\Sigma((\Sigma B + \Sigma BS)(\Sigma B + \Sigma SB))]^* \\
&\subseteq (\Sigma(\Sigma B(\Sigma B + \Sigma SB) + \Sigma BS(\Sigma B + \Sigma SB))]^* \\
&\subseteq (\Sigma(\Sigma B^2 + \Sigma BSB + \Sigma BSB + \Sigma BSSB)]^* \\
&\subseteq (\Sigma(\Sigma B^2 + \Sigma BSB)]^* \\
&= (\Sigma(\Sigma B^2) + \Sigma(\Sigma BSB)]^* \\
&= (\Sigma B^2 + \Sigma BSB]^* \\
&\subseteq ((\Sigma BSB]^* + \Sigma BSB]^* \\
&\subseteq (\Sigma BSB]^*
\end{aligned}
$$

.

(iii) $\Longrightarrow$ (iv) It is obvious according to the Theorem 3.4.

(iv) $\Longrightarrow$ (i) Let $a \in S$ then $Q^*(a) = (Q^*(a)SQ^*(a)]^*$. By Corollary 2 and Lemma 1.2 we have

$$\begin{aligned}
a &\in (\mathbb{N}a + ((Sa]^* \cap (aS]^*)]^* \\
&= (\Sigma((\mathbb{N}a + ((Sa]^* \cap (aS]^*)]^* S(\mathbb{N}a + ((Sa]^* \cap (aS]^*)]^*)]^* \\
&\subseteq (\Sigma((\mathbb{N}a + (aS]^*]^* S(\mathbb{N}a + (Sa]^*]^*)]^* \\
&\subseteq (\Sigma((\mathbb{N}a + aS]^* S(\mathbb{N}a + Sa]^*)]^* \\
&\subseteq (\Sigma(((\mathbb{N}a + aS)S]^*(\mathbb{N}a + Sa]^*)]^* \\
&\subseteq (\Sigma((aS]^*(\mathbb{N}a + Sa]^*)]^* \\
&\subseteq (\Sigma(aS(\mathbb{N}a + Sa))]^* \\
&\subseteq (\Sigma aSa]^* \\
&= (aSa]^*.
\end{aligned}$$

Therefore, by Lemma 3.2, S is regular.

□

Theorem 3.8. *Let S be a regular posemiring then the following statements are hold:*

(i) *Every ordered high- quasi- ideal Q of S can be written in the form $Q = R \cap L = (RL]^*$ for some right ordered ideal R and left ordered ideal L of S,*

(ii) $(Q^2]^* = (Q^3]^*$ *for every ordered high- quasi- ideal Q of S.*

Proof. (i) It is obvious by Theorem 3.3 and 3.7.

(ii) Let Q be an ordered high- quasi- ideal of S. Clearly, $((QQ)Q]^* \subseteq (QQ]^*$. Let $x \in (QQ]^*$ then $x \leq (q_1q_2)^n$ for some $q_1, q_2 \in Q$ and $n \in \mathbb{N}$ Since S is regular there exist $s \in S$ where $x \leq (q_1q_2)^n \leq (q_1q_2)^n s(q_1q_2)^n \leq q_1^n q_2^n s q_1^n q_2^n \leq (q_1q_2sq_1q_2)^n$ such that $q_1q_2sq_1q_2 \in QQSQQ$. Hence, $x \in (Q(QSQ)Q]^* \subseteq (QQQ]^*$. Therefore, $(Q^2]^* = (Q^3]^*$.

□

Definition 3.9. A posemiring S is said to be a *duo-posemiring* if every one-sided(right or left) ordered ideal of S is an ordered ideal of S.

We note that every multiplicatively commutative posemiring is a duo-posemiring, but the converse is not generally true.

Example 3.10. Let $S = \{a, b, c, d, e\}$. We define binary operations $+$ and . on S by the following equations:

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	c	c	c	c
c	c	c	c	c	c
d	d	c	c	c	c
e	e	c	c	c	c

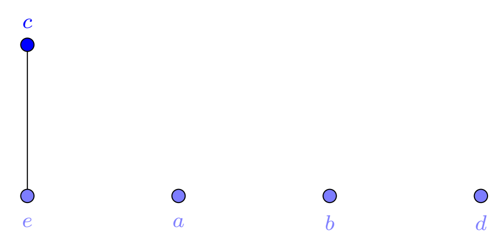
$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	c	e	c
c	a	c	c	c	c
d	a	c	c	e	c
e	a	c	c	c	c

then $(S, +, .)$ is an additively commutative semiring with an absorbing zero a. Also we define a binary ordering relation on S by:

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (e, c)\},$$

the covering relation and the figure of S is given by

$$\prec := \{(e, c)\}$$



since $bd \neq db$ so $(S, +, ., \leq)$ is a posemiring but is not multiplicatively commutative. We have all one-sided ordered ideals of S which are as follows:

$$\{a\}, \{a, c\}, \{ac, e\}, \{a, b, c, e\}, \{a, c, d, e\}, S.$$

It is not difficult to check that all of them are ordered ideals of S. This show that S is a duo-posemiring.

Lemma 3.11. *Let S be a posemiring then the following conditions are equivalent:*

(i) *S is a duo-posemiring,*

(ii) *$R(A) = L(A)$ for each $A \subseteq S$,*

(ii) *$R(a) = L(a)$ for each $a \in S$.*

Proof. (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are obvious.

(iii) $\Longrightarrow$ (i) Suppose L be a left ordered ideal of S and let $x \in L$, $s \in S$. By assumption (iii) we have $xs \in R(x)S \subseteq R(x) = L(x) \subseteq L(L) = L$. So L is a right ordered ideal of S. Similarly, every right ordered ideal of S is a left ordered ideal of S. Therefore, S is a duo-posemiring.

□

Theorem 3.12. *Let S be a duo-posemiring then S is regular if and only if $(\Sigma Q_1Q_2]^* = Q_1 \cap Q_2$ for every two ordered high- quasi- ideals Q_1 and Q_2 of S.*

Proof. Assume S is a regular posemiring. Let Q_1 and Q_2 be ordered high- quasi-ideals of S. By Theorem 3.3 Q_1 and Q_2 can be written in the forms $Q_1 = R_1 \cap L_1$ and $Q_2 = R_2 \cap L_2$ for some R_1, R_2 and L_1, L_2 which are right ordered ideals and left ordered ideals of S, respectively. Since S is a duo-posemiring and R_1, R_2, L_1 and L_2 are ordered ideals of S which implies that Q_1 and Q_2 are ordered ideals of S. By Theorem 3.7 we have $(\Sigma Q_1Q_2]^* = Q_1 \cap Q_2$.

Conversely, assume $(\Sigma Q_1Q_2]^* = Q_1 \cap Q_2$ for every two ordered high- quasi-ideals Q_1 and Q_2 of S. Let $A \subseteq S$, obviously $A \subseteq Q^*(A) \cap Q^*(A) = (\Sigma Q^*(A)Q^*(A)]^*$ and by Lemmas 1.2, 1.3 and 2.2 we have:

$$
\begin{aligned}
&A \subseteq (\Sigma((\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)]^*.(\Sigma A + ((\Sigma SA]^* \cap (\Sigma AS]^*)]^*)]^* \\
&\subseteq (\Sigma((\Sigma A + (\Sigma AS]^*]^*.(\Sigma A + (\Sigma SA]^*]^*)]^* \\
&\subseteq (\Sigma((\Sigma A + \Sigma AS]^*(\Sigma A + \Sigma SA]^*)]^* \\
&\subseteq (\Sigma((\Sigma A + \Sigma AS)(\Sigma A + \Sigma SA))]^* \\
&\subseteq (\Sigma(\Sigma A(\Sigma A + \Sigma SA) + \Sigma AS(\Sigma A + \Sigma SA))]^* \\
&\subseteq (\Sigma(\Sigma A^2 + \Sigma ASA + \Sigma ASA + \Sigma ASSA)]^* \\
&\subseteq (\Sigma(\Sigma A^2 + \Sigma ASA)]^* \\
&= (\Sigma(\Sigma A^2) + \Sigma(\Sigma ASA)]^* \\
&= (\Sigma A^2 + \Sigma ASA]^* \\
&\subseteq ((\Sigma ASA]^* + \Sigma ASA]^* \\
&\subseteq (\Sigma ASA]^*.
\end{aligned}
$$

Therefore, by Lemma 3.2 S is a regular posemiring.

□

Theorem 3.13. *Let S be a duo-posemiring then the following conditions are equivalent:*

(i) *S is regular,*

(ii) $(\Sigma L_1 L_2]^* = L_1 \cap L_2$ *and* $(\Sigma R_1 R_2]^* = R_1 \cap R_2$ *for every two left ordered ideals* L_1,L_2 *and right ordered ideals* R_1,R_2 *of* S*,*

(iii) $(\Sigma RL]^* = R \cap L = (\Sigma LR]^*$*, for every right ordered ideal* R *and left ordered ideal* L *of* S*.*

Proof. By considering Theorem 3.7, it is obvious. □

Definition 3.14. An element a of a posemiring S is said to be *left regular(right regular)* if $a \leq xa^2 (a \leq a^2 x)$ for some $x \in S$. A posemiring S is said to be *left regular(right regular)* if every element $a \in S$ is left regular(right regular).

Example 3.15. Let $S = \{a, b, c, d, e, f\}$. We define binary operations $+$ and $.$ on S by the following equations:

$+$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	b	b	b	b
c	c	c	c	c	c	c
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f

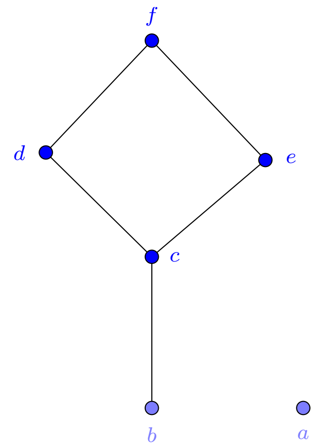
$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	b	b	b	b
d	a	b	b	d	b	d
e	a	e	e	e	e	e
f	a	e	e	f	e	f

then $(S, +, .)$ is a semiring with an absorbing zero a. We define a binary ordering relation on S by:

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (b,c), (b,e), (b,f), (c,d), (c,e), (c,f), (d,f), (e,f)\},$$

the covering relation and the figure of S is given by:

$$\prec := \{(b,c), (c,d), (c,e), (d,f), (e,f)\}$$



now, $(S, +, ., \leq)$ is a posemiring. Clearly, a, b, d, e and f are left regular. We consider $c \leq ec^2 = eb = e$ which implies that S is left regular. S is not regular, since there is not exist $x \in S$ such that $c \leq cxc$.

Example 3.16. Consider the posemiring $S = (\mathbb{N} \cup \{0\}, max, min, \leq)$ where $\leq$ is the natural order relation on numbers. Since, $n \leq min\{n, n, n\}$ for every $n \in \mathbb{N}$, then S is a regular, left regular and right regular posemiring.

The following lemmas are obvious:

Lemma 3.17. *Let S be a posemiring then the following statements are equivalent:*

(i) *S is left regular,*

(ii) *$A \subseteq (\Sigma SA^2]^*$ for each $A \subseteq S$,*

(iii) *$a \in (Sa^2]^*$ for each $a \in S$.*

Lemma 3.18. *Let S be a posemiring then the following statements are equivalent:*

(i) *S is right regular,*

(ii) *$A \subseteq (\Sigma A^2 S]^*$ for each $A \subseteq S$,*

(iii) *$a \in (a^2 S]^*$ for each $a \in S$.*

Definition 3.19. Let T be a non-empty subset of a posemiring S then T is said to be *semiprime* if for every $a \in S$, $a^2 \in T$ implies that $a \in T$.

We note that a non-empty subset T of a posemiring S is semiprime if and only if for every $\emptyset \neq A \subseteq S$, $A^2 \subseteq T$ implies that $A \subseteq T$. Considering that T is semiprime, $\emptyset \neq A^2 \subseteq T$ and $a \in A$, then $a^2 \in T$ and so $a \in T$ hence, $A \subseteq T$.

Theorem 3.20. *Let S be a posemiring then S is left regular and right regular if and only if every ordered high- quasi- ideal of S is semiprime.*

Proof. Suppose Q be an ordered high- quasi- ideal of S. Let A be a non-empty subset of S such that $A^2 \subseteq Q$. Since S is a left regular and right regular, by Lemmas 3.17 and 3.18 we have $A \subseteq (\Sigma SA^2]^*$ and $A \subseteq (\Sigma A^2S]^*$, respectively. Hence, $A \subseteq (\Sigma SA^2]^* \cap (\Sigma A^2S]^* \subseteq (\Sigma SQ]^* \cap (\Sigma QS]^* \subseteq Q$. Therefore, Q is semiprime. Conversely, assume that every ordered high- quasi- ideal Q of S is semiprime. Let $A \subseteq S$. Considering Lemma 2.2, we have, $Q^*(A^2) = (\Sigma A^2 + ((\Sigma SA^2]^* \cap (\Sigma A^2S]^*)]^*$. Since $A^2 \subseteq Q^*(A^2)$ and $Q^*(A^2)$ is semiprime so $A \subseteq Q^*(A^2) = (\Sigma A^2 + ((\Sigma SA^2]^* \cap (\Sigma A^2S]^*)]^*$ that implies

$$\begin{aligned}
&A \subseteq (\Sigma A^2 + ((\Sigma SA^2]^* \cap (\Sigma A^2S]^*)]^*\\
&\subseteq (\Sigma A^2 + (\Sigma SA^2]^*]^*\\
&\subseteq (\Sigma A^2 + \Sigma SA^2]^*.
\end{aligned}$$

So,

$$\begin{aligned}
&\Sigma A^2 \subseteq \Sigma A(\Sigma A^2 + \Sigma SA^2]^*\\
&\subseteq \Sigma(A\Sigma A^2 + A\Sigma SA^2]^*\\
&\subseteq \Sigma(\Sigma A^3 + \Sigma ASA^2]^*\\
&\subseteq \Sigma(\Sigma A^3 + \Sigma SA^2]^*\\
&\subseteq (\Sigma(\Sigma A^3) + \Sigma(\Sigma SA^2))^*\\
&\subseteq (\Sigma A^3 + \Sigma SA^2]^*\\
&\subseteq (\Sigma SA^2 + \Sigma SA^2]^*\\
&= (\Sigma SA^2]^*
\end{aligned}$$

and so $A \subseteq (\Sigma A^2 + \Sigma SA^2]^* \subseteq ((\Sigma SA^2]^* + \Sigma SA^2]^* \subseteq (\Sigma SA^2]^*$. Therefore, by Lemma 3.17, S is left regular. Also,

$$\begin{aligned}
&A \subseteq (\Sigma A^2 + ((\Sigma SA^2]^* \cap (\Sigma A^2S]^*)]^*\\
&\subseteq (\Sigma A^2 + (\Sigma A^2S]^*]^*\\
&\subseteq (\Sigma A^2 + \Sigma A^2S]^*.
\end{aligned}$$

So,

$$
\begin{aligned}
&\Sigma A^2 \subseteq \Sigma(\Sigma A^2 + \Sigma A^2 S]^* A \\
&\subseteq \Sigma(\Sigma A^2 A + (\Sigma A^2 S)A]^* \\
&\subseteq \Sigma(\Sigma A^3 + \Sigma A^2 SA]^* \\
&\subseteq \Sigma(\Sigma A^3 + \Sigma A^2 S]^* \\
&\subseteq (\Sigma(\Sigma A^3) + \Sigma(\Sigma A^2 S))]^* \\
&\subseteq (\Sigma A^3 + \Sigma A^2 S]^* \\
&\subseteq (\Sigma A^2 S + \Sigma A^2 S]^* \\
&= (\Sigma A^2 S]^*
\end{aligned}
$$

and so $A \subseteq (\Sigma A^2 + \Sigma A^2 S]^* \subseteq ((\Sigma A^2 S]^* + \Sigma A^2 S]^* \subseteq (\Sigma A^2 S]^*$.

Therefore, by Lemma 3.18, S is right regular.

□

References

[1] R. Chinram, A note on quasi-ideals in Γ-semirings, International Mathematical Forum. 3, **25** (2008), 1253-1259.

[2] R. Chinram, A note on quasi-ideals in regular Γ-semirings, International Journal of Contemporary Mathematical Sciences. 3, **35** (2008), 1725-1731.

[3] C. Donges, On quasi-ideals of semirings, International Journal of Mathematics and Mathematical Sciences. 17, **1** (1994), 47-58.

[4] A. P. Gan and Y. L. Jiang, On ordered ideals in ordered semirings, Journal of Mathematical Research and Exposition. 31, **6** (2011), 986-996.

[5] K. Iseki, Quasi-ideals in semirings without zero, Proceedings of the Japan Academy. 34, (1958), 79-81.

[6] R. D. Jagatap and Y. S. Pawar, Quasi-ideals and minimal quasi-ideals in Γ-semirings, Novi Sad Journal of Mathematics. 39, **2** (2009), 79-87.

[7] N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum 46, **3** (1993), 271-278.

[8] S. K. Lee and D. M. Lee, On intra-regular ordered semigroups, Kangweon-Kyungki Math. Jour. 14, **1** (2006), 95-100.

[9] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy Information and Engineering. 6, **1** (2014), 101-114.

[10] M. Munir and M. Habib, Characterizing semirings using their quasi and bi-ideals, Proceedings of the Pakistan Academy of Sciences. 53, **4** (2016), 469-475.

[11] M. Munir and A. Shafiq, A generalization of bi ideals in semirings, Bull. Int. Math. Virtual Inst. 8, **1** (2018), 123-133.

[12] M. Shabir, A. Ali and S. Batool, A note on quasi-ideals in semirings, Southeast Asian Bulletin of Mathematics. 27, **5** (2004), 923-928.

[13] O. Steinfeld, On quotients and prime ideals, Acta Math Acad. Sci. Hung. 4, (1953), 289-298.

[14] O. Steinfeld, Quasi-ideals in rings and semigroups, Akademiaikiado, Budapest, (1978).

[15] O. Steinfeld, Uber die quasi-ideals von halbgruppen, Publications Mathematicae Debrecen. 4, (1956), 262-275.