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A mathematical perspective on graph operations and modularity properties

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Abstract. Modularity is an important concept in social network analysis and plays a crucial role in understanding the structure of networks. One effective approach to studying the context graph of a network is to analyze the mathematical properties of graph structure and graph operations. In this study, we investigate the mathematical properties of modularity measures in graphs and focus on the interplay between modularity and graph operations. On this basis, we present formulas for the modularity of these graph operations. This paper can contribute to the current discourse on modularity and graphs based on graph structure.

Keywords: Modularity measure, Social networks, Graph operation.

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1. Introduction

Graph theory is a fundamental area of mathematics with applications in various fields such as computer science, operations research, social

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network analysis, and more. Understanding graphs and their properties enables the analysis and modeling of complex relationships and systems. A graph G consists in a pair $G = (V, E)$ of two sets together with a map $i : E \to V \times V$assigning to every $e \in E$ a pair (u, v) of elements of V. Elements of V are called vertices, elements of E are called edges. If $i(e) = (u, v)$, the vertices u and v are also called the endpoints of e [15]. Graphs provide a powerful framework for solving problems related to connectivity, networks, optimization, and more [1].

A graph can be used to explain the network because it can show the relationships between vertices at a suitable abstract level as a mathematical structure [2].

Social networks are structures composed of individuals (or entities) connected by various types of relationships, such as friendships, professional collabora tions, family ties, etc. These networks can be represented as graphs where vertices represent entities and edges represent relationships between pairs of entities. Analyzing social networks involves studying the patterns of connections and interactions between individuals to gain insights into social structures, behavior, information flow, and influence propagation. Social network analysis provides valuable insights into the relationships, structures, and dynamics of social systems, offering a powerful tool for studying complex interactions and behaviors in various contexts [3].

Let $G = (V, E)$ be a graph, and the set vertex is divided into the communities $C_1, C_2, ..., C_k$ so that C_i represents the community to which the vertex v_i belongs for $1 \leq i \leq k$. The modularity measure of G is defined as where Q is the modularity measure, $A = [A_{i,j}]$ is the edge weight between vertices v_i and v_j, k_i and k_j are the degrees of vertices v_i and v_j respectively, m is the total edge weight in the network, C_i and C_j are the community assignments of vertices v_i and v_j respectively, and $\delta(C_i, C_j)$ is the Kronecker delta function which equals 1 if $C_i = C_j$ and 0 otherwise [4].

$$Q = \frac{1}{2m} \sum_{i,j} (A_{i,j} - \frac{k_i k_j}{2m}) \delta(C_i, C_j). \tag{1.1}$$

Modularity score is a specific numerical value calculated based on the concept of modularity. The modularity score [13] of a community C

$$Q(C) = \sum_{\substack{v_i, v_j \in C \\ i \neq j}} (A_{i,j} - \frac{k_i k_j}{2m}). \tag{1.2}$$

The cumulative modularity score [13] of a partition of the network into p communities is the sum of the modularity scores of the individual communities, defined as:

$$Q = \sum_{l=1}^{p} \sum_{v_i, v_j \in c_l, i \neq j} (A_{i,j} - \frac{k_i k_j}{2m}). \tag{1.3}$$

In graph theory, there are several operations and transformations that are commonly employed to analyze and modifying graphs. These operations can be used in various applications of graph theory, such as network analysis, social network analysis, and computational biology, among others [5]. Product graphs are utilized in various computer science fields and applications such as distributed computing, network protocols, parallel computing, and communication networks. Understanding the structure and properties of product graphs can aid in solving combinatorial optimization problems efficiently [6]. On this basis, we use the graph operations tool to study the modularity behavior in the network.

2. Modularity of Networks

Newman's modularity is a concept in network theory that measures the strength of division of a network into modules or communities by comparing the number of edges inside modules with the expected number of edges in a random network with the same vertices and degree distribution. This concept helps analyze the structure of complex networks and understand the relationships between vertices in a network [7]. In 2004 [8], Newman introduced a quantitative measure called modularity to measure the quality of community detection algorithms. Modularity is a measure that helps to detect the strength of division of a network into modules or communities [9]. The main idea of defining modularity is that the edges within a community are more than the expected edges in that community [9]. In essence, modularity measures the quality of a division of a network into communities by comparing the number of edges within communities to the expected number of such edges [10]. Modularity is typically calculated using the formula (eq1). The goal is to find a partition that maximizes the modularity value, as high modularity suggests a well-defined community structure in the graph [11]. In the mathematical field of graph theory, graph operations are actions or transformations that can be applied to graphs to create new graphs or modify existing ones. Some common graph operations include union, intersection, products, composition and so on. In section (3), the modularity criteria of some graph operations are investigated. Products of graphs provide a structured way to create new graphs from existing ones by combining their vertex sets and edge sets. This construction can be useful in designing and analyzing complex networks or systems.

3. Main result

In this section, we will examine the modularity of some specific graphs and calculate the modularity value of graph operations.

Example 3.1. (1) $Q(P_n) = \frac{2m-1}{2m}$,
(2) $Q(K_n) = \frac{n-1}{2}$,
(3) $Q(\bar{K_n}) = 0$.

The following results for paths and completes on n vertices and m edges follow easily by direct calculations.

The $G - v$, removing the vertex (vertices) from the graph G and removing all incident edges with the removed vertex. The $G-e$, removing the edge only with it's ends [14].

Theorem 3.2. *Let G be a simple graph and $v \in V(G)$ and $e = uv \in E(G)$. The modularity value of $G - v$ and $G - e$ is equals to*

$$Q_{G-v} = Q_G - \frac{k_v^2}{2m}, \tag{3.1}$$

and

$$Q_{G-e} = Q_G - \frac{k_u k_v}{2m}, \tag{3.2}$$

respectively.

The union of two graphs G and H is a graph G, written by $G \cup H$ with vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. The intersection of two graphs G and H is a graph G, written by $G \cap H$ with vertex set $V(G) \cap V(H)$ and the edge set $E(G) \cap E(H)$. This fundamental concepts has various applications in studying network structures, connectivity, and relationships between different entities represented by vertices and edges in graphs [14].

Theorem 3.3. *Let G and H be two graphs. Then*

- $Q(G \cup H) \geq (Q(G), Q(H))$.
- $Q(G \cap H) \leq (Q(G), Q(H))$.

Proof.

- Let G and H be two graphs, according to the definition of union of two graphs, the number of vertices is equal to the union of vertices of two graphs and the number of edges is equal to the union of edges of two graphs, so the degree of vertices becomes larger in the union state and according to the definition of modularity, the value of modularity increases.
- Let G and H be two graphs, according to the definition of intersection two graphs, the number of vertices is equal to the intersection of the vertices of the two graphs and the number of edges is equal

to the intersection of edges from the two graphs. Therefore, the degree of the vertices in the intersection mode becomes smaller and according to the definition of modularity, the value of modularity decreases.

□

The Cartesian product $G \times H$ of two graphs G and H is a graph that has a vertex set $V(G) \times V(H)$, where each vertex is a pair (u, v) with u in $V(G)$ and v in $V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if either $u_1 = u_2$ and v_1v_2 is an edge in H, or u_1u_2 is an edge in G and $v_1 = v_2$. This product provides a mathematical tool for modeling and analyzing various systems in different fields of computer science and mathematics. It is obviously that the Cartesian product $G \times H$ will have $n_1 \times n_2$ vertices and $n_1m_2 + n_2m_1$ edges. Additionally, the degree of a vertex in the Cartesian product graph is the collected of the degrees of the corresponding vertices in the original graphs [12].

Theorem 3.4. *Let G and H be two simple graphs of oredr n_1 and n_2 also with the number of edges respectively m_1 and m_2. The modularity value of $Q(G \times H)$ is equals to*

$$Q(G \times H) = \frac{1}{n_1m_2 + m_1n_2}(n_1m_2Q(G) + n_2m_1Q(H) + 2m_1m_2).$$

Proof. $Q(G \times H) = \sum\limits_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (((A \otimes I_{n_2}) + (I_{n_1} \otimes A')) - \frac{k_{(i,i')}k_{(j,j')}}{2m})$.

According to the article [12] the adjacency matrix of the product $G \times H$ of order $n \times m$ is equal to $((A \otimes I_{n_2}) + (I_{n_1} \otimes A'))$ and the degree of a vertex in the Cartesian product graph is the collected of the degrees of the corresponding vertices in graphs G and H and the edges is equal to $m_2n_1 + n_1m_2$. Therefore, as much $m_2n_1 + n_1m_2$ positive score as $n(n-1)/2 - (m_2n_1 + n_1m_2)$ negative score is assigned to the modularity of Cartesian product.

$$Q = \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (((A \otimes I_{n_2}) + (I_{n_1} \otimes A')) - \frac{(k_i + k_{i'})(k_j + k_{j'})}{2m})$$

$$= \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (((A \otimes I_{n_2}) + (I_{n_1} \otimes A')) - \frac{(k_ik_j + k_ik_{j'} + k_{i'}k_j + k_{i'}k_{j'})}{2(m_2n_1 + n_1m_2)})$$

$$= \frac{1}{n_1m_2 + m_1n_2}(n_1m_2(\sum_{\substack{i,j \\ i\neq j}} A - \frac{k_ik_j}{2m_1}) + n_2m_1(\sum_{\substack{i',j' \\ i'\neq j'}} A' - \frac{k_{i'}k_{j'}}{2m_2}) + 2m_1m_2).$$

□

The Tensor product $G \otimes H$ of two graphs G and H is a graph that has a vertex set $V(G) \times V(H)$, where each vertex is a pair (u, v) with u in $V(G)$ and v in $V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if either u_1u_2 is an edge in G and v_1v_2 is an edge in H. This product provides a mathematical tool for modeling and analyzing various systems in different fields of computer science and mathematics. It is obviously that the Tensor product $G \otimes H$ will have $n_1 \times n_2$ vertices and $2m_1m_2$ edges. Additionally, the degree of a vertex in the Tensor product graph is the product of the degrees of the corresponding vertices in the original graphs [12].

Theorem 3.5. *Let G and H be two simple graphs of oredr n_1 and n_2 also with the number of edges respectively m_1 and m_2. The modularity value of $Q(G \otimes H)$ is equals to*

$$Q(G \otimes H) = 2Q(G)Q(H).$$

Proof. $Q(G \otimes H) = \sum\limits_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A \otimes A' - \frac{k_{(i,i')}k_{(j,j')}}{2m})$.

According to the article [12] on the adjacency matrix of the product $G \otimes H$ of order $n \times m$ is equal to $A \otimes A'$ and the degree of each vertex of the tensor product is equal to the product of the degrees of the vertices of the graphs G and H and the edges is equal to $2m_1m_2$.

Therefore, as much $2m_1m_2$ positive score as $n(n-1)/2 - 2m_1m_2$ negative score is assigned to the modularity of tensor multiplication.

$$\text{Q}= \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A \otimes A' - \frac{k_ik_{i'}k_jk_{j'}}{4m_1m_2}) = 2(\sum_{\substack{i,j \\ i\neq j}} A - \frac{k_ik_j}{2m_1})(\sum_{\substack{i',j' \\ i'\neq j'}} A' - \frac{k_{i'}k_{j'}}{2m_2}).$$ □

The Symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G \oplus H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \oplus H$ if and only if either u_1u_2 is an edge in G or v_1v_2 is an edge in H but not both. The number of edges of this graph operation is equals to $m_1n_2^2 + m_2n_1^2 - 4m_1m_2$ and the degree of arbitrary vertex (u, v) of $G \oplus H$ is equals to $n_2k_u + n_1k_v - 2k_uk_v$ [12].

Theorem 3.6. *Let G and H be two simple graphs of oredr n_1 and n_2 also with the number of edges respectively m_1 and m_2. The modularity* *value of $Q(G \oplus H)$ is equals to*

$$Q(G \oplus H) = \frac{1}{(m_1n_2^2 + m_2n_1^2 - 4m_1m_2)}((m_1n_1n_2^2 - 8m_1m_2n_2)Q(G) + (m_2n_2n_1^2 - 8m_1n_1m_2)Q(H) + 16m_1m_2Q(G)Q(H) + 2m_1m_2n_1n_2).$$

Proof. $Q(G \oplus H) = \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} ((A \oplus A' - \frac{k_{(i,i')}k_{(j,j')}}{2m})$.

According to the article [12] on the adjacency matrix of the product $G \oplus H$ of order $n \times m$ is equal to $A \oplus A'$ and the degree of arbitrary vertex (u, v) of $G \oplus H$ is equals to $n_2k_u + n_1k_v - 2k_uk_v$ and the edges is equal to $m_1n_2^2 + m_2n_1^2 - 4m_1m_2$.
Therefore, as much $m_1n_2^2+m_2n_1^2-4m_1m_2$ positive score as $n(n-1)/2-(m_1n_2^2+m_2n_1^2-4m_1m_2)$ negative score is assigned to the modularity of Symmetric difference.

$$\begin{aligned}
Q &= \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A \oplus A' - \tfrac{(n_2k_i+n_1k_{i'}-2k_ik_{i'})(n_2k_j+n_1k_{j'}-2k_jk_{j'})}{m_1n_2^2+m_2n_1^2-4m_1m_2}) \\
&= \tfrac{1}{(m_1n_2^2+m_2n_1^2-4m_1m_2)}((m_1n_1n_2^2 - 8m_1m_2n_2)(\sum_{\substack{i,j \\ i\neq j}} A - \tfrac{k_ik_j}{2m_1}) \\
&\quad +(m_2n_2n_1^2 - 8m_1n_1m_2)(\sum_{\substack{i',j' \\ i'\neq j'}} A' - \tfrac{k_{i'}k_{j'}}{2m_2}) + 2m_1m_2n_1n_2 \\
&\quad +16m_1m_2(\sum_{\substack{i,j \\ i\neq j}} A - \tfrac{k_ik_j}{2m_1})(\sum_{\substack{i',j' \\ i'\neq j'}} (A' - \tfrac{k_{i'}k_{j'}}{2m_2}).
\end{aligned}$$

□

The $G[H]$ of two graphs G and H is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G[H]$ if and only if either u_1u_2 is an edge in G or v_1v_2 is an edge in H and $u_1 = u_2$. The number of edges of this graph operation is equals to $n_2^2m_1 + m_2n_1$ and the degree of arbitrary vertex (u, v) of $G[H]$ is equals to $n_2k_u + k_v$ [12].

Theorem 3.7. *Let G and H be two simple graphs of oredr n_1 and n_2 also with the number of edges respectively m_1 and m_2. The modularity value of $Q(G[H])$ is equals to*

$$Q(G[H]) = \frac{1}{(n_2^2m_1 + m_2n_1)}(n_2^3m_1Q(G) + n_1m_2Q(H) + 2n_2m_1m_2).$$

Proof. $Q(G[H]) = \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A[A'] - \frac{k_{(i,i')}k_{(j,j')}}{2m})$.

According [12] the adjacency matrix of the product $G[H]$ of order $n \times m$ is equal to $A[A']$ and the degree of each vertex of the product is equal to the degree of arbitrary vertex (u, v) of $G[H]$ is equals to $n_2k_u + k_v$ and the edges is equal to $n_2^2 m_1 + m_2n_1$.

Therefore, as much $n_2^2 m_1 + m_2n_1$ positive score as $n(n-1)/2 - (n_2^2 m_1 + m_2n_1)$ negative score is assigned to the modularity of tensor multiplication.

$$Q = \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A[A']) - \frac{(n_2k_i + k_{i'})(n_1k_j + k_{j'})}{2(n_2^2 m_1 + m_2n_1)}$$

$$= \frac{1}{(n_2^2 m_1 + m_2n_1)} (n_2^3 m_1 (\sum_{\substack{i,j \\ i\neq j}} A - \frac{k_ik_j}{2m_1}) + n_1m_2(\sum_{\substack{i',j' \\ i'\neq j'}} A' - \frac{k_{i'}k_{j'}}{2m_2}) + 2n_2m_1m_2).$$

□

Theorem 3.8. *Let G and H be two graphs of order n_1 and n_2 then*

$$Q(G \otimes H) \leq \frac{(n_1 - 1)(n_2 - 1)}{2}, \tag{3.3}$$

equality holds if and only if G and H are isomorphic to K_n.

Proof.

$$\begin{aligned} Q(G \otimes H) &= \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A_{(i,i')(j,j')} - \frac{k_{(i,i')}k_{(j,j')}}{4m_1m_2}) \\ &= \sum_{v_iv_j \in E(G) \text{ and } v_{i'}v_{j'} \in E(H)} (1 - \frac{k_ik_{i'}k_jk_{j'}}{4m_1m_2}) \\ &+ \sum_{v_iv_j \notin E(G) \text{ and } v_{i'}v_{j'} \in E(H)} (0 - \frac{k_ik_{i'}k_jk_{j'}}{4m_1m_2}) \\ &+ \sum_{v_iv_j \in E(G) \text{ and } v_{i'}v_{j'} \notin E(H)} (0 - \frac{k_ik_{i'}k_jk_{j'}}{4m_1m_2}) \\ &+ \sum_{v_iv_j \notin E(G) \text{ or } v_{i'}v_{j'} \notin E(H)} (0 - \frac{k_ik_{i'}k_jk_{j'}}{4m_1m_2}) \\ &\leq 2(\frac{n_1 - 1}{2}\frac{n_2 - 1}{2}) = \frac{(n_1 - 1)(n_2 - 1)}{2}. \end{aligned}$$

□

If $n_1 = n_2$ *then*

$$Q(G \otimes H) \leq \frac{(n-1)^2}{2}. \tag{3.4}$$

Theorem 3.9. *Let* G *and* H *be two graphs of order* n_1 *and* n_2 *then*

$$Q(G \times H) \leq \frac{1}{(n_1-1)+(n_2-1)}((n_1-1)^2+(n_2-1)^2+((n_1-1)(n_2-1))), \tag{3.5}$$

equality holds if and only if G *and* H *are isomorphic to* K_n*.*

Proof.

$$\begin{aligned} Q(G \times H) &= \sum_{\substack{(i,i')(j,j') \\ i \neq j, i' \neq j'}} (A_{(i,i')(j,j')} - \frac{k_{(i,i')}k_{(j,j')}}{2m_1n_2 + 2n_1m_2}) \\ &= \sum_{v_iv_j \in E(G) \text{ and } v_{i'} = v_{j'}} (1 - \frac{(k_i + k_{i'})(k_j + k_{j'})}{2m_1n_2 + 2n_1m_2}) \\ &+ \sum_{v_i = v_j \text{ and } v_{i'}v_{j'} \in E(H)} (1 - \frac{(k_i + k_{i'})(k_j + k_{j'})}{2m_1n_2 + 2n_1m_2}) \\ &+ \sum_{v_i \neq v_j \text{ or } v_{i'}v_{j'} \notin E(H)} (0 - \frac{(k_ik_j + k_ik_{j'} + k_{i'}k_j + k_{i'}k_{j'})}{2m_1n_2 + 2n_1m_2}) \\ &+ \sum_{v_iv_j \notin E(G) \text{ or } v_{i'} \neq v_{j'}} (0 - \frac{(k_ik_j + k_ik_{j'} + k_{i'}k_j + k_{i'}k_{j'})}{2m_1n_2 + 2n_1m_2}) \\ &\leq \frac{1}{(n_1-1)+(n_2-1)}((n_1-1)^2 + (n_2-1)^2 + ((n_1-1)(n_2-1))). \end{aligned}$$

□

If $n_1 = n_2$ *then*

$$Q(G \times H) \leq \frac{3(n-1)}{2}. \tag{3.6}$$

Theorem 3.10. *Let* G *and* H *be two graphs of order* n_1 *and* n_2 *then*

$$Q(G[H]) \leq \frac{n_1n_2 - 1}{2}. \tag{3.7}$$

Proof.

$$
\begin{aligned}
Q(G[H]) &= \sum_{\substack{(i,i')(j,j')\\ i\neq j, i'\neq j'}} \Big(A_{(i,i')(j,j')} - \frac{(nk_i + k_{i'})(nk_j + k_{j'})}{2(n^2 m_1 + m_2 n)}\Big) \\
&= \sum_{v_i v_j \in E(G)} \Big(1 - \frac{(nk_i + k_{i'})(nk_j + k_{j'})}{2(n^2 m_1 + m_2 n)}\Big) \\
&+ \sum_{v_i = v_j \ \ and \ \ v_{i'} v_{j'} \in E(H)} \Big(1 - \frac{(nk_i + k_{i'})(nk_j + k_{j'})}{2(n^2 m_1 + m_2 n)}\Big) \\
&+ \sum_{v_i v_j \notin E(G) \ \ or \ \ v_i \neq v_j} \Big(0 - \frac{(nk_i + k_{i'})(nk_j + k_{j'})}{2(n^2 m_1 + m_2 n)}\Big) \\
&+ \sum_{v_{i'} v_{j'} \notin E(H)} \Big(0 - \frac{(nk_i + k_{i'})(nk_j + k_{j'})}{2(n^2 m_1 + m_2 n)}\Big) \\
&\leq \frac{1}{2(n_2(n_1 - 1) + (n_2 - 1))}((n_2(n_1 - 1)) + (n_2 - 1))^2 = \frac{n_1 n_2 - 1}{2}.
\end{aligned}
$$

□

If $n_1 = n_2$ then

$$Q(G[H]) \leq \frac{n^2 - 1}{2}. \tag{3.8}$$

Theorem 3.11. *Let G and H be two graphs of order n then*

$$Q(G \oplus H) \geq n - 1. \tag{3.9}$$

Proof.

$$
\begin{aligned}
Q(G \oplus H) &= \sum_{\substack{(i,i')(j,j') \\ i\neq j, i'\neq j'}} (A_{(i,i')(j,j')} - \frac{(nk_i + nk_{i'} - 2k_ik_{i'})(nk_j + nk_{j'} - 2k_jk_{j'})}{2(n^2m_1 + n^2m_2 - 4m_1m_2)}) \\
&= \sum_{v_iv_j \in E(G)} (1 - \frac{(nk_i + nk_{i'} - 2k_ik_{i'})(nk_j + nk_{j'} - 2k_jk_{j'})}{2(n^2m_1 + n^2m_2 - 4m_1m_2)} \\
&+ \sum_{v_{i'}v_{j'} \in E(H)} (1 - \frac{(nk_i + nk_{i'} - 2k_ik_{i'})(nk_j + nk_{j'} - 2k_jk_{j'})}{2(n^2m_1 + n^2m_2 - 4m_1m_2)}) \\
&+ \sum_{v_iv_j \in E(G) \ and \ v_{i'}v_{j'} \in E(H)} (0 - \frac{(nk_i + nk_{i'} - 2k_ik_{i'})(nk_j + nk_{j'} - 2k_jk_{j'})}{2(n^2m_1 + n^2m_2 - 4m_1m_2)} \\
&+ \sum_{v_iv_j \notin E(G) \ and \ v_{i'}v_{j'} \notin E(H)} (0 - \frac{(nk_i + nk_{i'} - 2k_ik_{i'})(nk_j + nk_{j'} - 2k_jk_{j'})}{2(n^2m_1 + n^2m_2 - 4m_1m_2)} \\
&\geq n - 1.
\end{aligned}
$$

□

Corollary 3.12. *Let G be a graph of order n and $H = K_2$, then the modularity of the Tensor product is equal to*

$$Q(G \otimes H) = Q(G).$$

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