Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **3**(1)(2014),123-130

A modification of Chebyshev-Halley method free from second derivatives for nonlinear equations

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Abstract. In this paper, we present a new modification of Chebyshev-Halley method, free from second derivatives, to solve nonlinear equations. The convergence analysis shows that our modification is third-order convergent. Every iteration of this method requires one function and two first derivative evaluations. So, its efficiency index is $3^{1/3} = 1.442$ that is better than that of Newton method. Several numerical examples are given to illustrate the performance of the presented method.

Keywords: Chebyshev-Halley method; Newton method; Nonlinear equations; Third-order convergence.

2000 Mathematics subject classification: 65H05, 49M15, 34A34.

1. INTRODUCTION

Solving nonlinear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : I \mapsto R$, for an open interval I, is a scalar function. Newton method is undoubtedly the most famous iterative method to find α by using the scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.1}$$

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Received: 15 April 2013
Revised: 6 November 2013
Accepted: 15 November 2013

that converges quadratically in some neighborhood of α [1].

The classical Chebyshev-Halley method [2] which improves Newton method is given by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\,\frac{L_f(x_n)}{1-\beta L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}\,, \tag{1.2}$$

in which

$$L_f(x_n) = \frac{f(x_n)f''(x_n)}{f'^2(x_n)}\,, \tag{1.3}$$

and β is a parameter. This family is known to converge cubically, and includes, as particular cases, the classical Chebyshev method ($\beta = 0$), Halley method ($\beta = 1/2$) and Supper-Halley method ($\beta = 1$). It is observed that the methods depend on the second derivatives in computing process, making its practical utility rigorously restricted, so that the Newton method is frequently used as an alternative in solving nonlinear equations.

To remove the second derivative from (1.3), recently, some variants of Chebyshev-Halley method have been obtained [3,4,5,6].

Chun [3] considers approximating the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ by the quadratic equation in x and y in the form $x^2 + ay^2 + bx + cy + d = 0$ and imposes the tangency conditions $y(x_n) = f(x_n)$, $y'(x_n) = f'(x_n)$, $y(w_n) = f(w_n)$, where $w_n = x_n - \frac{f(x_n)}{f'(x_n)}$, to obtain

$$f''(x_n) \approx y''(x_n) = 2\left(1 + af'^2(x_n)\right)\frac{f(w_n)f'^2(x_n)}{f^2(x_n) + af'^2(x_n)[f(w_n) - f(x_n)]^2}\,.$$

Therefore, he gets the approximation of

$$L_f(x_n) \approx \frac{2f(x_n)f(w_n)(1 + af'^2(x_n))}{f^2(x_n) + af'^2(x_n)[f(w_n) - f(x_n)]^2}\,, \tag{1.4}$$

in which a is a parameter.

Xiaojian [4], to remove the $f''(x_n)$ in (1.3), approximated the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ by the hyperbola form $axy + y + bx + c = 0$ and imposed the tangency conditions $y(x_n) = f(x_n)$, $y'(x_n) = f'(x_n)$ and $y(w_n) = f(w_n)$, where $w_n = x_n - \frac{f(x_n)}{f'(x_n)}$. So, he gets the approximation of

$$f''(x_n) \approx y''(x_n) = \frac{2f'^2(x_n)f(w_n)}{f^2(x_n) - f(x_n)f(w_n)},$$

that results in

$$L_f(x_n) \approx \frac{2f(w_n)}{f(x_n) - f(w_n)}\,. \tag{1.5}$$

Chun [5], to derive an approximation of $f''(x_n)$ in (1.3), considers the approximation $f(x) \approx g(x) := ax^3+bx^2+cx+d$, satisfying the conditions $f'(x_n) = g'(x_n)$ and $f'(w_n) = g'(w_n)$, in which $w_n = x_n - \frac{f(x_n)}{f'(x_n)}$. He then derived the approximation of

$$f''(x_n) \approx g''(x_n) = \frac{f'(w_n) - f'(x_n)}{w_n - x_n} - \lambda(w_n - x_n), \quad \lambda = 3a$$

that results in

$$L_f(x_n) \approx 1 - \frac{f'(w_n)}{f'(x_n)} + \lambda \frac{f^2(x_n)}{f'^3(x_n)}\,. \tag{1.6}$$

Jisheng, Yitian and Xiuhua [6] used Taylor expansion of $f(y_n)$ about x_n, in which $y_n = x_n - \theta\frac{f(x_n)}{f'(x_n)}$ and θ is a nonzero real parameter, and get the following approximation of

$$L_f(x_n) \approx \frac{f(y_n) + (\theta - 1)f(x_n)}{\theta^2 f(x_n)}\,. \tag{1.7}$$

It is proved that all of the above approximations of $L_f(x_n)$ applied to the Chebyshev-Halley method (1.2) lead to a cubically convergent method.

In this paper, using a new observation, we will approximate $L_f(x_n)$ by a finite difference between first derivatives. So, a new modification of Chebyshev-Halley method, free from second derivatives, is obtained. It is proved that our new modification has third-order convergence. Some examples are given to show the efficiency and superiority of the new method.

2. A NEW APPROXIMATION OF $L_f(x_n)$

To obtain a new modification for Chebyshev-Halley method, free from second derivatives, we will approximate $L_f(x_n)$ using first derivatives. To this end, notice that

$$\frac{f''(x)}{f'^2(x)} = -\left(\frac{1}{f'(x)}\right)'.$$

Using the well-known finite difference approximation $F'(x_n) \approx [F(x_n + \delta_n) - F(x_n)]/\delta_n$, in which $F(x) = 1/f'(x)$ and $\delta_n = \gamma f(x_n)$, we have

$$\left(\frac{1}{f'(x)}\right)' \approx \frac{1}{\gamma f(x_n)}\left[\frac{1}{f'(x_n + \gamma f(x_n))} - \frac{1}{f'(x_n)}\right].$$

So, we get the following approximation:

$$\frac{f''(x_n)}{f'^2(x_n)} \approx \frac{1}{\gamma f(x_n)}\left[\frac{1}{f'(x_n)} - \frac{1}{f'(x_n + \gamma f(x_n))}\right],$$

where $\gamma \neq 0$ is a parameter. Considering the above approximation, we will have

$$L_f(x_n) \approx \tilde{L}_f(x_n) := \frac{1}{\gamma}\left[\frac{1}{f'(x_n)} - \frac{1}{f'(x_n+\gamma f(x_n))}\right], \qquad (2.1)$$

that results in the following modification of the Chebyshev-Halley method, free from second derivatives:

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\,\frac{\tilde{L}_f(x_n)}{1-\beta\tilde{L}_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}. \qquad (2.2)$$

In the sequel, we prove that the modified Chebyshev-Halley method (2.2) is cubically convergent for any choices of the constant parameters γ and β. To do so, we need the following facts.

Definition 2.1. [4] *Let $f(x)$ be a real function with a simple root α and $\{x_n\}_{n\geq 0}$ be a sequence of real numbers, converging towards α. Then, we say that the order of convergence of the sequence is p, if there exists a constant real number $C \neq 0$, called the asymptotic error constant, such that*

$$\lim_{n\to\infty}\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C.$$

For p =1, 2, 3 the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Definition 2.2. [4] *Let $e_n = x_n - \alpha$ be the error in the n-th iteration. We call the relation*

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}),$$

as the error equation.

If we can obtain the error equation for any iterative method, then the value of p is its order of convergence and C is the asymptotic error constant.

Definition 2.3. [4] *Let r be the number of new pieces of information required by a method. A "piece of information" is typically any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index and is defined by*

$$\rho = p^{1/r},$$

where p is the order of the method.

The following theorem is the main result of the manuscript.

Theorem 2.4. *Let $\alpha \in I$ be a simple root (that is $f'(\alpha) \neq 0$) of a sufficiently differentiable function $f : I \mapsto R$ for an open interval I.*

If x_0 is sufficiently close to α, then for any choices of the parameters $\gamma \neq 0$ and β, the modified Chebyshev-Halley method (2.2) has third-order convergence, satisfying the error equation

$$e_{n+1} = \left(2\left(1-\beta+\gamma f'(\alpha)\right)c_2^2 - \left(1+1.5\gamma f'(\alpha)\right)c_3\right)e_n^3 + O(e_n^4), \quad (2.3)$$

where $e_n = x_n - \alpha$ and $c_k = \dfrac{f^{(k)}(\alpha)}{k!f'(\alpha)}$.

Proof. Using Taylor expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right), \quad (2.4)$$

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)\right). \quad (2.5)$$

Dividing (2.4) by (2.5) gives

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \quad (2.6)$$

Using the relation (2.5), we have also

$$\frac{1}{f'(x_n)} = \frac{1}{f'(\alpha)}\left[1 - 2c_2e_n + (4c_2^2 - 3c_3)e_n^2 + (12c_2c_3 - 8c_2^3 - 4c_4)e_n^3 + O(e_n^4)\right]. \quad (2.7)$$

Take $z_n = x_n + \gamma f(x_n)$. Then,

$$z_n - \alpha = (1+\gamma f'(\alpha))\,e_n + \gamma f'(\alpha)\left(c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right)$$

and the Taylor expansion of $f'(z_n)$ about α can be read as

$$\begin{aligned} f'(z_n) &= f'(\alpha)\left(1 + 2c_2(z_n-\alpha) + 3c_3(z_n-\alpha)^2 + 4c_4(z_n-\alpha)^3 + O(e_n^4)\right) \\ &= f'(\alpha)\left(1 + 2c_2(1+\gamma f'(\alpha))e_n + ke_n^2 + me_n^3 + O(e_n^4)\right), \end{aligned}$$

in which

$$\begin{aligned} k &= 2c_2^2\gamma f'(\alpha) + 3c_3(1+\gamma f'(\alpha))^2, \\ m &= 2c_2c_3\gamma f'(\alpha)(4+3\gamma f'(\alpha)) + 4c_4(1+\gamma f'(\alpha))^3. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{f'(z_n)} = \frac{1}{f'(\alpha)}&\left[1 - 2c_2(1+\gamma f'(\alpha))e_n\right. \\ &\left.+ \left(4c_2^2(1+\gamma f'(\alpha)^2 - k\right)e_n^2 + Ne_n^3 + O(e_n^4)\right], \end{aligned} \quad (2.8)$$

where

$$N = 4kc_2(1+\gamma f'(\alpha)) - 8c_2^3(1+\gamma f'(\alpha))^3 - m.$$

Using relations (2.7) and (2.8), we obtain the following Taylor expansion of $\tilde{L}_f(x_n)$ about α:

$$\tilde{L}_f(x_n) = \frac{1}{\gamma f'(\alpha)}\left[2c_2\gamma f'(\alpha)e_n + Se_n^2 + Me_n^3 + O(e_n^4)\right], \tag{2.9}$$

in which

$$\begin{aligned} S &= 4c_2^2 - 3c_3 - 4c_2^2(1+\gamma f'(\alpha))^2 + k, \\ M &= 12c_2c_3 - 8c_2^3 - 4c_4 - N. \end{aligned}$$

Thus,

$$1-\beta\tilde{L}_f(x_n) = \frac{1}{\gamma f'(\alpha)}\left[\gamma f'(\alpha) - 2c_2\gamma\beta f'(\alpha)e_n - \beta Se_n^2 - \beta Me_n^3 + O(e_n^4)\right]. \tag{2.10}$$

Dividing (2.9) by (2.10) gives

$$\begin{aligned} \frac{\tilde{L}_f(x_n)}{1-\beta\tilde{L}_f(x_n)} = 2c_2e_n &+ \frac{1}{\gamma f'(\alpha)}\left(4c_2^2\gamma\beta f'(\alpha) + S\right)e_n^2 \\ &+ \frac{1}{\gamma f'(\alpha)}\left(4c_2\beta S + 8c_2^3\beta^2\gamma f'(\alpha) + M\right)e_n^3 + O(e_n^4). \end{aligned} \tag{2.11}$$

Therefore, using (2.6) and (2.11), we have

$$e_{n+1} = \left(2c_3 - c_2^2 - 2c_2^2\beta - \frac{1}{2\gamma f'(\alpha)}S\right)e_n^3 + O(e_n^4).$$

By substitution the value of S in this relation and application of some simplifications, we obtain

$$e_{n+1} = \left(2\left(1-\beta+\gamma f'(\alpha)\right)c_2^2 - \left(1+1.5\gamma f'(\alpha)\right)c_3\right)e_n^3 + O(e_n^4),$$

which indicates that the method defined by (2.2) is cubically convergent. □

It is obvious that each iteration of the modified Chebyshev-Halley method (2.2) requires one evaluation of the function and two of its first derivative. So, according to the Definition 2.3, this method has the efficiency index equal to $\sqrt[3]{3} \approx 1.442$, which is better than that of Newton method $\sqrt{2} \approx 1.414$. Thus, the new method is preferable if the computational cost of the first derivative is not more than that of the function itself.

3. NUMERICAL EXAMPLES

Now, we compare the methods of Newton (1.1) (N), Halley (1.2) with $\beta = 0.5$ (H), Xiaojian (1.5) (X), Chun (1.6) with $\lambda = -1$ (C), Jisheng etc. (1.7) with $\theta = -1$ (JYX), modified Halley (2.2) with $\beta = 0.5$ and $\gamma = 0.2$ (MH) obtained in this paper to solve some nonlinear equations.

All computations were done using MATLAB 6.5 with the format of long floating point arithmetics. We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criterion for computer programs: $|x_{n+1} - x_n| < \varepsilon$ or the maximum number of iterations is maxiter. So, when the stopping criterion is satisfied, $x^* := x_{n+1}$ is taken as the exactly computed root α. For numerical illustrations in this section, we used the fixed stopping criterion $\varepsilon = 10^{-15}$ and maxiter$=$ 250. The following test functions displayed the approximate zeros x^* round up to 16th decimal places.

$$
\begin{aligned}
&f_1(x) = x^3 + 4x^2 - 10, && x^* = 1.365230013414097\\
&f_2(x) = (x+2)e^x - 1, && x^* = -0.442854401002388\\
&f_3(x) = x^4 + 9x^3 + 11x^2 + 19x - 41, && x^* = 1.013772500077165\\
&f_4(x) = e^x \sin x + \ln(x^2+1), && x^* = 0\\
&f_5(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, && x^* = -1.207647827130919\\
&f_6(x) = e^{x^2+7x-30} - 1, && x^* = 3\\
&f_7(x) = \sin^2 x - x^2 + 1, && x^* = 1.4044916482153411\\
&f_8(x) = 1 - x + 2\sin x, && x^* = 2.380061273139339
\end{aligned}
$$

Numerical results are shown in Table 1. The quantities therein denote the number of iterations, and the asterisk means that the number of iterations is more than maxiter. By taking Table 1 into consideration, we note that the number of iterations of our method (2.2) is comparable with that of obtained by using the other procedures.

Table 1. Comparison of various cubically convergent methods and the Newton method

	N	H	X	C	JYX	MH
$f_1,\ x_0 = -0.1$	80	73	29	*	20	25
$f_2,\ x_0 = -1.2$	8	5	4	6	6	5
$f_3,\ x_0 = 0$	8	4	4	4	4	4
$f_4,\ x_0 = 1$	7	5	4	5	5	4
$f_5,\ x_0 = 2$	220	6	31	22	*	14
$f_6,\ x_0 = 3.3$	9	5	4	5	4	4
$f_7,\ x_0 = 0.1$	16	9	9	*	28	6
$f_8,\ x_0 = 0.1$	32	33	28	6	11	6

4. CONCLUSIONS

In this work, we presented a new modification for Chebyshev-Halley method, free from second derivatives, to solve nonlinear equations. Every iteration of the new method requires one function and two first derivative evaluations. Hence, its efficiency index is $3^{1/3} = 1.442$ that is better than that of Newton method. Order of convergence of this method is three. Numerical experiments show that our method is comparable with other third-order methods.

Acknowledgement. The authors thanks to the referee for his important suggestions which essentially improved the first version of the paper.

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