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Finite difference and local discontinuous Galerkin methods for fourth-order time-fractional partial integro-differential equation: Computational approach for one-dimensional case

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Abstract. Our focus in this paper is on numerically solving fourth-order time-fractional integro-differential equations with weakly singular kernels. L1 and quadrature formulas are used to discretize the temporal and memory terms. For spatial discretization, a high-order local discontinuous Galerkin method is employed. Finally, the numerical optimal convergence rate for the proposed scheme is demonstrated by the use of numerical results.

Keywords: L1 formula, Quadrature formula, Local discontinuous Galerkin method, Fourth-order PIDEs, Memory term.

2000 Mathematics subject classification: 65M60, 45K05.

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1. Introduction

Partial differential equations (PDEs) are not capable of capturing memory effects in the mathematical modeling of a system. To address this issue and simulate some history dependence, models have utilized fractional operators [19, 37, 38] and integral terms. Mathematicians are influenced by this and end up working with Volterra-type partial integro-differential equations (PIDEs). viscoelastic materials [22, 25, 39] are the subject of these models.

Numerous studies have been carried out on numerical solutions to these equations. McLean et al. utilized various strategies, including Crank-Nicolson and backward Euler methods [31], and a parallelizable method built upon Laplace transforms [26, 28, 29, 30]. In addition, they used the piecewise-linear finite element method to discretize space. The main goal of [10] is to investigate the numerical behavior of second-order PIDEs arising from viscoelasticity. An iteration technique was employed by Dehghan and Shakeri in their paper [12] to describe heat conduction in materials that have memory.

There are many physical concepts described by fourth-order models, such as the energy field in fluid mechanics [24], simulating thin beams [15]. Due to the non-local nature of the fractional derivative, these derivatives play a significant role in the modeling of sub-diffusion processes as a result of their non-locality [40].

The main concern of this paper is to solve the following fourth order time-fractional (TF) PIDE:

$$\begin{aligned} {}_0^C\mathrm{D}_t^{\alpha} u + \frac{\partial u}{\partial x} - \int_0^t (t-\xi)^{\beta_1 - 1}\, \Delta u \, d\xi \\ + \int_0^t (t-\xi)^{\beta_2 - 1}\, \Delta^2 u \, d\xi = f, \ (x,t) \in \Omega \times (0,T] \end{aligned} \tag{1.1}$$

subject to the initial condition $u(x,0) = u_0(x)$ and and the periodic boundary conditions (BCs), where α, β_1 and $\beta_2 \in (0,1)$, $f \in L^2(\Omega)$ and ${}_0^{\mathrm{C}}\mathrm{D}_{\mathrm{t}}^{\alpha}$ is the Caputo fractional derivative and defined as

$${}_0^{\mathrm{C}}\mathrm{D}_{\mathrm{t}}^{\alpha} w(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{w'(\xi)}{(t-\xi)^{\alpha}}\, d\xi, & 0 < \alpha < 1, \\ \dfrac{\partial w(t)}{\partial t}, & \alpha = 1. \end{cases} \tag{1.2}$$

$\Delta = \dfrac{\partial^2}{\partial x^2}$ and $\Delta^2 = \dfrac{\partial^4}{\partial x^4}$ are the Laplacian and biharmonic operators, respectively.

There are many numerical techniques available for solving a fourth-order PIDEs with a positive-type memory term. Fakhar-Izadi el al.

introduced spectral-Galerkin methods [16, 17]. Using quasi-wavelets, Yang et al. [42, 43] solved fourth-order PIDEs with a weakly singular kernel (WSK). In [35], Mohammadi et al. analyzed the stability of backward FD and local discontinuous Galerkin (LDG) method.

Discontinuous Galerkin (DG) method was introduced by Reed and Hill [41] for solving a variety of problems. Many extensions of the DG method exist, such as symmetric interior penalty DG (SIPG), non-symmetric interior penalty DG (NIPG) and local DG (LDG) methods. In [1], Abbaszadeh and Dehghan analyzed the interior penalty DG (IPDG) methods for generalized Sobolev equations. In [34], Mohammadi et al. studied and analyzed the IPDG and enriched Galerkin (EG) methods for solving TF parabolic-type equations.

Second- and higher-order PDEs can be numerically solved using the LDG method [9]. For the numerical approximation of fourth- and higher-order PDEs, the LDG method has been used and analyzed in [14]. LDG method has been used over the past decade for various integer and fractional ODEs [3, 13, 32] and PDEs [2, 18]. Baccouch investigated the superconvergence error analysis of various fourth-order PDEs [4, 5, 6, 7]. Based on variational multiscale element free Galerkin (VMEFG) and LDG methods, the author of [11] solved the Brusselator equation. Fouladi and Dahaghin, in [21] , have combined the LDG method with the locally one-dimensional strategy to solve diffusion equations. Authors of [20] recovered the numerical solution of the Riesz space distributed-order (DO) Sobolev equation. To solve the DO TF equations, the authors of [33] combined LDG method and Laplace transforms (LTM-LDG) together. In [36] the authors used LTM-LDG scheme for the fourth-order TF PIDEs with weakly singular kernels.

The rest of the paper, is organized as follows: Section 2, is dedicated to some backgrounds and preliminaries of temporal discretization, quadrature formula and spatial discretization. In Section 3, we present the semi-discrete LDG and full-discrete FD-LDG schemes. Finally some experimental results in Section 4 numerically confirm the convergence of the proposed scheme.

2. Preliminaries and backgrounds

2.1. **Time discretization: L1 formula for Caputo derivative.** Consider the time domain $I = [0, T]$ with the partition $0 = t_0 < t_1 < \cdots < t_{N-1} = t_N = T$. We define $I_n = [t_{n-1}, t_n]$, so $[0, T] \approx \mathcal{T}_h = \cup_{n=1}^{N} I_n$. We aim to discretize the TF Caputo derivative ${}_0^C\mathrm{D}_t^{\alpha} w(t_n)$ using the $L1$

formula [23] as

$$
\begin{aligned}
{}_0^C\mathrm{D}_t^\alpha w(t_{n+1}) = \frac{1}{\tau}\Bigg[b_0^{(\alpha)} w(t_{n+1}) - \sum_{s=1}^{n}\left(b_{n-s}^{(\alpha)} - b_{n+1-s}^{(\alpha)}\right) w(t_s) \\
- b_n^{(\alpha)} w(t_0)\Bigg] + \mathcal{R}_1^{n+1}(w),
\end{aligned}
\tag{2.1}
$$

in which τ is the time step size and b_s's are

$$
b_s^{(\alpha)} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\Big[(s+1)^{1-\alpha} - s^{1-\alpha}\Big], \qquad s = 0, 1, \ldots, n, \tag{2.2a}
$$

$$
|\mathcal{R}_1^n(w)| \leq C \max_{0\leq t\leq t} |w_{tt}(t)| \tau^{2-\alpha}. \tag{2.2b}
$$

So, denoting $w(t_n)$ by w^n, we define the approximation operator ${}_0^C\mathcal{D}_t^\alpha w^{n+1}$ as

$$
{}_0^C\mathcal{D}_t^\alpha w^{n+1} = \frac{1}{\tau}\left[b_0^{(\alpha)} w^{n+1} - \sum_{s=1}^{n}\left(b_{n-s}^{(\alpha)} - b_{n+1-s}^{(\alpha)}\right) w^s - b_n^{(\alpha)} w^0\right]. \tag{2.3}
$$

2.2. **Quadrature formula: Integral terms.** In this part, we aim to approximate the integral term

$$
\int_0^{t_{n+1}} (t_{n+1} - \xi)^{\beta-1} w(\xi)\, d\xi. \tag{2.4}
$$

Following [27, 44], let $\Pi_{0,s+1} w(t) = w(t_{s+1})$ be the the piecewise constant approximation of $w(\xi)$. From the interpolation theory, for $s = 0, \ldots, N-1$, we have

$$
w(t) - \Pi_{0,s+1} w(t) = w^{'}(\eta_{s+1})(t - t_{s+1}),\ t \in [t_s, t_{s+1}],\ \eta_{s+1} \in (t_s, t_{s+1}). \tag{2.5}
$$

Thus, we arrive at

$$
\begin{aligned}
\int_0^{t_{n+1}} (t_{n+1} - \xi)^{\beta-1} w(\xi)\, d\xi &= \sum_{s=0}^{n} \int_{t_s}^{t_{s+1}} (t_{n+1} - \xi)^{\beta-1} w(\eta)\, d\xi \\
&= \frac{\tau^\beta}{\beta} \sum_{s=0}^{n} d_s^{(\beta)} w^{n-s+1} + \mathcal{R}_2^{n+1}(w),
\end{aligned}
\tag{2.6}
$$

where $d_s^{(\beta)} = (s+1)^\beta - s^\beta$, $0 < \beta < 1$, $0 \leq s$ and $\mathcal{R}_2^{n+1}(w) = O(\tau^{1+\beta})$. Then, we use the following quadrature operator [23]:

$$
\mathcal{I}^\beta w^{n+1} = \frac{\tau^\beta}{\beta} \sum_{s=0}^{n} d_s^{(\beta)} w^{n-s+1}. \tag{2.7}
$$

Lemma 2.1 ([8, 27]). *Let us define the quadrature error as*

$$\varepsilon^{n+1}(w) = \frac{\tau^\beta}{\beta}\sum_{s=0}^{n} d_s^{(\beta)} w^{n-s+1} - \int_0^{t_{n+1}} (t_{n+1}-\xi)^{\beta-1} w(\xi)\, d\xi, \qquad (2.8)$$

such that $w_t \in L_1(0,T;L_2)$, then, the global quadrature error holds

$$\sum_{n=0}^{N-1} \|\varepsilon^{n+1}(w)\| \le C_T \int_0^{t_{n+1}} \|w_t\|\, d\xi, \qquad t_N \le T, \qquad (2.9)$$

where C_T is a constant.

2.3. **Spatial discretization.** Consider the spatial domain $\Omega = [a,b]$ with the partition $a = x_0 < x_1 < \cdots < x_{M-1} = x_M = b$. We define $E_i = [x_{i-1}, x_i]$, so $\Omega \approx \mathcal{E}_h = \cup_{i=1}^{M} E_i$, $h_i = x_i - x_{i-1}$ and $h = \Delta x = \max\limits_{1\le i\le M} h_i$.

Let e be the intersection of elements E_i and E_{i+1}, we denote

$$w_i^{\pm} = w(x_i^{\pm}) = \lim_{\varrho\to 0^{\pm}} w(x_i+\varrho),\ \forall (x_i) \in e, \qquad (2.10)$$

the left and right limits of $w(x_i)$. Next, in order to formulate the numerical scheme, we define the finite element subspace V_h as

$$V_h = \left\{ v:\ \mathcal{E}_h \to \mathbb{R} \;\middle|\; v\big|_{E_i} \in \mathcal{P}^k(E_i), i = 1,\ldots,M \right\}, \qquad (2.11)$$

where $\mathcal{P}^k(E_i)$ is the set of all polynomials of degree less than or equal k, $k \ge 1$ and then restrict the trial and test functions to V_h.

3. Numerical scheme

This section is divided in two parts. First, utilizing LDG method, we present the semi-discrete LDG method for the numerical solution of (1.1). Next, considering the FD scheme (L1 formula) for temporal terms, the full discrete FD-LDG (L1-LDG) method is presented.

3.1. **Semi-discrete LDG method.** In order to introduce the LDG formulation, at first we define some auxiliary variables as

$$p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial p}{\partial x}, \qquad r = \frac{\partial q}{\partial x}, \qquad (3.1)$$

then rewrite Eq. (1.1) as

$$\begin{cases} {}_0^C\mathrm{D}_t^{\alpha} u + \dfrac{\partial u}{\partial x} - \displaystyle\int_0^t (t-\tau)^{\beta_1 - 1} \dfrac{\partial p}{\partial x}\, d\tau \\ \qquad\qquad + \displaystyle\int_0^t (t-\tau)^{\beta_2 - 1} \dfrac{\partial r}{\partial x}\, d\tau = f, \\ p = \dfrac{\partial u}{\partial x}, \\ q = \dfrac{\partial p}{\partial x}, \\ r = \dfrac{\partial q}{\partial x}, \end{cases} \tag{3.2}$$

Multiplying equations (3.2) by some smooth functions v, w, ψ and χ and integrating by parts over an element $E_i \in \mathcal{E}_h$, result the following formulation that holds for (u, p, q, r)

$$\begin{aligned} &\left({}_0^C\mathrm{D}_t^{\alpha} u, v\right)_{E_i} + \left[\left[uv\right]_{x_{i-1}}^{x_i} - \left(u, \frac{\partial v}{\partial x}\right)_{E_i} \right] \\ &- \int_0^t (t-\xi)^{\beta_1 - 1} \left[\left[pv\right]_{x_{i-1}}^{x_i} - \left(p, \frac{\partial v}{\partial x}\right)_{E_i} \right] d\xi \qquad (3.3a) \\ &+ \int_0^t (t-\xi)^{\beta_2 - 1} \left[\left[rv\right]_{x_{i-1}}^{x_i} - \left(r, \frac{\partial v}{\partial x}\right)_{E_i} \right] d\xi = (f, v)_{E_i}, \\ &(p, w)_{E_i} = \left[\left[uw\right]_{x_{i-1}}^{x_i} - \left(u, \frac{\partial w}{\partial x}\right)_{E_i} \right], \qquad (3.3b) \\ &(q, \psi)_{E_i} = \left[\left[p\psi\right]_{x_{i-1}}^{x_i} - \left(p, \frac{\partial \psi}{\partial x}\right)_{E_i} \right], \qquad (3.3c) \\ &(r, \chi)_{E_i} = \left[\left[q\chi\right]_{x_{i-1}}^{x_i} - \left(q, \frac{\partial \chi}{\partial x}\right)_{E_i} \right], \qquad (3.3d) \end{aligned}$$

Now, the LDG scheme is defined as: find $(u_h, p_h, q_h, r_h) \in V_h \times V_h \times V_h \times V_h$ as the approximation of (u, p, q, r) such that (u_h, p_h, q_h, r_h) satisfies:

$$\begin{aligned}
&\left({}_0^C\mathrm{D}_t^{\alpha} u_h, v\right)_{E_i} + \left[\left[\hat{u}_h v\right]_{x_{i-1}}^{x_i} - \left(u_h, \frac{\partial v}{\partial x}\right)_{E_i}\right] \\
&- \int_0^t (t-\xi)^{\beta_1 - 1} \left[\left[\hat{p}_h v\right]_{x_{i-1}}^{x_i} - \left(p_h, \frac{\partial v}{\partial x}\right)_{E_i}\right] d\xi \qquad (3.4a) \\
&+ \int_0^t (t-\xi)^{\beta_2 - 1} \left[\left[\hat{r}_h v\right]_{x_{i-1}}^{x_i} - \left(r_h, \frac{\partial v}{\partial x}\right)_{E_i}\right] d\xi = (f, v)_{E_i},
\end{aligned}$$

$$(p_h, w)_{E_i} = \left[\left[\hat{u}_h w\right]_{x_{i-1}}^{x_i} - \left(u_h, \frac{\partial w}{\partial x}\right)_{E_i}\right], \qquad (3.4b)$$

$$(q_h, \psi)_{E_i} = \left[\left[\hat{p}_h \psi\right]_{x_{i-1}}^{x_i} - \left(p_h, \frac{\partial \psi}{\partial x}\right)_{E_i}\right], \qquad (3.4c)$$

$$(r_h, \chi)_{E_i} = \left[\left[\hat{q}_h \chi\right]_{x_{i-1}}^{x_i} - \left(q_h, \frac{\partial \chi}{\partial x}\right)_{E_i}\right], \qquad (3.4d)$$

The notations $\hat{u}$, $\hat{p}$, $\hat{q}$ and $\hat{r}$ denote the numerical fluxes which are defined at interior interfaces (edges) as [14]

$$\hat{u} = u^-, \qquad \hat{p} = p^+, \qquad \hat{q} = q^-, \qquad \hat{r} = r^+, \qquad (3.5a)$$

and for the boundary faces

$$\begin{aligned}
&\hat{u}\big|_{x_0} = \hat{u}\big|_{x_M} = u^-\big|_{x_M}, \qquad \hat{p}\big|_{x_0} = \hat{p}\big|_{x_M} = p^+\big|_{x_0}, \\
&\hat{q}\big|_{x_0} = \hat{q}\big|_{x_M} = q^-\big|_{x_M}, \qquad \hat{r}\big|_{x_0} = \hat{r}\big|_{x_M} = r^+\big|_{x_0}.
\end{aligned} \qquad (3.5b)$$

These (alternative) numerical fluxes ensure stability and optimal convergence rate. It should be noted that this choice is not unique and other numerical fluxes such as the central numerical flux can also be used.

3.2. **Full-discrete FD-LDG method.** Using (2.3) for temporal discretization and quadrature rules (2.7) in semi-discrete scheme (3.4), the full-discrete FD-LDG method follows as: Find $\left(u_h^{n+1}, p_h^{n+1}, q_h^{n+1}, r_h^{n+1}\right) \in V_h \times V_h \times V_h \times V_h$ as the approximation of $\left(u^{n+1}, p^{n+1}, q^{n+1}, r^{n+1}\right)$ such

that (u_h, p_h, q_h, r_h) satisfies:

$$b_0^{(\alpha)}\left(u_h^{n+1}, v\right)_{E_i} - \sum_{s=0}^{n}\left(b_{n-s}^{(\alpha)} - b_{n+1-s}^{(\alpha)}\right)\left(u_h^s, v\right)_{E_i} - b_n^{(\alpha)}\left(u_h^0, v\right)_{E_i}$$
$$+\tau\left[\left[\hat{u}_h^{n+1}v\right]_{x_{i-1}}^{x_i} - \left(u_h^{n+1}, \frac{\partial v}{\partial x}\right)_{E_i}\right]$$
$$-\mu_1\sum_{s=0}^{n} d_s^{(\beta_1)}\left[\left[\hat{p}_h^{n-s+1}v\right]_{x_{i-1}}^{x_i} - \left(p_h^{n-s+1}, \frac{\partial v}{\partial x}\right)_{E_i}\right]$$
$$+\mu_2\sum_{s=0}^{n} d_s^{(\beta_2)}\left[\left[\hat{r}_h^{n-s+1}v\right]_{x_{i-1}}^{x_i} - \left(r_h^{n-s+1}, \frac{\partial v}{\partial x}\right)_{E_i}\right] = \tau\left(f^{n+1}, v\right)_{E_i}, \tag{3.6a}$$

$$\left(p_h^s, w\right)_{E_i} = \left[\left[\hat{u}_h^s w\right]_{x_{i-1}}^{x_i} - \left(u_h^s, \frac{\partial w}{\partial x}\right)_{E_i}\right], \tag{3.6b}$$

$$\left(q_h^s, \psi\right)_{E_i} = \left[\left[\hat{p}_h^s \psi\right]_{x_{i-1}}^{x_i} - \left(p_h^s, \frac{\partial \psi}{\partial x}\right)_{E_i}\right], \tag{3.6c}$$

$$\left(r_h^s, \chi\right)_{E_i} = \left[\left[\hat{q}_h^s \chi\right]_{x_{i-1}}^{x_i} - \left(q_h^s, \frac{\partial \chi}{\partial x}\right)_{E_i}\right], \tag{3.6d}$$

for $s = 0, \ldots, n+1$ and all v, w, ψ and $\chi \in V_h$.

4. Numerical experiments

In this section, we present some numerical results to show the accuracy of the proposed scheme. Assume that u and u_h be the exact and approximation solution obtained by FD-LDG methods, respectively. Let us define the L_2 norm of the error, $e_h = u - u_h$, as

$$\|u - u_h\|_{L^2} = \sqrt{\int_{\mathcal{E}_h} (u - u_h)^2 \, dx}, \tag{4.1}$$

and the computational order (CO) is defined as bellow

$$CO = \frac{\ln(\frac{\|e_h\|_{L_2}\big|_{h_1}}{\|e_h\|_{L_2}\big|_{h_2}})}{\ln(\frac{h_1}{h_2})}. \tag{4.2}$$

Example 4.1. Consider the following fourth order time-fractional PIDEs:

$$\begin{aligned} {}_0^C\mathrm{D}_t^{\alpha} u + \frac{\partial u}{\partial x} - \int_0^t (t-\xi)^{\beta_1 - 1}\, \Delta u \, d\xi \\ + \int_0^t (t-\xi)^{\beta_2 - 1}\, \Delta^2 u \, d\xi = f, \ (x,t) \in [0,1] \times (0,1] \end{aligned} \tag{4.3}$$

subject to zero initial condition and periodic BCs. The source term, f, is a given function, so that the exact solution of this problem is as follows

$$u(x,t) = t^{3-\alpha} \cos(2\pi x). \tag{4.4}$$

The errors and computational order of the proposed scheme is reported in Table 1 for $k = 1, 2$ for different values of fractional orders of α, β_1 and β_2. The Convergence plots of the scheme is presented in figure 1.

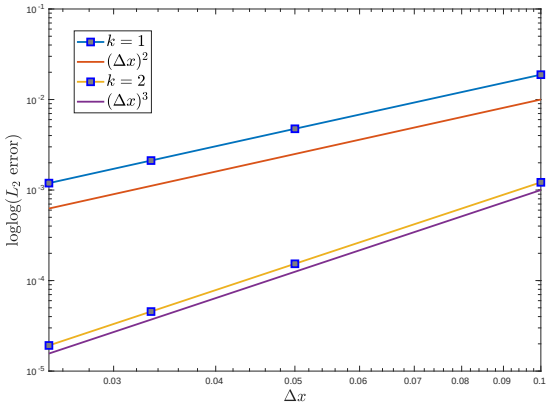

FIGURE 1. Convergence plots of FD-LDG method with $k = 1, 2$ and $\alpha = \beta_1 = \beta_2 = 0.5$: Example 4.1.

The results show the accuracy of the scheme and the optimal convergence rate, $O(h^{k+1})$, in the discrete L_2 norm, where k represents the degree of approximation polynomials.

5. Conclusion

Our proposed method for solving fourth-order time-fractional PIDEs uses a finite difference and local discontinuous Galerkin method, which can be extended for high-dimensional cases and is suitable for convection-dominated equations. The temporal term was discreted using the finite difference method (L1 formula) and the memory term was

TABLE 1. Errors and CO of the FD-LDG methods (3.6) for Eq. (4.3) with $k = 1, 2$: Example 4.1.

		$k=1$		$k=2$	
$(\alpha\ , \beta_1, \beta_2)$	h	$\|e_h\|_{L_2}$	CO	$\|e_h\|_{L_2}$	CO
$\alpha = 0.5$	$\frac{1}{10}$	1.8916e-02	–	1.2201e-03	–
$\beta_1 = 0.25$	$\frac{1}{20}$	4.7608e-03	1.9903	1.5310e-04	2.9944
$\beta_2 = 0.75$	$\frac{1}{30}$	2.1195e-03	1.9958	4.5416e-05	2.9971
	$\frac{1}{40}$	1.1932e-03	1.9973	1.9172e-05	2.9978
$\alpha = 0.5$	$\frac{1}{10}$	1.8804e-02	–	1.2171e-03	–
$\beta_1 = 0.5$	$\frac{1}{20}$	4.7494e-03	1.9853	1.5319e-04	2.9900
$\beta_2 = 0.5$	$\frac{1}{30}$	2.1170e-03	1.9928	4.5466e-05	2.9959
	$\frac{1}{40}$	1.1924e-03	1.9953	1.9195e-05	2.9976
$\alpha = 0.5$	$\frac{1}{10}$	1.8143e-02	–	1.1403e-03	–
$\beta_1 = 0.75$	$\frac{1}{20}$	4.6088e-03	1.9769	1.4700e-04	2.9555
$\beta_2 = 0.25$	$\frac{1}{30}$	2.0626e-03	1.9828	4.4074e-05	2.9709
	$\frac{1}{40}$	1.1650e-03	1.9857	1.8714e-05	2.9776

discreted using quadrature respectively. Finally, numerical experiments have been carried out and they have demonstrated that the rate of convergence is optimal.

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