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Topological Action on a Discrete Semi-group

Mohammad Taghi Heydari [1] and Hamid Rezaei
Department of Mathematics, College of Sciences, Yasouj University,
Yasouj-75918-74831, Iran

Abstract. Given a discrete semi-group S, a topological action θ of S on a locally compact space X is defined. Additionally, there is an action α of S on the C^*-algebra $C_0(X)$, which is introduced in relation to θ. We explore the topological independence of θ and the impact of θ on α. Finally, has been discussed the concept of a semi-partial dynamical system $(C_0(X), S, \alpha)$ and examine some of its properties.

Keywords: Partial automorphism, Partial action, Partial homeomorphism, Topological action.

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1. Introduction

A unital inverse semigroup is a semigroup S with an element e such that for every s in S, there exists a unique element s^* in S with the following properties:
(i) $ss^*s = s$;
(ii) $s^*ss^* = s^*$.

[1]Corresponding author: heydari@yu.ac.ir
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It is clear to see that the map $s \to s^*$ is an involution. For instance, let X be a set. A partially defined map on X refers to a bijective map $\varphi : A \to B$, where $A = dom(\varphi)$ and $B = rang(\varphi)$, and both A and B are subsets of X. The set of all partial maps on X is denoted by $I(X)$. The multiplication on $I(X)$ is given by the composition of partial maps in the largest domain where it is meaningful. That is, if φ and ϕ are elements of $I(X)$, then $dom(\varphi\phi) = \phi^{-1}(rang(\phi) \cap dom(\varphi))$ and $rang(\varphi\phi) = \varphi(rang(\phi) \cap dom(\varphi))$.

Every inverse semigroup is isomorphic to a semigroup of $I(X)$ for some X [4]. Additionally, let H be a Hilbert space and $P(H)$ represent the set of all *partial isometries* on H. Note that $T \in B(H)$ is a partial isometry if and only if T^*T is a projection on $(KerT)^{\perp}$. For $T \in P(H)$, let T^* be the Hilbert adjoint of T. Owing to the properties of partial isometries, we have $TT^*T = T$ and $T^*TT^* = T^*$. In fact, if $T \in P(H)$, then the initial space of T is the range of T^*T.

Let I and J be closed ideals in the C^*-algebra A, and let

$$\alpha : I \to J$$

be an isomorphism of the C^*-algebras. The triple (α, I, J) is referred to as a partial automorphism of A. The set of all partial automorphisms of the C^*-algebra A is denoted by $Pa(A)$. Then, $Pa(A)$ is a unital inverse semigroup with the identity. It is well known that the adjoint is the inverse and the product is the composition with the largest possible domain.

In conjunction with the concept of partial actions, the idea of crossed products emerged as a generalization in this theory (see [2], [3], [1], [5], [6], and [8]). Following these works, an action of a unitary inverse semigroup on the C^*-algebra is defined. Moreover, we consider a C^*-algebra and subsequently the partial crossed product of this C^*-algebra by that action.

Definition 1.1. [7] If A is a C^*-algebra and S is a unital inverse semigroup, an action of S on A is a semigroup isomorphism:

$$\beta : \quad S \to Pa(A), \quad s \to (\beta_s, E_{s^*}, E_s)$$

such that $E_e = A$, where e is the identity of S. If $s \in S$ and $s^2 = s$, then s is called an idempotent element. If $s^2 = s$ for all $s \in S$, then S is an idempotent semigroup.

Lemma 1.2. *Let S be a unital inverse semigroup, A be a C^*-algebra, β be an action of S on A, and $s \in S$. Then $\beta_{s^*} = \beta_s^{-1}$, β_e is the identity map on A, and if s is an idempotent, then β_s is the identity map on $E_{s^*} = E_s$.*

Proof. Because β is a homomorphism, therefore

$$\beta_s = \beta(s) = \beta(ss^*s) = \beta(s)\beta(s^*)\beta(s) = \beta_s\beta_{s^*}\beta_s$$

On the other hand, $\beta_s^{-1} = \beta_s\beta_s^{-1}\beta_s$. So, by the uniqueness of the inverse in inverse semigroups, it follows that $\beta_{s^*} = \beta_s^{-1}$. Moreover,

$$\beta_e\beta_s = \beta_{es} = \beta_s = \beta_{se} = \beta_s\beta_e$$

therefore $\beta e = i_A$. If s is an idempotent, since $s^2 = s$, we have $sss = s^2 = s$ and $ss^*s = s$. So, by the uniqueness of the inverse of s, it follows that $s = s^*$ and $\beta_s = \beta_{s^*}$. On the other hand, $(\beta_s)^2 = \beta_s\beta_s = \beta_s\beta_{s^*} = \beta_e = i$. □

Lemma 1.3. *If β is an action of the unital inverse semigroup S on a C^*-algebra A, then $\beta_t(E_{t^*}E_s) = E_{ts}$ for all $s, t \in S$.*

Proof. Since E_{t^*} and E_s are ideals in the C^*-algebra A, we have $E_{t^*}E_s = E_{t^*} \cap E_s$. So,

$$\begin{aligned} \beta_t(E_{t^*}E_s) = \beta_t(E_{t^*} \cap E_s) &= image(\beta_t\beta_s) \\ &= image(\beta(t)\beta(s)) \\ &= image(\beta(ts)) \end{aligned}$$

□

Remark 1.4. Let β be an action of the unital inverse semigroup S on the C^*-algebra A. Consider $L = \{x \in \ell^1(S, A) : x(s) \in E_s\}$, the closed subspace of $\ell^1(S, A)$. Define multiplication and involution on L by

$$(x * y)(s) = \sum_{rt=s} \beta_r[\beta_{r^*}(x(r))y(t)] \quad \textit{and} \quad x^*(s) = \beta_s[x(s^*)^*].$$

Note that L is closed with respect to the above operations, since by Lemma 1.3, elements of the form $(x*y)(s)$ are in E_s for every $s \in S$ and, as a consequence, $x*y \in L$. Also, for a given $x \in L$, since $x(s^) \in E_{s^*}$ and E_{s^*} is an ideal of A, we have $(x(s^*))^* \in E_{s^*}$. Therefore, $\beta_s(x(s^*)^*) \in E_s$, so $x^* \in L$. Simple computations show that $\|xy\| \leq \|x\|\|y\|$ and $\|x^*\| = \|x\|$, where $\|.\|$ denotes the norm of L inherited from $\ell^1(S, A)$.

The notion of partial group C^*-algebra of a discrete group introduced by R. Exel in [3] is generalized to an idempotent unital inverse semigroup, and the partial inverse semigroup C^*-algebra is defined. By using the algebras of multipliers of ideals of an associative algebra, It can be proved some theorems in the C^*-algebra context without using the identity.

Definition 1.5. Let A be a C^*-algebra and S be a unital inverse semigroup with the identity e. A partial action of S on A is a collection $\{(\alpha_s, D_s, D_s) : s \in S\}$ of partial automorphisms (denoted by α or by (A, S, α)) such that

- $D_e = A$;
- α_{st} extends $\alpha_s\alpha_t$, that is, $\alpha_{st}|\alpha_t^*(Ds^*) = \alpha_s\alpha_t$ for all $s, t \in S$.

Proposition 1.6. *If β is a partial action of S on A, then,*

- *β_e is the identity map i on A;*
- *$\beta_t = \beta_t$ for all $t \in S$;*
- *$\beta_t(D_tD_s) = D_tD_{ts}$, $\forall t, s \in S$;*
- *$\beta_t(D_tD_{s_1}D_{s_2}...D_{s_n}) = D_tD_{ts_1}D_{ts_2}...D_{ts_n}$, $\forall t, s_1, s_2, ..., s_n \in S$.*

Proof. Similar to Proposition 2.3, Lemma 2.4, Lemma 2.5, and Theorem 2.6 of [7] □

Definition 1.7. Let β be a partial action of S on A. A *covariant representation* of β is a triple (π, u, H), where H is a Hilbert space and $\pi : A \to B(H)$ is a non-degenerate representation, and for each $s \in S$, u_s is a partial isometry on H with initial space $\pi(D_s)H$ and final space $\pi(D_s)H$, such that

- $u_s\pi(a)u_s = \pi(\beta_s(a))$ for all $a \in D_s$;
- $u_{st}h = u_su_th$ for all $h \in \pi(D_tD_{ts})H$.

The class of all covariant representations of (A, S, β) is denoted by $CovRep(A, S, \beta)$.

Definition 1.8. [2] Let $(\pi, v, H) \in CovRep(A, S, \beta)$. We define the operator $\pi \times v : L \to B(H)$ by

$$(\pi \times v)(x) = \int_S \pi(x(s))v_s ds.$$

where ds denotes the Haar measure on S. If S is a discrete group, then,

$$(\pi \times v)(x) = \sum_{s \in S} \pi(x(s))v_s.$$

Definition 1.9. [7] Let A be a C^*-algebra and β be an action of the unital inverse semigroup S on A. We define a seminorm $\|\cdot\|_c$ on L by

$$\|x\|_c = \sup\{\|(\pi \times v)(x)\| : (\pi, v) \in CovRep(A, S, \beta)\}.$$

Let $I = \{x \in L : \|x\|c = 0\}$. The *crossed product* $A \times_\beta S$ of the C^*-algebra A and the semigroup S by the action β is the C^*-algebra obtained by completing the quotient L/I with respect to $\|\cdot\|_c$.

It is clear that the quotient map $x \mapsto x + I$ is contractive. In fact, $\|x + I\| \leq \|x\|$.

2. Main result

In the last two decades, the notion of a partial crossed product of a C^*-algebra by a discrete group was defined by McClanahan [5] as a generalization of Exel's definition in [1]. The more well-established notion of the crossed product of a C^*-algebra by an action of a group uses a homomorphism into the automorphism group of the C^*-algebra. The idea of a partial action is to replace the automorphism group by the inverse semigroup of partial automorphisms. A partial automorphism is an isomorphism between two closed ideals of a C^*-algebra. Of course we cannot talk about a homomorphism from a group into an inverse semigroup; a partial action is an appropriate generalization. In this section a detailed discussion of action on a discrete semigroup will be presented.

A semipartial dynamical system is a triple (A, S, α) consisting of a C^*-algebra A, a unital inverse semigroup S, and an action α of S on A. In this section, we will focus on the specific case of $(C_0(X), S, \alpha)$, where X is a locally compact Hausdorff space, and α is the action of $C_0(X)$ arising from partial homeomorphisms of X. For every $s \in S$, there exists an open subset U_s of X and a homeomorphism $\theta : U_{s^*} \to U_s$ such that $U_e = X$, and θ_e is the identity map on X. The action α of S on $C_0(X)$ corresponding to the partial homeomorphism θ is given by the formula:

$$\alpha_s(f)(x) = f(\theta_{s^*}(x)),\ s \in S,\ f \in C_0(U_{s^*}).$$

The above facts are summarized with the following definition:

Definition 2.1. Let S be a unital inverse semigroup, and X be a locally compact Hausdorff space. A *topological action* of S on X is a pair $\theta = (\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$, where for each $s \in S$, U_s is an open subset of X, $\theta_s : U_{s^*} \to U_s$ is a homeomorphism, and $U_e = X$ and θ_e is the identity map on X.

Given a topological action $(\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of a semigroup S on a locally compact Hausdorff space X, we can identify each $E_s = C_0(U_s)$, in the usual way, with the ideal of functions in $C_0(X)$ vanishing on $X - U_s$. Using this identification, we can define the action α of S on $C_0(X)$ corresponding to the topological action θ as follows:

$$\alpha_s(f)(x) := f(\theta_{s^*}(x)), f \in C_0(U_{s^*})$$

where s^* denotes the inverse of s in S.

Now the notion of a topologically free action is defined as follows:

Definition 2.2. The topological action θ of S on X is said to be *topologically free* if the set of fixed points for the partial homeomorphism associated with each non-trivial semigroup element has empty interior.

In summary, an action of a semigroup on a locally compact Hausdorff space induces an action on the space of continuous functions vanishing at infinity, and we say that the original action is topologically free if the fixed point sets of its non-trivial semigroup elements have empty interior. Note that $F_e = \{x \in X : \theta_e(x) = x\} = \{x \in X : I(x) = x\} = X$ and $X^o = X$, so F_e hasn't empty interior. Therefore, in Definition 2.2, $s \neq e$.

Theorem 2.3. *The topological action θ of a unital inverse semigroup S on X is topologically free if and only if for every $s \in S - \{e\}$, the set F_s is nowhere dense.*

Proof. The "if" part is trivial. For the "only if" part, assume that θ is topologically free and let $x \in U_{s^*}$ be a limit point of F_s relative to U_{s^*}. Then, there is a net $x_i \subset F_s$ such that $x_i \to^{cl} x$. By continuity of θ_s, we have $x_i = \theta_s(x_i) \to \theta_s(x)$, and so $\theta_s(x) = x$. Therefore, $x \in F_s$. Since F_s is closed in X relative to U_{s^*}, it follows that $F_s = C \cap U_{s^*}$ for some closed subset C of X. If V is open and $V \subset \overline{F_s}$, then

$$V \cap U_{s^*} \subset \overline{F_s} \cap U_* = (\overline{C \cap U_{s^*}}) \cap U_{s^*} \subseteq \overline{C} \cap U_{s^*} = C \cap U_{s^*} = F_s.$$

Since F_s is nowhere dense in X and $V \cap U_*$ is open in X, we see that $V \cap U_* = \emptyset$. Thus, the open sets U_{s^*} and V are separated. Moreover, since $V \subset \overline{F_s} = (\overline{C \cap U_{s^*}}) \subseteq C \cap \overline{U_{s^*}} \subset U_{s^*}$, we see that $V = \emptyset$. Therefore, F_s is nowhere dense in X. □

The following equivalent formulation of topological freeness better suits our objectives:

Corollary 2.4. *The topological action θ of a unital inverse semigroup S on a space X exhibits topological freeness if and only if, for every finite subset $\{s_1, s_2, ..., s_n\}$ of $S - e$, the union $\bigcup_{i=1}^n F_{s_i}$ has an empty interior.*

In the remainder of this work, we denote by $\delta_s, s \in S$ the function in L_A that takes the value 1 at s and zero for all other elements of S.

Theorem 2.5. *Let $s \in S - e$, $f \in E_s = C_0(U_s)$, and $x_0 \notin F_s$. For every $\epsilon > 0$, there exists $h \in C_0(X)$ such that:*

(i) $h(x_0) = 1$;
(ii) $\|h(f\delta_s)h\| \leq \epsilon$, *and*
(iii) $0 \leq h \leq 1$.

Proof. By embedding $C_0(X) \hookrightarrow C_0(X) \times S$ with $h \longmapsto h\delta_e$, we consider $h(f\delta_s)h = (h\delta_e)(f\delta_s)(h\delta_e)$ as an element of $C_0(X) \times S$. Thus, $h(f\delta_s)h = (hf)\delta_s.(h\delta_e) = \alpha_s(\alpha_{s^*}(hf)h)\delta_s$. Additionally, $\alpha_{s^*}(hf)(x) = (hf)(\theta_s(x)) = h(\theta_s(x))f(\theta_s(x))$. If $x \in \text{support}(\alpha_{s^*}(hf))$, then, $\theta_s(x) \in \text{support}(h)$. □

Definition 2.6. If A is a C^*-algebra and B is a C^*-subalgebra of A, then by a *conditional expectation* from A to B, we mean a continuous positive projection P of A onto B that satisfies the conditional expectation property

$$P(ba) = bP(a) \text{ and } P(ab) = P(a)b,\ a \in A,\ b \in B.$$

Example 2.7. Let A be a C^*-algebra with identity 1 and ρ be a state on A. Set $B = \mathbb{C}.1$ and define $P : A \to B$ by $P(a) = \rho(a).1$. Then, P is a conditional expectation. Furthermore, if G is a compact group that acts continuously as a group of automorphisms of A, and M is the subalgebra of elements that are invariant under the action of G, let $P : A \to M$ be defined by $P(a) := \int_G x(a)dx$. Then P is a conditional expectation. For an explicit example of a conditional expectation, let $B = iAi$ for a self-adjoint idempotent $i \in A$, and define $P : A \to B$ by $P(a) = iai$.

We can consider $C_0(X)$ as a C^*-subalgebra of the partial crossed product $C_0(X) \times_\alpha S$. Therefore, the conditional expectation from $C_0(X) \times_\alpha S$ onto $C_0(X)$, which is denoted by E, is meaningful.

Definition 2.8. A semi-partial dynamical system (A, S, α) is said to be *topologically free* if, for every $s \in S - e$, the set $F_s := \{x \in U_{s^*} : \theta_s(x) = x\}$ has an empty interior.

It is well-known that (see [9]) a crossed product by a partial action is a graded C^*-algebra. Since the conditional expectation $E : C_0(X) \times_\alpha S \to C_0(X)$ is contractive, we can state and prove the following theorem.

Theorem 2.9. *If $(C_0(X), S, \alpha)$ is a topologically free semi-partial dynamical system, then for every $c \in C_0(X) \times_\alpha S$ and $\epsilon > 0$, there exists $h \in C_0(X)$ such that:*

(i) $\|hE(c)h\| \geq \|E(c)\| - \epsilon$;
(ii) $\|hE(c)h - hch\| \leq \epsilon$, *and*
(iii) $0 \leq h \leq 1$.

Proof. Let c be a finite linear combination of the form $\sum_{t \in T} a_t \delta_t$, where T denotes a finite subset of S. Define $E(s) = a_e$ if $e \in T$ and $E(s) = 0$ if $e \notin T$. Since

$$\|a_e\| = \sup\{|a_e(x)| : x \in X\},$$

for given $\epsilon > 0$, the set $V = \{x \in X : |a_e(x)| \geq \|a_e\| - \epsilon\}$ is a non-empty open set. Since the topological action α is topologically free, there exists $x_0 \in V$ such that $x_0 \notin F_t$ for every $t \in T$. Take $f_t = a_t\delta_t \in D_t$, for $\frac{\epsilon}{n(T)}$, there exist functions h_t such that

$$h_t(x_0) = 1,\ \|h_t(a_t\delta_t)h_t\| \leq \frac{\epsilon}{n(T)} \text{ and } 0 \leq h_t \leq 1.$$

Let $h = \prod_{t \in T-e} h_t$. Obviously $0 \le h \le 1$, which means (iii) holds. Also (i) holds, simply because $x_0 \in V$ and

$$\begin{aligned} \|ha_e h\| &= \sup h(x)a_e(x)h(x) : x \in X \\ &\ge |h(x_0)a_e(x_0)h(x_0)| \\ &= |a_e(x_0)| > \|a_e\| - \epsilon. \end{aligned}$$

In order to prove (ii), we have

$$\begin{aligned} \|ha_e h - hch\| &= \|ha_e h - \sum_{t \in T} ha_t \delta_t h\| \\ &= \| \sum_{t \in T-e} ha_t \delta_t h\| \\ &\le \sum_{t \in T-e} \|ha_t \delta_t h\| < n(T). \frac{\epsilon}{n(T)} = \epsilon. \end{aligned}$$

For an arbitrary element c, since c is the limit of a net in $C_0(X) \times_\alpha S$ and E is contractive, a standard approximation argument finishes the proof. □

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