Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2024.26628.1679
Caspian J Math Sci. **13**(2)(2024), 279-295 (Research Article)

A note on invariant submanifolds of hyperbolic Sasakian manifolds

Sunil Kumar Yadav [1], Shyamal Kumar Hui [2] and Rajendra Prasad[3]
[1] Department of Applied Science and Humanities, United College of Engineering & Research, A-31, UPSIDC Institutional Area, Naini-211010, Prayagraj, India.
Email. prof_sky16@yahoo.com
[2] Department of Mathematics, The University of Burdwan, Golapbag, Burdwan, West Bengal, India.
Email.skhui@math.buruniv.ac.in
[3] Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India.
Email.rp.manpur@rediffmail.com

Abstract. This article aims to investigate some geometric conditions for an invariant submanifold of a hyperbolic Sasakian manifold. likewise, we take into account a generalized quasi-conformal curvature tensor on an invariant submanifold of a hyperbolic Sasakian manifold to be totally geodesic with certain geometric restrictions. We additionally investigate the properties of the conformal η-Ricci-Yamabe soliton on such a submanifold. To validate our findings, we also construct an example.

Keywords: Hyperbolic Sasakian manifolds, invariant and totally geodesic submanifolds, concircular vector field, conformal η-Ricci-Yamabe soliton.

2000 Mathematics subject classification: Primary 53C30, 53C40; Secondary 53C50.

[1]Corresponding author: prof_sky16@yahoo.com
Received: 21 January 2024
Revised: 05 August 2024
Accepted: 19 August 2024
How to Cite: Yadav, Sunil Kumar; Hui, Shyamal Kumar; Prasad, Rajendra. A note on invariant submanifolds of hyperbolic Sasakian manifolds, Casp.J. Math. Sci.,**13**(2)(2024), 279-295.

1. Introduction

In a Riemannian manifold $(\widetilde{\mathbb{V}}^n, g)$, the concept of conformal Ricci flow is defined as a generalisation of the classical Ricci flow [12]

$$\frac{\partial g}{\partial t} = -2(\widetilde{Ric} + \frac{g}{n}) - \tilde{p}g, \quad \tau(g) = -1, \tag{1.1}$$

where $\tilde{p}$ define a time dependent non-dynamical scalar field (also called the conformal pressure), g is the Riemannian metric, and τ and $\widetilde{Ric}$ denote the scalar curvature and the Ricci tensor of $\widetilde{\mathbb{V}}^n$, respectively.
A conformal Ricci soliton on $(\widetilde{\mathbb{V}}^n, g)$ is defined as follows [5]:

$$\mathfrak{L}_{\mathcal{E}} g + 2\widetilde{Ric} = [\frac{1}{n}(\tilde{p}n + 2) - 2\mu]g, \tag{1.2}$$

where $\mathfrak{L}_{\mathcal{E}}$, $\mathcal{E}$ and μ indicate the Lie-derivative operator, a smooth vector field, and a constant, respectively. A new class of geometric flow called Ricci-Yamabe flow of type (κ, l), which is a scalar combination of Ricci and Yamabe flow is given by [14]:

$$\frac{\partial}{\partial t} g(t) = 2\kappa\widetilde{Ric}(g(t)) - l\tau(t)g(t), \quad g(0) = g_0, \tag{1.3}$$

for some scalars κ and l.

Definition 1.1. g, the Riemannian metric is named the Ricci-Yamabe soliton of type (κ, l) (briefly, (RYS))[9] if

$$\mathfrak{L}_{\mathcal{E}} g + 2\kappa\widetilde{Ric} + (2\mu - l\tau)g = 0, \tag{1.4}$$

where $l, \kappa, \mu \in \mathbb{R}$.

Definition 1.2. g, the Riemannian metric is said to be the conformal Ricci-Yamabe soliton (briefly, (CRYS))[33] if

$$\mathfrak{L}_{\mathcal{E}} g + 2\kappa\widetilde{Ric} + [2\mu - l\tau - \frac{1}{n}(\tilde{p}n + 2)]g = 0, \tag{1.5}$$

A Riemannian manifold $(\widetilde{\mathbb{V}}^n, g)$ is said to be conformal η-Ricci-Yamabe soliton (briefly, (CERYS)) if

$$\mathfrak{L}_{\mathcal{E}} g + 2\kappa\widetilde{Ric} + [2\mu - l\tau - \frac{1}{n}(\tilde{p}n + 2)]g + 2\nu\eta \otimes \eta = 0, \tag{1.6}$$

where $l, \kappa, \mu, \nu \in \mathbb{R}$ and and η is a 1-form on $\widetilde{\mathbb{V}}^n$.
A CRYS (or gradient CRYS) is said to be shrinking, steady or expanding if $\mu < 0, = 0$ or > 0, respectively. A CERYS (or gradient CERYS) reduces to (i) conformal η-Ricci soliton if κ=1, l=0, (ii) conformal η-Yamabe soliton if κ=0, l=1, (iii) conformal η-Einstein soliton if κ=1, l=-1.
In [27], introduced the notion of an almost hyperbolic contact (f, g, η, ξ)-structure. A $(2n+1)$-dimensional differentiable manifold of class $\mathcal{C}^\infty$ equipped with the structure

(f, g, η, ξ) is known as an almost hyperbolic contact manifold. Further, it was studied by number of authors ([2], [4], [22]). Let $\mathcal{T}_p(\widetilde{\mathbb{V}})$ denote the tangent space of the almost hyperbolic contact manifold $\widetilde{\mathbb{V}}$ at point p. Then a vector field $\upsilon \in \mathcal{T}_p(\widetilde{\mathbb{V}})$, $\upsilon \neq 0$, is said to be time-like (resp., null, space-like, and non-space-like) if it satisfies $g_p(\upsilon, \upsilon) < 0$ $(resp., = 0, > 0,$ and $\leq 0)$ ([8],[24]). If $\{e_1, e_2, ..., e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal basis of $\widetilde{\mathbb{V}}$, then the Ricci tensor $\widetilde{Ric}$ and scalar curvature τ of an almost hyperbolic contact metric manifold are defined as follows:

$$\widetilde{Ric}(\mathcal{E}, \mathcal{F}) = \sum_{i=1}^{2n+1} \epsilon_i g(\widetilde{\mathcal{R}}(e_i, \mathcal{E})\mathcal{F})e_i = \sum_{i=1}^{2n} \epsilon_i g(\widetilde{\mathcal{R}}(e_i, \mathcal{E})\mathcal{F})e_i - g(\widetilde{\mathcal{R}}(\xi, \mathcal{E})\mathcal{F})\xi \quad (1.7)$$

$$\tau = \sum_{i=1}^{2n+1} \epsilon_i \widetilde{Ric}(e_i, e_i) = \sum_{i=1}^{2n} \epsilon_i \widetilde{Ric}(e_i, e_i) - \widetilde{Ric}(\xi, \xi), \quad (1.8)$$

for all $\mathcal{E}, \mathcal{F} \in \mathcal{T}\widetilde{\mathbb{V}}$, where $\epsilon_i = g(e_i, e_i)$, ξ is a unit time-like vector field, $\widetilde{R}$ denote the curvature tensor of $\widetilde{\mathbb{V}}$ and $\mathcal{T}\widetilde{\mathbb{V}}$ denotes the tangent bundle of $\widetilde{\mathbb{V}}$.

2. Hyperbolic Sasakian manifolds and their submanifolds

Let $(\widetilde{\mathbb{V}}^n, g)$, $(n=2m+1)$ be a differentiable manifold. Then the structure (ϕ, ξ, η) satisfying

$$\phi^2 = \mathcal{I} + \eta(\xi), \ \ \eta \circ \phi = 0, \quad (2.1)$$

is said to be an almost hyperbolic contact structure [27], where $\mathcal{I}$ denotes the identity transformation and ϕ, η and ξ are the tensor fields of type $(1, 1)$, $(0, 1)$ and $(1, 0)$. The manifold $\widetilde{\mathbb{V}}$ admits the structure (ϕ, ξ, η) is called an almost hyperbolic contact manifold. Also from (2.1), we have

$$\phi\xi = 0, \ \ \eta(\xi) = -1 \quad \text{and} \quad rank(\phi) = n - 1. \quad (2.2)$$

If the semi-Riemannian metric g of $\widetilde{\mathbb{V}}$ satisfies

$$g(\mathcal{E}, \xi) = \eta(\mathcal{E}), \quad g(\phi\mathcal{E}, \phi\mathcal{F}) = -g(\mathcal{E}, \mathcal{F}) - \eta(\mathcal{E})\eta(\mathcal{F}), \quad (2.3)$$

for all $\mathcal{E}, \mathcal{F} \in \mathcal{T}\widetilde{\mathbb{V}}$, then the structure (ϕ, ξ, η) is called an almost hyperbolic contact metric structures and the $\widetilde{\mathbb{V}}^n$ with the structure (ϕ, ξ, η) is known as an almost hyperbolic contact metric manifold. An almost hyperbolic contact metric manifold is said to be an almost hyperbolic Sasakian manifold if the 2-form defined as $\Pi(\mathcal{E}, \mathcal{F}) = g(\phi\mathcal{E}, \mathcal{F})$ satisfies $-2\Pi = d\eta$, which is equivalent to

$$(\widetilde{\nabla}_{\mathcal{E}}\phi) = g(\mathcal{E}, \mathcal{F})\xi - \eta(\mathcal{F})\mathcal{E}. \text{ Then } \ \widetilde{\nabla}_{\mathcal{E}}\xi = -\phi\mathcal{E}, \quad (2.4)$$

for all $\mathcal{E}, \mathcal{F} \in \mathcal{T}\widetilde{\mathbb{V}}$ and $\widetilde{\nabla}$ indicate the Levi-Civita connection of $\widetilde{\mathbb{V}}$.
In a hyperbolic Sasakian manifold we have:

$$\widetilde{\mathcal{R}}(\mathcal{E}, \mathcal{F})\xi = \eta(\mathcal{F})\mathcal{E} - \eta(\mathcal{E})\mathcal{F}, \quad (2.5)$$

$$\eta(\widetilde{\mathcal{R}}(\mathcal{E}, \mathcal{F})\mathcal{G}) = g(\mathcal{F}, \mathcal{G})\eta(\mathcal{E}) - g(\mathcal{E}, \mathcal{G})\eta(\mathcal{F}), \quad (2.6)$$

$$\widetilde{\mathcal{R}}(\xi,\mathcal{E})\mathcal{F} = g(\mathcal{E},\mathcal{F})\xi - \eta(\mathcal{F})\mathcal{E}, \tag{2.7}$$

$$g(\phi\mathcal{E},\mathcal{F}) = -g(\mathcal{E},\phi\mathcal{F}), \tag{2.8}$$

for all $\mathcal{E},\mathcal{F},\mathcal{G} \in \mathcal{T}\widetilde{\mathbb{V}}$. Let $\widetilde{\mathbb{N}}$ be an m-dimensional submanifold of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ $(m < n)$ with induced metric g on $\widetilde{\mathbb{N}}$. Also let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle $\mathcal{T}\widetilde{\mathbb{N}}$ and the normal bundle $\mathcal{T}^{\perp}\widetilde{\mathbb{N}}$ of $\widetilde{\mathbb{N}}$ respectively. Then the Gauss and Weingarten formulae are given by

$$\widetilde{\nabla}_{\mathcal{E}}\mathcal{F} = \nabla_{\mathcal{E}}\mathcal{F} + \hbar(\mathcal{E},\mathcal{F}) \tag{2.9}$$

and

$$\widetilde{\nabla}_{\mathcal{E}}\mathcal{Z}^{\flat} = -\mathcal{A}_{\mathcal{Z}^{\flat}}\mathcal{E} + \nabla^{\perp}_{\mathcal{E}}\mathcal{Z}^{\flat}, \tag{2.10}$$

for all $\mathcal{E},\mathcal{F} \in (\mathcal{T}\widetilde{\mathbb{N}})$ and $\mathcal{Z}^{\flat} \in (\mathcal{T}^{\perp}\widetilde{\mathbb{N}})$, where $\hbar$ and $\mathcal{A}_{\mathcal{Z}^{\flat}}$ are second fundamental form and the shape operator (corresponding to the normal vector field $\mathcal{Z}^{\flat}$) respectively for the immersion of $\widetilde{\mathbb{N}}$ into $\widetilde{\mathbb{V}}$. The second fundamental form $\hbar$ and the shape operator $\mathcal{A}_{\mathcal{Z}^{\flat}}$ are related by

$$g(\hbar(\mathcal{E},\mathcal{F}),\mathcal{Z}) = g(\mathcal{A}_{\mathcal{Z}^{\flat}}\mathcal{E},\mathcal{F}), \tag{2.11}$$

for all $\mathcal{E},\mathcal{F} \in (\mathcal{T}\widetilde{\mathbb{N}})$ and $\mathcal{Z}^{\flat} \in (\mathcal{T}^{\perp}\widetilde{\mathbb{N}})$. We note that $\hbar(\mathcal{E},\mathcal{F})$ is bilinear and since $\nabla_{f\mathcal{E}}\mathcal{F}=f\nabla_{\mathcal{E}}\mathcal{F}$, for any smooth function f on a manifold, we have

$$\hbar(f\mathcal{E},\mathcal{F}) = f\hbar(\mathcal{E},\mathcal{F}). \tag{2.12}$$

A submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is said to be totally umbilical if

$$\hbar(\mathcal{E},\mathcal{F}) = g(\mathcal{E},\mathcal{F})\mathcal{H}, \tag{2.13}$$

where $\mathcal{E},\mathcal{F} \in T\mathbb{N}$ and the mean curvature vector $\mathcal{H}$ on $\widetilde{\mathbb{N}}$ is given by $\mathcal{H} = \frac{1}{m}\sum_{i=1}^{m}\hbar(e_i,e_i)$, where $\{e_1,e_2,....e_m\}$ is a local orthonormal frame of vector fields on $\widetilde{\mathbb{N}}$. Moreover if $\hbar(\mathcal{E},\mathcal{F})$=0 for all $\mathcal{E},\mathcal{F} \in \mathcal{T}\widetilde{\mathbb{N}}$, then $\widetilde{\mathbb{N}}$ is said to be totally geodesic and if $\mathcal{H}$=0 then $\widetilde{\mathbb{N}}$ is minimal in $\widetilde{\mathbb{V}}$. The covariant derivative of $\hbar$ is

$$(\nabla_{\mathcal{E}}\hbar)(\mathcal{F},\mathcal{G}) = \nabla^{\perp}_{\mathcal{E}}(\hbar(\mathcal{F},\mathcal{G})) - \hbar(\nabla_{\mathcal{E}}\mathcal{F},\mathcal{G}) - \hbar(\mathcal{F},\nabla_{\mathcal{E}}\mathcal{G}), \tag{2.14}$$

for any vector field $\mathcal{E},\mathcal{F},\mathcal{G}$ tangent to $\widetilde{\mathbb{N}}$. Then $\nabla\hbar$ is a normal bundle valued tensor of type $(0,3)$ and is said to be third fundamental form of $\widetilde{\mathbb{N}}$, ∇ is called the Vander–Waerden–Bortolotti connection of $\widetilde{\mathbb{V}}$, i.e., ∇ is the connection in $\mathcal{T}\widetilde{\mathbb{N}} \oplus \mathcal{T}^{\perp}\widetilde{\mathbb{N}}$ built with ∇ and $\nabla^{\perp}$. If $\nabla\hbar$=0, then $\widetilde{\mathbb{N}}$ is said to have parallel second fundamental form [29]. A submanifold $\widetilde{\mathbb{N}}$ is said to be semiparallel [22] (resp. 2-semi-parallel, see [3]) if

$$\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\cdot\hbar = 0, \ \text{(resp. } \widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\cdot\widetilde{\nabla}\hbar = 0\text{)}, \text{ for all } \mathcal{E},\mathcal{F} \in \widetilde{\mathbb{V}} \tag{2.15}$$

On a Riemannian manifold $\widetilde{\mathbb{V}}$, for a $(0,k)$-type tensor field $\mathcal{T}$, $(k > 1)$ and a $(0,2)$-type tensor field $\mathcal{A}$, we denote the $\widehat{\mathcal{Q}}(\mathcal{A},\mathcal{T})$ as a $(0,k+2)$-type tensor field [28], defined as follows:

$$\begin{aligned}\widehat{\mathcal{Q}}(\mathcal{A},\mathcal{T})(\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_k;\mathcal{E},\mathcal{F}) &= -\mathcal{T}((\mathcal{E}\wedge_{\mathcal{A}}\mathcal{F})(\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_k)\\ &- \mathcal{T}(\mathcal{E}_1,\mathcal{E}\wedge_{\mathcal{A}}\mathcal{F})\mathcal{E}_2,\mathcal{E}_k)\\ &- ......-\mathcal{T}(\mathcal{E}_1,\mathcal{E}_2,....,\mathcal{E}_{k-1}(\mathcal{E}\wedge_{\mathcal{A}}\mathcal{F})\mathcal{E}_k)\end{aligned} \tag{2.17}$$

where $(\mathcal{E}\wedge_{\mathcal{A}}\mathcal{F})\mathcal{G}$=$\mathcal{A}(\mathcal{F},\mathcal{G})\mathcal{E}$-$\mathcal{A}(\mathcal{E},\mathcal{G})\mathcal{F}$.
A submanifold $\widetilde{\mathbb{N}}$ is said to be pseudo-parallel [3] if

$$\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\cdot\hbar = f_{\widehat{\mathcal{Q}}}(g,\hbar), \tag{2.18}$$

holds for any vector fields $\mathcal{E}$ and $\mathcal{F}$ tangent to $\widetilde{\mathbb{V}}$ and a smooth function f. Similarly, a submanifold $\widetilde{\mathbb{N}}$ is said to be 2-pseudo-parallel if $\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\cdot\widetilde{\nabla}\hbar$=$f_{\widehat{\mathcal{Q}}}(\widetilde{Ric},\widetilde{\nabla}\hbar)$ and it is Ricci generalized pseudo-parallel [23] if $\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\cdot\hbar$=$f_{\widehat{\mathcal{Q}}}(\widetilde{Ric},\hbar)$ for any $\mathcal{E},\mathcal{F}\in\widetilde{\mathbb{V}}$.

3. Invariant submanifold of a hyperbolic Sasakian manifold

A submanifold $\widetilde{\mathbb{N}}$ of an n-dimensional hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is said to be invariant if the structure vector field ξ is tangent to $\widetilde{\mathbb{N}}$ at every point of $\widetilde{\mathbb{N}}$ and $\phi\mathcal{E}$ is tangent to $\widetilde{\mathbb{N}}$ for every vector field $\mathcal{E}$ tangent to $\widetilde{\mathbb{N}}$ at every point of $\widetilde{\mathbb{N}}$. i.e., $\phi(\mathcal{T}\widetilde{\mathbb{N}})\subset\mathcal{T}\widetilde{\mathbb{N}}$ at every point of $\widetilde{\mathbb{N}}$.(see [7], [17], [26], [32], [18], [19], [20]).
In an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ [7], we have

$$\widetilde{\nabla}_{\mathcal{E}}\xi = -\phi\mathcal{E}, \tag{3.1}$$

$$\hbar(\mathcal{E},\xi)=0,\quad \hbar(\mathcal{E},\phi\mathcal{F})=\hbar(\phi\mathcal{E},\mathcal{F})=\phi\hbar(\mathcal{E},\mathcal{F}), \tag{3.2}$$

$$(\widetilde{\nabla}_{\mathcal{E}}\phi)(\mathcal{F})=(\nabla_{\mathcal{E}}\phi)(\mathcal{F})+\hbar(\mathcal{E},\phi\mathcal{F})-\phi\hbar(\mathcal{E},\mathcal{F}), \tag{3.3}$$

$$\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\xi=\eta(\mathcal{F})\mathcal{E}-\eta(\mathcal{E})\mathcal{F}, \tag{3.4}$$

$$\widetilde{\mathcal{R}}(\xi,\mathcal{E})\mathcal{F}=g(\mathcal{E},\mathcal{F})\xi-\eta(\mathcal{F})\mathcal{E}, \tag{3.5}$$

$$\widetilde{\mathcal{Q}}\xi=(n-1)\xi,\quad \widetilde{Ric}(\mathcal{E},\xi)=(n-1)\eta(\mathcal{E}). \tag{3.6}$$

In light of the aforementioned outcome, we aim to study the invariant submanifold of a hyperbolic Sasakian manifolds satisfying some geometric conditions such as $\widehat{\mathcal{Q}}(\hbar,\widetilde{\mathcal{R}})$=0, $\widehat{\mathcal{Q}}(\widetilde{Ric},\hbar)$=0, $\widehat{\mathcal{Q}}(\widetilde{Ric},\nabla\hbar)$=0, $\widehat{\mathcal{Q}}(\widetilde{Ric},\widetilde{\mathcal{R}}\cdot\hbar)$=0, $\widehat{\mathcal{Q}}(g,\mathcal{Z}\cdot\hbar)$=0 and $\widehat{\mathcal{Q}}(\widetilde{Ric},\mathcal{Z}\cdot\hbar)$=0. Sections 5 and 6 concern with the study of conformal η-Ricci-Yamabe solitons on invariant submanifolds of hyperbolic Sasakian manifolds and obtain some interesting results. Next we have investigate invariant submanifold whose second fundamental form $\hbar$ satisfies [29]

$$(\nabla_{\mathcal{E}}\hbar)(\mathcal{F},\mathcal{G})=\epsilon_1(\mathcal{E})\hbar(\mathcal{F},\mathcal{G})+\epsilon_2(\mathcal{F})\hbar(\mathcal{E},\mathcal{G})+\epsilon_3(\mathcal{G})\hbar(\mathcal{E},\mathcal{F}), \tag{3.7}$$

where ϵ_1, ϵ_2 and ϵ_3 are non-zero 1-forms defined by $\epsilon_1(\mathcal{E})$=$g(\mathcal{E},\theta_1)$, $\epsilon_2(\mathcal{E})$=$g(\mathcal{E},\theta_2)$ and $\epsilon_2(\mathcal{E})$=$g(\mathcal{E},\theta_3)$ in section 7. Finally, in section 8 we construct a non-trivial example of an invariant submanifold of hyperbolic Sasakian manifold which verify the result.

Theorem 3.1. *An invariant submanifold of a hyperbolic Sasakian manifold satisfies* $\widehat{\mathcal{Q}}(\hbar,\widetilde{\mathcal{R}})$=0 *if and only if it is totally geodesic.*

Proof. The condition $\widehat{\mathcal{Q}}(\hbar,\widetilde{\mathcal{R}})(\mathcal{E},\mathcal{F},\mathcal{G};\mathcal{U},\mathcal{V})$=0 on $\widetilde{\mathbb{N}}$ with the help of (**??**) implies

$$\widetilde{\mathcal{R}}((\mathcal{U}\wedge_{\hbar}\mathcal{V})\mathcal{E},\mathcal{F})\mathcal{G}-\widetilde{\mathcal{R}}(\mathcal{E},(\mathcal{U}\wedge_{\hbar}\mathcal{V})\mathcal{F})\mathcal{G}-\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})(\mathcal{U}\wedge_{\hbar}\mathcal{V})\mathcal{G}=0, \tag{3.8}$$

where $(\mathcal{U}\wedge_{\hbar}\mathcal{V})$ is the endomorphism and it is defined by

$$(\mathcal{U}\wedge_{\hbar}\mathcal{V})\mathcal{W}=\hbar(\mathcal{V},\mathcal{W})\mathcal{U}-\hbar(\mathcal{U},\mathcal{W})\mathcal{V}. \tag{3.9}$$

With the help of (3.9) and (3.8), we get

$$\begin{aligned}
&- \hbar(\mathcal{V},\mathcal{E})\widetilde{\mathcal{R}}(\mathcal{U},\mathcal{F})\mathcal{G}+\hbar(\mathcal{U},\mathcal{E})\widetilde{\mathcal{R}}(\mathcal{V},\mathcal{F})\mathcal{G}\\
&- \hbar(\mathcal{V},\mathcal{F})\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\mathcal{G}+\hbar(\mathcal{U},\mathcal{F})\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{V})\mathcal{G}\\
&- \hbar(\mathcal{V},\mathcal{G})\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\mathcal{G}+\hbar(\mathcal{U},\mathcal{G})\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\mathcal{G}=0.
\end{aligned} \tag{3.10}$$

For fix $\mathcal{V}$=$\mathcal{G}$=ξ in (3.10), using (3.2) and (3.5), we have

$$\hbar(\mathcal{U},\mathcal{E})[\eta(\mathcal{F})\xi+\mathcal{F}]+\hbar(\mathcal{U},\mathcal{F})[\eta(\mathcal{E})\xi-\mathcal{E}]=0. \tag{3.11}$$

After contracting (3.11) over $\mathcal{F}$, we yields $\hbar(\mathcal{U},\mathcal{E})$=0. Thus the manifold is totally geodesic. Conversely, if $\hbar(\mathcal{E},\mathcal{F})$=0, for any vector fields $\mathcal{E}$ and $\mathcal{F}$ on $\widetilde{\mathbb{N}}$, then it follows from (3.10) that $\widehat{\mathcal{Q}}(\hbar,\widetilde{\mathcal{R}})$=0. This proves the theorem. □

Theorem 3.2. *An invariant submanifold of a hyperbolic Sasakian manifold satisfies* $\widehat{\mathcal{Q}}(\widetilde{Ric},\hbar)$=0 *if and only if it is totally geodesic.*

Proof. Let the invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ satisfies $\widehat{\mathcal{Q}}(Ric,\hbar)$=0. Then we have

$$\begin{aligned}
0 &= \widehat{\mathcal{Q}}(\widetilde{Ric},\hbar)(\mathcal{E},\mathcal{F};\mathcal{U},\mathcal{V})\\
&= -\hbar((\mathcal{U}\wedge_{\widetilde{Ric}}\mathcal{V})\mathcal{E},\mathcal{F})-\hbar(\mathcal{E},(\mathcal{U}\wedge_{\widetilde{Ric}}\mathcal{V})\mathcal{F}),
\end{aligned} \tag{3.12}$$

where $(\mathcal{U}\wedge_{\widetilde{Ric}}\mathcal{V})\mathcal{W}$ is defined as

$$(\mathcal{U}\wedge_{\widetilde{Ric}}\mathcal{V})\mathcal{W}=\widetilde{Ric}(\mathcal{V},\mathcal{W})\mathcal{U}-\widetilde{Ric}(\mathcal{U},\mathcal{W})\mathcal{V}. \tag{3.13}$$

By virtue of (3.13), we get from (3.12), that

$$\begin{aligned}
&- \widetilde{Ric}(\mathcal{V},\mathcal{E})\hbar(\mathcal{U},\mathcal{F})+\widetilde{Ric}(\mathcal{U},\mathcal{E})\hbar(\mathcal{V},\mathcal{F})\\
&- \widetilde{Ric}(\mathcal{V},\mathcal{F})\hbar(\mathcal{E},\mathcal{U})+\widetilde{Ric}(\mathcal{U},\mathcal{F})\hbar(\mathcal{E},\mathcal{V})=0.
\end{aligned} \tag{3.14}$$

After taking $\mathcal{U}$=$\mathcal{F}$=ξ in (3.14), using (3.2) and (3.6) we yields

$$(n-1)\hbar(\mathcal{E},\mathcal{V})=0, \tag{3.15}$$

which implies that $\hbar(\mathcal{E},\mathcal{V})$=0, for any vector fields $\mathcal{E}$ and $\mathcal{E}$ on $\widetilde{\mathbb{V}}$. So, $\widetilde{\mathbb{N}}$ is totally geodesic. Conversely, statement is obvious. This proves the theorem. □

Theorem 3.3. *An invariant submanifold of a hyperbolic Sasakian manifold is totally geodesic if and only if* $\widehat{\mathcal{Q}}(\widetilde{Ric}, \widetilde{\nabla}\hbar)$=0.

Proof. We assume that $\widehat{\mathcal{Q}}(\widetilde{Ric}, \widetilde{\nabla}\hbar)$=0, on $\widetilde{\mathbb{N}}$ implies that

$$\widehat{\mathcal{Q}}(\widetilde{Ric}, \widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F}) \cdot \hbar)(\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}) = 0. \tag{3.16}$$

Using (**??**), equation (3.16) can be written as

$$\begin{aligned} - \quad & (\nabla_{\mathcal{E}}\hbar)(\widetilde{Ric}(\mathcal{W},\mathcal{G})\mathcal{V},\mathcal{U}) + (\nabla_{\mathcal{E}}\hbar)(\widetilde{Ric}(\mathcal{V},\mathcal{G})\mathcal{W},\mathcal{U}) \\ - \quad & (\nabla_{\mathcal{E}}\hbar)(\mathcal{G},\widetilde{Ric}(\mathcal{W},\mathcal{U})\mathcal{V}) + (\nabla_{\mathcal{E}}\hbar)(\mathcal{G},\widetilde{Ric}(\mathcal{V},\mathcal{U})\mathcal{W}) = 0. \end{aligned} \tag{3.17}$$

For fix, $\mathcal{G}$=$\mathcal{U}$=$\mathcal{V}$=ξ in (3.17), using (3.1) we get

$$2m\hbar(\mathcal{U}, \phi\mathcal{E}) = 0, \tag{3.18}$$

which implies that $\hbar(\mathcal{U}, \phi\mathcal{E})$=0. That is, $\widetilde{\mathbb{N}}$ is totally geodesic. So, the proof is finished. □

Theorem 3.4. *An invariant submanifold of a hyperbolic Sasakian manifold is totally geodesic if and only if* $\mathcal{Q}(\widetilde{Ric}, \widetilde{\mathcal{R}} \cdot \hbar)$=0.

Proof. It follows from the condition $\widehat{\mathcal{Q}}(Ric, \widetilde{\mathcal{R}} \cdot \hbar)$=0 on $\widetilde{\mathbb{N}}$ that

$$\widehat{\mathcal{Q}}(\widetilde{Ric}, \widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F}) \cdot \hbar)(\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}) = 0, \tag{3.19}$$

for any vector fields $\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}$ on $\widetilde{\mathbb{N}}$. In view of (**??**), above equation can be written as

$$\begin{aligned} - \quad & (\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\hbar(\widetilde{Ric}(\mathcal{W},\mathcal{G})\mathcal{V},\mathcal{U}) + (\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\hbar(\widetilde{Ric}(\mathcal{V},\mathcal{G})\mathcal{W},\mathcal{U}) \\ - \quad & (\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\hbar(\mathcal{G},\widetilde{Ric}(\mathcal{W},)\mathcal{U}),\mathcal{V}) + (\widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\hbar(\mathcal{G},\widetilde{Ric}(\mathcal{V},)\mathcal{U}),\mathcal{W}). \end{aligned} \tag{3.20}$$

After taking $\mathcal{F}$=$\mathcal{G}$=$\mathcal{U}$=$\mathcal{W}$=ξ in (3.20), we have

$$\widetilde{Ric}(\xi,\xi)\hbar(\widetilde{\mathcal{R}}(\mathcal{E},\xi)\xi,\mathcal{V}) = 0. \tag{3.21}$$

Using (3.4) and (3.6), we get from (3.21) that $2m\,\hbar(\mathcal{E},\mathcal{V})$=0, which implies that $\hbar(\mathcal{E},\mathcal{V})$=0. This complete the proof. □

4. Invariant submanifolds admitting $\widehat{\mathcal{Q}}(g, \mathcal{Z} \cdot \hbar)$=0 and $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{Z} \cdot \hbar)$=0

The notion of generalized quasi-conformal curvature (briefly,GQC) tensor [30] and is defined on an $(2n+1)$-dimensional manifold $\widetilde{\mathbb{V}}$

$$\begin{aligned} \mathcal{Z}(\mathcal{E},\mathcal{F})\mathcal{G} \;=\; & \frac{2n-1}{2n+1}\left[(1+2na-b) - \{1+2n(a+b)\}c\right]\mathcal{C}(\mathcal{E},\mathcal{F})\mathcal{G} \\ + \; & [1-b+2na]\mathcal{D}(\mathcal{E},\mathcal{F})\mathcal{G} + 2n(b-a)\mathcal{P}(\mathcal{E},\mathcal{F})\mathcal{G} \\ + \; & \frac{2n-1}{2n+1}(c-1)[1+2n(a+b)]\mathcal{C}^{\perp}(\mathcal{E},\mathcal{F})\mathcal{G}, \end{aligned} \tag{4.1}$$

for all vector fields $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ on $\widetilde{\mathbb{V}}$ and $a, b, c \in \mathbb{R}$. The GQC curvature tensor is the generalization of Riemann curvature tensor $\mathcal{R}$ for a=b=c=0; Conformal curvature tensor $\mathcal{C}$[11] for a=b=$-\frac{1}{2n-1}$, c=1; Conharmonic curvature tensor $\mathcal{C}^{\perp}$[21] for a=b=$-\frac{1}{2n-1}$, c=0; Concircular curvature tensor $\mathcal{D}$[31] for a=b=0, c=1; Projective curvature tensor $\mathcal{P}$[31] for a=-$\frac{1}{2n}$, b=0, c=0 and m-projective curvature tensor $\mathcal{M}$[25], for a=b=$-\frac{1}{4n}$, c=0.
After simplification (4.1) on $\widetilde{\mathbb{V}}$ takes the form

$$\begin{aligned} \mathcal{Z}(\mathcal{E},\mathcal{F})\mathcal{G} &= \widetilde{\mathcal{R}}(\mathcal{E},\mathcal{F})\mathcal{G} + a[\widetilde{Ric}(\mathcal{F},\mathcal{G})\mathcal{E} - \widetilde{Ric}(\mathcal{E},\mathcal{G})\mathcal{F}] \\ &+ b[g(\mathcal{F},\mathcal{G})\widetilde{\mathcal{Q}}\mathcal{E} - g(\mathcal{E},\mathcal{G})\widetilde{\mathcal{Q}}\mathcal{F}] \\ &- \frac{c\tau}{2n+1}(\frac{1}{2n} + a + b)[g(\mathcal{F},\mathcal{G})\mathcal{E} - g(\mathcal{E},\mathcal{G})\mathcal{F}], \end{aligned} \tag{4.2}$$

where τ being the scalar curvature of the manifold. So, first we lead to the following:

Theorem 4.1. *An invariant submanifold of a hyperbolic Sasakian manifold is totally geodesic if and only if* $\widehat{\mathcal{Q}}(g, \mathcal{Z}\cdot\hbar)$=0*, provided* $\tau \neq \frac{2m(2m+1)}{c}$.

Proof. We suppose that $\widehat{\mathcal{Q}}(g, \mathcal{Z}\cdot\hbar)$=0 on $\widetilde{\mathbb{N}}$ implies that

$$\widehat{\mathcal{Q}}(g, \mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}) = 0,$$

for any vector fields $\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}$ on $\widetilde{\mathbb{V}}$. With the help of (**??**), above equation can be written as

$$\begin{aligned} -g(\mathcal{W},\mathcal{G})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{V},\mathcal{U}) &+ g(\mathcal{V},\mathcal{U})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{W},\mathcal{U}) \\ &- g(\mathcal{W},\mathcal{U})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{G},\mathcal{V}) \\ &+ g(\mathcal{V},\mathcal{U})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{G},\mathcal{W}) = 0. \end{aligned} \tag{4.3}$$

On substituting $\mathcal{F}$=$\mathcal{U}$=$\mathcal{G}$=$\mathcal{V}$=ξ in (4.3), we have

$$\widetilde{Ric}(\xi,\xi)\hbar(\mathcal{Z}(\mathcal{E},\xi)\xi,\mathcal{W}) = 0, \tag{4.4}$$

which implies with the help of (4.2) that

$$\left[\frac{c\tau}{2m+1}(\frac{1}{2m} + a + b) - 2m(a+b) - 1\right]\hbar(\mathcal{E},\mathcal{W}) = 0.$$

We obtain the statement of Theorem 4.1. □

As per above consequence, we can state the following corollaries

Corollary 4.2. *An invariant submanifold of a hyperbolic Sasakian manifold is totally geodesic for each of (i)* $\widehat{\mathcal{Q}}(g, \mathcal{C}\cdot\hbar)$=0*,(ii)*$\widehat{\mathcal{Q}}(g, \mathcal{D}\cdot\hbar)$=0*, provided* $\tau \neq 2m(2m+1)$.

Corollary 4.3. *Let* $\widetilde{\mathbb{N}}$ *be an invariant submanifold of a hyperbolic Sasakian manifold* $\widetilde{\mathbb{V}}$*. Then for each of (i)* $\widehat{\mathcal{Q}}(g, \mathcal{C}^{\perp}\cdot\hbar)$=0*, (ii)* $\widehat{\mathcal{Q}}(g, \mathcal{P}\cdot\hbar)$=0 *and (iii)*$\widehat{\mathcal{Q}}(g, \mathcal{M}\cdot\hbar)$=0*,* $\widetilde{\mathbb{N}}$ *is not totally geodesic.*

Now we prove our next result.

Theorem 4.4. *An invariant submanifold of a hyperbolic Sasakian manifold is totally geodesic if and only if* $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{Z}\cdot\hbar)$=0*, provided* $\tau \neq \frac{2m(2m+1)}{c}$.

Proof. Assuming that $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{Z}\cdot\hbar)$=0, we have

$$\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}) = 0,$$

for any vector fields $\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{U},\mathcal{V},\mathcal{W}$ on $\mathbb{V}$. By virtue of (**??**), above equation can be written as

$$\begin{aligned} -\widetilde{Ric}(\mathcal{W},\mathcal{G})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{V},\mathcal{U}) &+ \widetilde{Ric}(\mathcal{V},\mathcal{U})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{W},\mathcal{U}) \\ &- \widetilde{Ric}(\mathcal{W},\mathcal{U})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{G},\mathcal{V}) \\ &+ \widetilde{Ric}(\mathcal{V},\mathcal{U})(\mathcal{Z}(\mathcal{E},\mathcal{F})\cdot\hbar)(\mathcal{G},\mathcal{W}) = 0. \end{aligned} \tag{4.5}$$

After replacing $\mathcal{F}$=$\mathcal{U}$=$\mathcal{G}$=$\mathcal{V}$=ξ in (4.5), we obtain

$$\widetilde{Ric}(\xi,\xi)\hbar(\mathcal{Z}(\mathcal{E},\xi)\xi,\mathcal{W}) = 0, \tag{4.6}$$

which implies with the help of (4.2) that

$$\left[\frac{c\tau}{2m+1}(\frac{1}{2m}+a+b) - 2m(a+b) - 1\right]\hbar(\mathcal{E},\mathcal{W}) = 0.$$

This finished the proof. □

On behalf of theorem 4.4, we state the following corollaries

Corollary 4.5. *An invariant submanifold of a hyperbolic Sasakian manifold is totally geodesic for each of (i)* $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{C}\cdot\hbar)$=0,*(ii)*$\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{D}\cdot\hbar)$=0*, provided* $\tau \neq 2m(2m+1)$.

Corollary 4.6. *Let* $\widetilde{\mathbb{N}}$ *be an invariant submanifold of a hyperbolic Sasakian manifold* $\widetilde{\mathbb{V}}$*. Then for each of (i)* $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{C}^{\perp}\cdot\hbar)$=0*, (ii)* $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{P}\cdot\hbar)$=0 *and (iii)* $\widehat{\mathcal{Q}}(\widetilde{Ric}, \mathcal{M}\cdot\hbar)$=0*,* $\widetilde{\mathbb{N}}$ *is not totally geodesic.*

5. Conformal η-Ricci-Yamabe solitons on invariant submanifold

Let (g,ξ,κ,l,μ,ν) be the conformal η-Ricci-Yamabe soliton of type (κ,l) on invariant submanifold of a hyperbolic Sasakian manifold. Then from (1.6), we have

$$(\mathfrak{L}_\xi g)(\mathcal{E},\mathcal{F}) + 2\kappa\widetilde{Ric}(\mathcal{E},\mathcal{F}) + [2\mu - l\tau - \frac{1}{n}(\tilde{p}n+2)]g(\mathcal{E},\mathcal{F}) + 2\nu\eta(\mathcal{E})\eta(\mathcal{F}) = 0. \tag{5.1}$$

Since $\widetilde{\mathbb{N}}$ is invariant in $\widetilde{\mathbb{V}}$, then $-\phi\mathcal{E},\xi \in \mathcal{T}\widetilde{\mathbb{N}}$, then using (3.1) and (3.2), we get

$$(\mathfrak{L}_\xi g)(\mathcal{E},\mathcal{F}) = g(\widetilde{\nabla}_{\mathcal{E}}\xi,\mathcal{F}) + g(\mathcal{E},\widetilde{\nabla}_{\mathcal{F}}\xi) = 0. \tag{5.2}$$

Adopting (5.1) and (5.2) we yields

$$\widetilde{Ric}(\mathcal{E},\mathcal{F}) = [\frac{1}{2n\kappa}(n\tilde{p}+2) + \frac{l\tau}{2\kappa} - \frac{\mu}{\kappa}]g(\mathcal{E},\mathcal{F}) - \frac{\nu}{\kappa}\eta(\mathcal{E})\eta(\mathcal{F}), \tag{5.3}$$

which implies that $\widetilde{\mathbb{N}}$ is η-Einstein. Also from (2.13) and (3.2), $\eta(\mathcal{E})\mathcal{H}$=0, i.e., $\mathcal{H}$=0 and therefore $\widetilde{\mathbb{N}}$ is minimal in $\widetilde{\mathbb{V}}$. So, we turn up the result:

Theorem 5.1. *If $(g, \xi, \kappa, l, \mu, \nu)$ be the conformal η-Ricci-Yamabe soliton of type (κ, l) on invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$. Then we have*
(i) $\widetilde{\mathbb{N}}$ is η-Einstein,
(ii) $\widetilde{\mathbb{N}}$ is minimal and
(iii) ξ is a killing vector field in $\widetilde{\mathbb{V}}$.

We obtain the following results for different value of (κ=1, l=0, κ=0, l=1, and κ=1, l=1 (cf. [6],[15],[16]):

Corollary 5.2. *If $(g, \xi, l, \kappa, \mu, \nu)$ be as the conformal η-Ricci soliton or conformal η-Yamabe soliton or conformal η-Einstein soliton on invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$. Then we obtain the following*
(i) $\widetilde{\mathbb{N}}$ is η-Einstein,
(ii) $\widetilde{\mathbb{N}}$ is minimal and
(iii) ξ is a killing vector field in $\widetilde{\mathbb{V}}$.

Also from (5.3) we turn up

$$\kappa\,\widetilde{Ric}(\xi, \xi) = \mu - \frac{1}{2n}(n\tilde{p} + 2) - \frac{l\tau}{2} - \nu. \tag{5.4}$$

Equating (5.4) and (3.6), we get

$$\mu - \nu = \frac{1}{2n}(n\tilde{p} + 2) + \frac{l\tau}{2} - \kappa(n - 1). \tag{5.5}$$

In particular, if we put ν=0, then conformal η-Ricci-Yamabe soliton of type (κ, l) becomes conformal Ricci-Yamabe soliton of type (κ, l) with μ=$\frac{1}{2n}(n\tilde{p}+2)$+$\frac{l\tau}{2}$-$\kappa(n-1)$. So, we state the result:

Theorem 5.3. *Let $(g, \xi, \kappa, l, \mu, \nu)$ be the conformal η-Ricci-Yamabe solitons of type (κ, l) on invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$. Then μ=$\frac{1}{2n}(n\tilde{p} + 2)$+$\frac{l\tau}{2}$+$\nu$-$\kappa(n - 1)$. Also, the conformal Ricci-Yamabe solitons on $\widetilde{\mathbb{N}}$ is steady, expanding or shrinking according as τ=$\frac{1}{l}[2\kappa(n - 1) - (\tilde{p} + \frac{2}{n})]$, $\tau > \frac{1}{l}[2\kappa(n - 1) - (\tilde{p} + \frac{2}{n})]$, or $\tau < \frac{1}{l}[2\kappa(n - 1) - (\tilde{p} + \frac{2}{n})]$, respectively.*

Also, in view of Theorem 5.3, one can state the followings corollaries.

Corollary 5.4. *A conformal Ricci soliton $(g, \xi, \mu, 1, 0)$ on an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is steady, expanding or shrinking according as $\tilde{p}$=$[2(n + 1) - \frac{2}{n}]$, $\tilde{p} > [2(n + 1) - \frac{2}{n}]$ or, $\tilde{p} < [2(n + 1) - \frac{2}{n}]$,respectively.*

Corollary 5.5. *A Ricci soliton $(g, \xi, \mu, 1, 0)$ on an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is always shrinking.*

Corollary 5.6. *A conformal Yamabe soliton $(g, \xi, \mu, 0, 1)$ on an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is steady, expanding or shrinking according as* τ=-$(\tilde{p} + \frac{2}{n})$, $\tau > -(\tilde{p} + \frac{2}{n})$ *or,* $\tau < -(\tilde{p} + \frac{2}{n})$,*respectively.*

Corollary 5.7. *A Yamabe soliton $(g, \xi, \mu, 0, 1)$ on an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is steady, expanding or shrinking depending upon the sign of scalar curvature τ.*

Corollary 5.8. *A conformal Einstein soliton $(g, \xi, \mu, 1, -1)$ on an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is steady, expanding or shrinking according as* τ=$[2(n-1) + (\tilde{p} + \frac{2}{n})]$, $\tau > [2(n-1) + (\tilde{p} + \frac{2}{n})]$ *or,* $\tau < [2(n-1) + (\tilde{p} + \frac{2}{n})]$,*respectively.*

Corollary 5.9. *An Einstein soliton $(g, \xi, \mu, 1, -1)$ on an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is steady, expanding or shrinking according as* τ=$-2(n-1)$, $\tau > -2(n-1)$ *or,* $\tau < -2(n-1)$,*respectively.*

6. Conformal η-Ricci-Yamabe soliton with concircular vector field on invariant submanifolds

A vector field π on a (semi-) Riemannian manifold $\mathbb{N}$ is said to be a concircular vector field (or, geodesic fields) [13] if it satisfies

$$\nabla_{\mathcal{E}}\pi = \psi\mathcal{E} \tag{6.1}$$

for any $\mathcal{E} \in T\mathbb{N}$, where ∇ denotes the Levi-Civita connection of the metric g and ψ is a non-trivial smooth function on $\mathbb{N}$. Recently, Chen [16] studied the properties of Ricci solitons on submanifolds of a Riemannian manifold equipped with a concircular vector field. Particularly, if we choose ψ=1 in equation (6.1), then the concircular vector field π is called concurrent vector field. Then from Lemma 4.1[chaubey], we can write

$$\pi = \pi^t + \pi^{\perp}, \tag{6.2}$$

where $\pi \in T\mathbb{N}$, $\pi^t \in \mathcal{D}$ and $\pi \in \mathcal{D}^{\perp}$. Now, for a concircular vector field π on $\widetilde{\mathbb{V}}$, from (6.1) we have

$$\psi\mathcal{E} = \widetilde{\nabla}_{\mathcal{E}}\pi^t + \widetilde{\nabla}_{\mathcal{E}}\pi^{\perp}, \tag{6.3}$$

for any $\mathcal{E} \in \mathcal{D}$. Using (2.9) and (2.10) and comparing the tangential and normal components , we yields

$$\hbar(\mathcal{E}, \pi^t) = -\nabla^{\perp}_{\mathcal{E}}\pi^{\perp}, \quad \nabla_{\mathcal{E}}\pi^t = \psi\mathcal{E} - \mathcal{A}_{\pi^{\perp}}\mathcal{E}. \tag{6.4}$$

Now, we can state the following

Theorem 6.1. *Let $\widetilde{\mathbb{N}}$ be an invariant submanifold of $\widetilde{\mathbb{V}}$ admitting conformal η-Ricci-Yamabe soliton with concircular vector field π. Then the Ricci tensor $\widetilde{Ric}_{\mathcal{D}}$ on the* *invariant distribution $\mathcal{D}$ is given by*

$$\begin{aligned}\widetilde{Ric}_{\mathcal{D}}(\mathcal{E},\mathcal{F}) &= -\frac{1}{\kappa}\{(\psi+\mu-\frac{l\tau}{2}-\frac{1}{2n}(n\tilde{p}+2))g(\mathcal{E},\mathcal{F}) \\ &- g(\hbar(\mathcal{E},\mathcal{F}),\pi^{\perp})+\nu\eta(\mathcal{E})\eta(\mathcal{F})\} \qquad (6.5)\end{aligned}$$

for any vector field $\mathcal{E}$,$\mathcal{F} \in \mathcal{D}$.

Proof. Adopting (6.4) together with the definition of Lie-derivative, we have

$$(\mathfrak{L}_{\pi^t} g)(\mathcal{E},\mathcal{F}) = 2\psi g(\mathcal{E},\mathcal{F}) - 2g(\hbar(\mathcal{E},\mathcal{F}),\pi^{\perp}). \qquad (6.6)$$

Let the invariant submanifold $\widetilde{\mathbb{N}}$ admits conformal η-Ricci-Yamabe soliton, so from (1.6) we yields

$$(\mathfrak{L}_{\pi^t} g)(\mathcal{E},\mathcal{F})+2\kappa\widetilde{Ric}_{\mathcal{D}}(\mathcal{E},\mathcal{F})+[2\mu-l\tau-\frac{1}{n}(\tilde{p}n+2)]g(\mathcal{E},\mathcal{F})+2\nu\eta(\mathcal{E})\eta(\mathcal{F}) = 0. \quad (6.7)$$

Therefore, using (6.6) and (6.7), we can easily get the required result (6.5). □

Particularly, if we choose π is a concurrent vector field and (g,ξ,κ,l,μ,ν) is conformal η-Ricci-Yamabe soliton in $\widetilde{\mathbb{N}}$ of $\widetilde{\mathbb{V}}$. Then by same fashion as the above consequence, we can state:

Corollary 6.2. *If an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ admits conformal η-Ricci-Yamabe soliton with concurrent vector field π. Then the invariant distribution $\mathcal{D}$ of $\widetilde{\mathbb{N}}$ is an η-Einstein, provided the invariant distribution $\mathcal{D}$ of $\widetilde{\mathbb{N}}$ is $\mathcal{D}$-geodesic.*

Corollary 6.3. *Let $\widetilde{\mathbb{N}}$ be an invariant submanifold of $\widetilde{\mathbb{V}}$ admitting conformal η-Ricci-Yamabe soliton with concurrent vector field π. Then the Ricci tensor $\widetilde{Ric}_{\mathcal{D}}$ on the invariant distribution $\mathcal{D}$ is given by*

$$\begin{aligned}\widetilde{Ric}_{\mathcal{D}}(\mathcal{E},\mathcal{F}) &= -\frac{1}{\kappa}\{(1+\mu-\frac{l\tau}{2}-\frac{1}{2n}(n\tilde{p}+2))g(\mathcal{E},\mathcal{F}) \\ &- g(\hbar(\mathcal{E},\mathcal{F}),\pi^{\perp})+\nu\eta(\mathcal{E})\eta(\mathcal{Y})\} \qquad (6.8)\end{aligned}$$

for any vector field $\mathcal{E}$,$\mathcal{F} \in \mathcal{D}$.

Corollary 6.4. *Assume that an invariant submanifold $\widetilde{\mathbb{N}}$ of $\widetilde{\mathbb{V}}$ admits conformal η-Ricci-Yamabe soliton with a concurrent vector field π. If the invariant distribution $\mathcal{D}$ of $\widetilde{\mathbb{N}}$ is $\mathcal{D}$-geodesic, then the invariant distribution $\mathcal{D}$ is η-Einstein.*

Finally, with the help of Theorem 6.1, we obtain the following corollaries for different value of (κ=1, l=0; κ=0, l=1 and κ=1, l=1):

Corollary 6.5. *If an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ admits conformal η-Ricci-Yamabe soliton with a concircular vector field π, then the*

Ricci tensor $\widetilde{Ric}_{\mathcal{D}}$ on the invariant distribution $\mathcal{D}$ is given by

$$\widetilde{Ric}_{\mathcal{D}}(\mathcal{E},\mathcal{F}) = -\left\{(\psi+\mu-\frac{1}{2n}(n\tilde{p}+2))g(\mathcal{E},\mathcal{F}) - g(\hbar(\mathcal{E},\mathcal{F}),\pi^{\perp}) + \nu\eta(\mathcal{E})\eta(\mathcal{F})\right\}, \tag{6.9}$$

for any vector field $\mathcal{E}$,$\mathcal{F} \in \mathcal{D}$.

Corollary 6.6. *If an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ admits conformal η-Yamabe soliton with a concircular vector field π, then the scalar curvature on the invariant distribution $\mathcal{D}$ is given by*

$$\tau = 2(\psi+\mu-\nu) - (\tilde{p}+\frac{2}{n}). \tag{6.10}$$

Corollary 6.7. *If an invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ admits conformal η-Einstein soliton with a concircular vector field π, then the Ricci tensor $\widetilde{Ric}_{\mathcal{D}}$ on the invariant distribution $\mathcal{D}$ is given by*

$$\widetilde{Ric}_{\mathcal{D}}(\mathcal{E},\mathcal{F}) = -\left\{(\psi+\mu+\frac{\tau}{2}-\frac{1}{2n}(n\tilde{p}+2))g(\mathcal{E},\mathcal{F}) - g(\hbar(\mathcal{E},\mathcal{F}),\pi^{\perp}) + \nu\eta(\mathcal{E})\eta(\mathcal{F})\right\} \tag{6.11}$$

for any vector field $\mathcal{E},\mathcal{F} \in \mathcal{D}$.

7. Invariant submanifold whose second fundamental form $\hbar$ is weakly symmetric type

In this section, we assume that invariant submanifold $\widetilde{\mathbb{V}}$ has parallel second fundamental form. Then from (2.14), we have

$$(\nabla_{\mathcal{E}}\hbar)(\mathcal{F},\mathcal{G}) = \nabla^{\perp}_{\mathcal{E}}(\hbar(\mathcal{F},\mathcal{G})) - \hbar(\nabla_{\mathcal{E}}\mathcal{F},\mathcal{G}) - \hbar(\mathcal{F},\nabla_{\mathcal{E}}\mathcal{G}), \tag{7.1}$$

For fix, $\mathcal{G}$=ξ in (7.1), using (3.1) and (3.2), we get

$$(\nabla_{\mathcal{E}}\hbar)(\mathcal{F},\xi) = \phi\hbar(\mathcal{E},\mathcal{F}). \tag{7.2}$$

In view of (3.7) and (3.2) one can easily bring out

$$(\nabla_{\mathcal{E}}\hbar)(\mathcal{F},\xi) = \epsilon_3(\xi)\hbar(\mathcal{E},\mathcal{F}). \tag{7.3}$$

Equating (7.2) and (7.3), we yields

$$[\phi - \epsilon_3(\xi)]\hbar(\mathcal{E},\mathcal{F}) = 0. \tag{7.4}$$

Thus, we turn up to the following:

Theorem 7.1. *An invariant submanifold $\widetilde{\mathbb{N}}$ of a hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}$ is totally geodesic if second fundamental form $\hbar$ is of the types (i) symmetric, (ii) recurrent, (iii) pseudo symmetric, (iv) almost pseudo symmetric and (v) weakly pseudo symmetric, provided $\phi \neq \epsilon_3(\xi)$.*

8. Example

Let $\Re^n$ be an n-dimensional space of real number and we define $\widetilde{\mathbb{V}}^5$=$\{(x,y,z,u,v)\in \Re^5\}$. Let $\{e_1,e_2,e_3,e_4,e_5\}$ be a set of linearly independent vector fields of $\widetilde{\mathbb{V}}^5$ given by

$$e_1=-\alpha y\frac{\partial}{\partial z},\ e_2=\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z},\ e_4=-\alpha v\frac{\partial}{\partial z}+\frac{\partial}{\partial u},\ e_5=\frac{\partial}{\partial v}.$$

We define the metric $\widetilde{g}$ of $\widetilde{\mathbb{V}}^5$ by the following relation

$$\widetilde{g}(e_2,e_2)=\widetilde{g}(e_3,e_3)=\widetilde{g}(e_5,e_5)=-1,\ \widetilde{g}(e_1,e_1)=\widetilde{g}(e_4,e_4)=1.$$

Let η be the 1-form defined by $\eta(\mathcal{E})$=$g(\mathcal{E},e_3)$, for any $\mathcal{E}\in\widetilde{\mathbb{V}}^5$ and the $(1,1)$-tensor field ϕ of $\widetilde{\mathbb{V}}^5$ as

$$\phi e_1=e_2,\ \phi e_2=e_1,\ \phi e_3=0,\ \phi e_4=e_5,\ \phi e_5=e_4.$$

By the use of linearity properties of ϕ and g, we have

$$\phi^2e_i=e_i+\eta(e_i)\xi,\ \ \eta(e_3)=-1,$$

hold for i=$1,2,3,4,5$ and ξ=e_3. Also, for ξ=e_3, $\widetilde{\mathbb{V}}^5$ satisfies $g(e_i,e_3)$=$\eta(e_i)$, $g(\phi e_i,e_j)$=-$g(e_i,\phi e_j)$ and $g(\phi e_i,\phi e_j)$=$g(\phi e_i,e_j)+\eta(e_i)\eta(e_j)$, where i,j=$1,2,3,4,5$. Now, we can easily compute

$$[e_i,e_j]=\begin{cases}\alpha e_3, & \text{if } i=1,\ j=2.\\ \alpha e_3, & \text{if } i=4,\ j=5.\\ 0, & \text{otherwise.}\end{cases}$$

Using Koszul's formula, we obtain

$$\begin{aligned}
&\widetilde{\nabla}_{e_1}e_1=0,\ \ \widetilde{\nabla}_{e_1}e_2=-\frac{\alpha}{2}e_3,\ \ \widetilde{\nabla}_{e_1}e_3=\frac{\alpha}{2}e_2,\ \ \widetilde{\nabla}_{e_1}e_4=0,\ \ \widetilde{\nabla}_{e_1}e_5=0,\\
&\widetilde{\nabla}_{e_2}e_1=\frac{\alpha}{2}e_3,\ \ \widetilde{\nabla}_{e_2}e_2=0,\ \ \widetilde{\nabla}_{e_2}e_3=\frac{\alpha}{2}e_1\ \ \widetilde{\nabla}_{e_2}e_4=0,\ \ \widetilde{\nabla}_{e_2}e_5=0,\\
&\widetilde{\nabla}_{e_3}e_1=\frac{\alpha}{2}e_2,\ \ \widetilde{\nabla}_{e_3}e_2=\frac{\alpha}{2}e_1,\ \ \widetilde{\nabla}_{e_3}e_3=0,\ \ \widetilde{\nabla}_{e_3}e_4=\frac{\alpha}{2}e_5,\ \ \widetilde{\nabla}_{e_3}e_5=\frac{\alpha}{2}e_4,\\
&\widetilde{\nabla}_{e_4}e_1=0,\ \ \widetilde{\nabla}_{e_4}e_2=0,\ \ \widetilde{\nabla}_{e_4}e_3=\frac{\alpha}{2}e_5\ \ \widetilde{\nabla}_{e_4}e_4=o,\ \ \widetilde{\nabla}_{e_4}e_5=\frac{\alpha}{2}e_3,\\
&\widetilde{\nabla}_{e_5}e_1=0,\ \ \widetilde{\nabla}_{e_5}e_2=0,\ \ \widetilde{\nabla}_{e_5}e_3=\frac{\alpha}{2}e_4\ \ \widetilde{\nabla}_{e_5}e_4=\frac{\alpha}{2}e_3,\ \ \widetilde{\nabla}_{e_5}e_5=0.
\end{aligned}$$

Thus for e_3=ξ and α=-2, it can be easily verifies that $\nabla_{\mathcal{E}}\xi$=-$\phi\mathcal{E}$ for all $\mathcal{E}\in\mathcal{T}\widetilde{\mathbb{V}}^5$. So, the manifold $\widetilde{\mathbb{V}}^5$ equipped with the structure (ϕ,ξ,η,g) is an almost hyperbolic Sasakian manifold of dimension 5.

Let $\widetilde{f}$ be an isometric immersion from $\mathbb{N}$ to $\widetilde{\mathbb{V}}$ defined by $\widetilde{f}(x,y,z)$=$(x,y,z,0,0)$. Then we define $\mathbb{N}$=$\{(x,y,z)\in\Re^3: z\neq 0\}$, where (x,y,z) are the standard coordinates in $\Re^3$. Let $\{e_1,e_2,e_3\}$ be linearly independent global frame on $\mathbb{N}$ given by

$$e_1=-\alpha y\frac{\partial}{\partial z},\ e_2=\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}.$$

Thus we have

$$\widetilde{g}(e_1,e_1)=1,\ \widetilde{g}(e_2,e_2)=\widetilde{g}(e_3,e_3)=-1.$$

Also, the $(1,1)$-tensor field ϕ of $\widetilde{\mathbb{N}}^3$ as follows

$$\phi e_1=e_2,\ \phi e_2=e_1,\ \phi e_3=0.$$

Adopting the linearity properties of ϕ and g, we have

$$\phi^2 e_i=e_i+\eta(e_i)\xi,\ \eta(\xi)=-1,$$

holds for i=1, 2, 3 and ξ=e_3. Again, for ξ=e_3, $\widetilde{\mathbb{N}}^3$ satisfies

$$g(\phi e_i,\phi e_j)=-g(e_i,e_j)-\eta(e_i)\eta(e_j),$$

where i,j=1, 2, 3. Next, one can easily get

$$[e_1,e_2]=\alpha e_3,\quad [e_1,e_3]=0,\quad [e_2,e_3]=0.$$

By the use of Koszul's formula, we have

$$\widetilde{\nabla}_{e_1}e_1=0,\quad \widetilde{\nabla}_{e_1}e_2=-\frac{\alpha}{2}e_3,\quad \widetilde{\nabla}_{e_1}e_3=\frac{\alpha}{2}e_2,\quad \widetilde{\nabla}_{e_2}e_1=\frac{\alpha}{2}e_3,\quad \widetilde{\nabla}_{e_2}e_2=0,$$
$$\widetilde{\nabla}_{e_2}e_3=\frac{\alpha}{2}e_1,\quad \widetilde{\nabla}_{e_3}e_1=\frac{\alpha}{2}e_2,\quad \widetilde{\nabla}_{e_3}e_2=\frac{\alpha}{2}e_1,\quad \widetilde{\nabla}_{e_3}e_3=0.$$

It is obvious that for e_3=ξ and α=-2, it satisfies $\nabla_{\mathcal{E}}\xi$=-$\phi\mathcal{E}$ for all $\mathcal{E}\in\mathcal{T}\widetilde{\mathbb{N}}^3$. Thus, the manifold $\widetilde{\mathbb{N}}^3$ equipped with the structure (ϕ,ξ,η,g) is a hyperbolic Sasakian manifold of dimension 3. We define the tangent space $\mathcal{T}\widetilde{\mathbb{N}}$ of $\widetilde{\mathbb{N}}^3$ takes the form

$$T\widetilde{\mathbb{V}}=\mathcal{D}\oplus\mathcal{D}^{\perp}\oplus<\xi>,$$

where $\mathcal{D}$=$<e_1>$ and $\mathcal{D}^{\perp}$=$<e_2>$. Then we notice that ϕe_1=-$e_2\in\mathcal{D}^{\perp}$, for $e_1\in\mathcal{D}$ and ϕe_2=-$e_1\in\mathcal{D}$, for $e_2\in\mathcal{D}^{\perp}$. Hence, we can say that $\widetilde{\mathbb{N}}^3$ under consideration is an invariant submanifold of $\widetilde{\mathbb{V}}^5$. Also, from (2.9) we have $\hbar(e_i,e_j)=\widetilde{\nabla}_{e_i}e_j-\nabla_{e_i}e_j$. Thus from the values of $\widetilde{\nabla}_{e_i}e_j$ and $\nabla_{e_i}e_j$, we notice that $\hbar(e_i,e_j)$=0, $\forall\, i,\ j=1,2,3$. This leads to the submanifold is totally geodesic. So, Theorems 3.1, 3.2, 3.3, 3.4, 4.1 and 4.4 are verified.

The non-vanishing components of the curvature tensor $\mathcal{R}$ using the preceding relations

$$\widetilde{\mathcal{R}}(e_1,e_2)e_1=-\frac{\alpha^2}{4}e_2,\ \widetilde{\mathcal{R}}(e_1,e_3)e_1=-\frac{\alpha^2}{4}e_3,\ \widetilde{\mathcal{R}}(e_1,e_2)e_2=-\frac{\alpha^2}{4}e_1,$$
$$\widetilde{\mathcal{R}}(e_2,e_3)e_2=\frac{\alpha^2}{4}e_3,\ \widetilde{\mathcal{R}}(e_1,e_3)e_3=-\frac{\alpha^2}{4}e_1,\ \widetilde{\mathcal{R}}(e_2,e_3)e_3=-\frac{\alpha^2}{4}e_2$$

Also the Ricci tensor $\mathcal{S}$ and scalar curvature τ as:

$$\widetilde{Ric}(e_1,e_1)=\frac{\alpha^2}{2},\ \widetilde{Ric}(e_2,e_2)=\widetilde{Ric}(e_3,e_3)=-\frac{\alpha^2}{2}\ \text{and}\ \tau=-\frac{\alpha^2}{2}. \tag{8.1}$$

Since, $\widetilde{\mathbb{N}}^3$ in invariant on $\widetilde{\mathbb{V}}^5$. Then, we set $\mathcal{E}$=$\mathcal{F}$=e_3 into the identity (5.2), get $(\mathfrak{L}_\xi g)(e_3, e_3)$=0. So from (5.1) and (8.1) which provides

$$\mu - \nu = \frac{1}{2}(\tilde{p} + \frac{2}{3}) - \frac{l\alpha^2}{4} - \frac{\kappa\alpha^2}{2}. \quad (8.2)$$

Hence the above equation proves that, μ and ν satisfies our result (5.5) for n=3 and α=-2 and g gives a conformal η-Ricci–Yamabe soliton of type (κ, l) on the 3-dimensional hyperbolic Sasakian submanifold $\widetilde{\mathbb{N}}^3$ of the 5-dimensional hyperbolic Sasakian manifold $\widetilde{\mathbb{V}}^3$. Thus, we can conclude that the Theorem 5.3 and Corollary5.5,Corollary5.6, Corollary5.7, Corollary5.8, Corollary5.9 hold on $\widetilde{\mathbb{N}}^3$.

References

[1] K. Arslan, U. Lumiste, C. Murathn and C. Ozgür, 2-semiparallel surfaces in space forms, *I: two particular cases, Proc. Estonian Acad. Sci. Phys. Math.* **49***3 (2000), 139–148.*

[2] M. Ahmad, S. A. Khan and T. Khan, On non-invariant hypersurfaces of a nearly hyperbolic Sasakian manifold, *Int. J. Math.* **28** *(8) (2017) 1750064.*

[3] A. C. Asperti, G.A. Lobos and F. Mercuri, Pseudo-parallel immersions in space forms, Mat. Contemp. 17 (1999). 59–70.

[4] L. Bhatt and K. K. Dube, On CR-submanifolds of trans hyperbolic Sasakian manifold, *Acta Cienc. Indica* **29** *(1) (2003), 91–96.*

[5] N. Basu and A. Bhattacharyya, Conformal Ricci soliton in Kenmotsu manifold, *Global Journal of Advanced Research on Classical and Modern Geometries.* **4***, (2015), 15-21.*

[6] G. Catino and L. Mazzieri, Gradient Einstein solitons, *Nonlinear Anal.,* **132** *(2016), 66-94.*

[7] B.-Y. Chen, Geometry of Submanifolds, Dekker, New York, 1973.

[8] B.-Y. Chen, Pseudo-Riemannian Geometry, δ-Invariants and Applications, World Scientific: Hackensack, NJ, USA, 2011.

[9] U. C. De, A. Sardar, and K. De, Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds, *Turkish Journal of Mathematics,* **46***, (3), (2022), 1078–1088.*

[10] J. Deprez, Semi-parallel surfaces in Euclidean space, *J. Geom.,* **25***(2) (1985), 192–200.*

[11] L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1949.

[12] A. E. Fischer, An introduction to conformal Ricci flow, *Classical and Quantum Gravity,* **21***, (3), (2004), 171–218.*

[13] A. Fialkow, Conformal geodesic, Trans. Ammr. Math. Soc., **45**(3) (1939), 443–473.

[14] S. Guler and M. Crasmareanu, Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy, *Turkish Journal of Mathematics,* **43***, (5), (2019), 2631-2641.*

[15] R. S. Hamilton, Three Manifold with positive Ricci curvature, *J. Differ. Geom.,* **17** *(1982), 255–306.*

[16] R.S. Hamilton, The Ricci flow on surfaces, *Contemp. Math.,* **71** *(1988), 237–261.*

[17] S. K. Hui, S. Uddin, A. H. Alkhaldi and P. Mandal, Invariant submanifolds of generalized Sasakian-space-forms, *Int. J. Geom. Methods Mod. Phys.,* **15** *(09) (2018).*

[18] S. K. Hui and J. Roy, Invariant and anti-invariant submanifolds of special quasi-Sasakian manifolds, *Journal of Geometry,* **109** *(2) 16 pages, (2018).*

[19] S. K. Hui, L. I. Piscoran and T. Pal, Invariant submanifolds of $(\mathcal{LCS})_n$-manifolds with respect to quarter symmetric metric connection, *Acta Mathematica Universitatis Comenianae,* **87** *(2018), 205-221.*

[20] S. K. Hui, S. Kishor and J. Roy, Chaki-pseudo parallel invariant submanifold of Sasakian manifolds with respect to certain connections, *Ganita,* **68** *(2) (2018), 19-29.*

[21] Y.Ishii, On conharmonic transformations, *Tensor (N.S.),* **7** *(1957), 73–80.*

[22] N. K. Joshi and K. K. Dube, Semi invariant submanifolds of an almost r-contact hyperbolic metric manifold, *Demonstr. Math.* **34***(1) (2001), 135-143.*

[23] C. Murathan, K. Arslan and E. Ezentas, Ricci generalized pseudo-parallel symmetric immersions, *Differential geometry and its applications, pp. 99–108, Matfyzpress, Prague, 2005.*

[24] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[25] G.P. Pokhariyal and R. S. Mishra, Curvatur tensors and their relativistics significance I, *Yokohama Mathematical Journal,* **18***, (1970), 105–108.*

[26] A. Sarkar, M. Sen, On invariant submanifolds of LP-Sasakian manifolds, *Extracta Mathematicae* **27***(1) (2012), 145–154.*

[27] M. D. Upadhyay and K. K. Dube, Almost contact hyperbolic-(f, g, η, ξ) structure, *Acta Math. Acad. Sci. H.* **28***, (1-2) (1976), 1-4.*

[28] L. Verstraelen, Comments on pseudo-symmetry in the sense of Ryszard Deszcz, in: Geometry and Topology of Submanifolds, VI (Leuven, 1993/Brussels, 1993), pp. 199–209, World Scientific, River Edge, NJ, 1994.

[29] K. Yano, and M. Kon, Structures on manifolds, Series in Pure Mathematics, World Scientific, Singapore, 1984.

[30] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Diff. Geom.,* **2***,(1968), 161–184.*

[31] K. Yano and S. Bochner, Curvature and Betti numbers, *Annals of Mathematics Studies 32, Princeton University Press, 1953.*

[32] A. Yildiz and C. Murathan, Invariant submanifolds of Sasakian-space-forms, *J. Geom.* **95** *(2009), 135–150.*

[33] P. Zhang, Y. Li, S. Roy, S. Dey and A. Bhattacharyya, Geometrical structure in a perfect fluid spacetime with conformal Ricci-Yamabe soliton, *Symmetry,* **14***, (3), (2022), p. 594.*