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Co-identity join graph of lattices

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Abstract. Let $\pounds$ be a lattice with 1 and 0. The co-identity join graph of $\pounds$, denoted by $\mathbb{CG}(\pounds)$, is an undirected simple graph whose vertices are all nontrivial elements (i.e. different from 1 and 0) of $\pounds$ and two distinct elements x and y are adjacent if and only if $x \vee y \neq 1$. The basic properties and possible structures of this graph are studied and the interplay between the algebraic properties of $\pounds$ and the graph-theoretic structure of $\mathbb{CG}(\pounds)$ is investigated.

Keywords: Lattice; Small element; co-identity join graph.

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1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded.

Recently, a lot of study of algebraic structures has been explored via the graph theoretic approach. A basic question about this representation is, what graphs can represent algebraic structures? Attempts to

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answer this question involve looking at graph properties such as the chromatic number and maximal clique size to find rules about possible graph structures. In 1988, Beck [1] proposed the study of commutative rings by representing them as graphs, called zero divisor graph. These zero divisor graphs marked the beginning of an approach to studying commutative rings with graphs. Similarly, there is several graphs assigned to rings, modules and lattices. [3, 8-13]. One of the most important graphs which have been studied is the intersection graph. Bosak [2] in 1964 defined the intersection graph of semigroups. In 1969, Csakany and Pollak [6] studied the graph of subgroups of a finite group. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [3]. In 2016, co-intersection graph of submodules of a module has been investigated by Mahdavi and Talebi [10] (also see [8, 9, 11]). Motivated by such graphs, The aim of this paper is to introduce a graph associated to a lattice $\pounds$ called the co-identity join graph of lattices. This will result in characterization of lattices in terms of some specific properties of those graphs.

For a given bounded lattice $\pounds$ the co-identity join graph of $\pounds$ is a simple graph $\mathbb{CG}(\pounds)$ whose vertices are nontrivial elements and two distinct vertices x and y are adjacent if and only if $x \vee y \neq 1$. Among many results in this paper, the first, introductional section contains elementary observations needed later on. Section 2 concentrates on lattices $\pounds$ such that the associated graphs are not connected. Theorem 2.12 shows that the graph $\mathbb{CG}(\pounds)$ is not connected if and only if $1 = a_1 \oplus a_2$ for some atoms a_1 and a_2. An information about the structures of lattices $\pounds$ such that $\mathbb{CG}(\pounds)$ is a connected graph are also given. We characterize all of lattices for which the co-identity join graph of lattices are connected. Also the diameter and the girth of $\mathbb{CG}(\pounds)$ are determined.

Section 3 concentrates on lattices $\pounds$ such that the associated graphs contains vertices of small degree. Theorems 3.6 shows that $\mathbb{CG}(\pounds)$ is finite if and only if there exists an atom element a such that $\deg(a) < \infty$. The remaining part of this section is mainly devoted to investigation of lattices $\pounds$ such that $\mathbb{CG}(\pounds)$ contains a vertex of degree 1. Theorem 3.8 offers necessary and sufficient condition for an element c of $\pounds$ to be of degree 1 as a vertex in $\mathbb{CG}(\pounds)$.

Section 4 is mainly devoted to investigation clique number, chromatic number and domination number of $\mathbb{CG}(\pounds)$. We also study the condition under which the chromatic number of $\mathbb{CG}(\pounds)$ is finite (Theorem 4.7). It is shown in Theorem 4.10 that $\deg(a) < \infty$ for some atom element a of $\pounds$ if and only if $\mathbb{CG}(\pounds)$ is a finite graph if and only if $\omega(\mathbb{CG}(\pounds)) < \infty$.

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. Let G be a simple graph. The vertex set of G is denoted by $\mathcal{V}(G)$, $\deg_G(v)$ stands for the degree of $v \in \mathcal{V}(G)$, i.e. the cardinality of the set of all vertices which are adjacent to v. The maximum and minimum degrees of the graph G are the maximum and minimum degree of its vertices and are denoted by $\Delta(G)$ and $\delta(G)$, respectively. By a null graph, we mean a graph with no edges. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices. Let $u, v \in \mathcal{V}(G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(u, v) = \infty$. The diameter of G is $\mathrm{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\mathrm{gr}(G) = \infty$.

A tree is a connected graph which does not contain a cycle. A star graph is a tree consisting of one vertex adjacent to all the others. By a clique in a graph G, we mean a complete subgraph of G and the number of vertices in a largest clique of G, is called the clique number of G and is denoted by $\omega(G)$. A subset $S \subseteq \mathcal{V}(G)$ is independence if no two vertices of S are adjacent. An independence number of G, written $\alpha(G)$, is the maximum size of an independence set. For a graph G, let $\chi(G)$, denote the chromatic number of G, i.e., the minimum number of colors which can be assigned to the vertices of G such that every two adjacent vertices have different colors. By a dominating set D in a graph G, we mean a subset D of the vertex set $\mathcal{V}(G)$ such that every vertex in $\mathcal{V}(G) \setminus D$ is adjacent to at least one vertex in D. The domination number of G, written $\gamma(G)$, is the smallest cardinality of the cardinalities of the dominating sets of G. For terminology and notation not defined here, the reader is referred to [15].

By a lattice we mean a poset $(\pounds, \leq)$ in which every couple elements x, y has a greatest lower bound (called the meet of x and y, and written $x \wedge y$) and a least upper bound (called the join of x and y, and written $x \vee y$). A lattice $\pounds$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\pounds$. Setting $X = \pounds$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1). A lattice $\pounds$ is called modular if $(c \wedge b) \vee a = (c \vee a) \wedge b$ for all $a, b, c \in \pounds$ with

$a \leq b$. Let for $a, b \in \pounds$, $[a, b] = \{x \in \pounds : a \leq x \leq b\}$. Obviously, $[a, b]$ is a sublattice of $\pounds$ and $\pounds = [0, 1]$.

If $\pounds$ is a lattice, then $\pounds$ is Noetherian (resp. Artinian) if any non-empty set of elements of $\pounds$ has a maximal member (resp. minimal member) with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on elements of $\pounds$. A composition chain between a and b is a chain $a = a_0 < a_1 < \cdots < a_n = b$ which has no refinement, except by introducing repitition of the given elements a_i. The integer n is the length of the chain. A modular lattice $\pounds$ is of finite length (denoted by $l(\pounds) < \infty$) if there is a composition chain between 0 and 1. In this case, $\pounds$ is finite length if and only if it is both Noetherian and Artinian [14, Proposition 3.13].

We say that an element x in a lattice $\pounds$ is an atom (resp. coatom) if there is no $y \in \pounds$ such that $0 < y < x$ (resp. $x < y < 1$). The set of all coatom (resp. atom) elements of $\pounds$ is denoted by $\mathcal{CA}(\pounds)$ (resp. $\mathcal{A}(\pounds)$). The radical of $\pounds$ is the meet of all coatom elements of $\pounds$, and is denoted as $\mathrm{rad}(\pounds)$ (i.e. $\mathrm{rad}(\pounds) = \bigwedge_{c \in \mathcal{CA}(\pounds)} c$).

In a lattice with 1 an element $a \in \pounds$ is called small, denoted by $a \ll \pounds$, if $a \vee b \neq 1$ holds for every $b \neq 1$. A lattice $\pounds$ is called hollow if every non-zero element in $\pounds$ is small.

A nonzero element x of a lattice $\pounds$ is called semisimple, if for each element y of $\pounds$ with $y < x$, there exists an element z of $\pounds$ such that $x = y \vee z$ and $y \wedge z = 0$. In this case, we say that y is a direct join of x, and we write $x = y \oplus z$. A lattice $\pounds$ is called semisimple if 1 is semisimple in $\pounds$. Notice that if every chain of a non-empty poset A has an upper bound, then A has at least one maximal element (Zorn's Lemma). For terminology and notation not defined here, the reader is referred to [4, 14].

2. Basic properties of $\mathbb{CG}(\pounds)$

Let us begin this section with the following easy observation:

Lemma 2.1. *Let $\pounds$ be a complete lattice. If a is a vertex of the graph $\mathbb{CG}(\pounds)$, then there exists a coatom c of $\pounds$ such that $a \leq c$.*

Proof. Set $\Sigma = \{b : b$ is an element of £ with $a \leq b < 1\}$. Then $\Sigma \neq \emptyset$ since $a \in \Sigma$. Moreover, $(\Sigma, \leq)$ is a partial order. Clearly, Σ is closed under taking joins of chains and so the result follows by Zorn's Lemma. □

Proposition 2.2. *Let $\pounds$ be a complete lattice with the connected graph $\mathbb{CG}(\pounds)$. If $\mathrm{rad}(\pounds) \neq 0$, then $\mathbb{CG}(\pounds)$ has a universal vertex.*

Proof. In order to establish this result, consider $\mathrm{rad}(\pounds)$. It is easy to see that $\mathrm{rad}(\pounds) \neq 1$. We claim that $\mathrm{rad}(\pounds) \ll \pounds$. Assume to the contrary, that $\mathrm{rad}(\pounds)$ is not small in $\pounds$. Then there exists a nontrivial element b of $\pounds$ such that $b \vee \mathrm{rad}(\pounds) = 1$. By Lemma 2.1, there is a coatom element c of $\pounds$ with $b \leq c$ which gives $1 \leq c \vee \mathrm{rad}(\pounds) = c$, a contradiction. So $\mathrm{rad}(\pounds) \ll \pounds$. It follows that $\mathrm{rad}(\pounds)$ is a nontrivial element of $\pounds$ and for each nontivial element x of $\pounds$, we have $x \vee \mathrm{rad}(\pounds) \neq 1$. Hence, $\mathbb{CG}(\pounds)$ has the vertex $\mathrm{rad}(\pounds)$ which is adjacent to every other vertex. □

Lemma 2.3. *Let $a_1, \cdots, a_n \in \pounds$. Then the following hold:*

(1) If $a_1 \leq a_2$ and $a_2 \ll \pounds$, then $a_1 \ll \pounds$;

(2) If $a_1 \ll \pounds, \cdots, a_n \ll \pounds$, then $\bigwedge_{i=1}^n a_i, \bigvee_{i=1}^n a_i \ll \pounds$;

(3) If $\pounds$ is modular and $a_1 \ll [0, a_2]$, then $a_1 \ll \pounds$;

(4) If $a_1 \ll [a_2, 1]$ and $a_2 \ll \pounds$, then $a_1 \ll \pounds$.

Proof. (1) Let $a_1 \vee c = 1$ for some $c \in \pounds$. Then $1 = a_1 \vee c \leq a_2 \vee c$ gives $c = 1$ since a_2 is small in $\pounds$. So $a_1 \ll \pounds$.

(2) Since $\bigwedge_{i=1}^n a_i \leq a_1$, $\bigwedge_{i=1}^n a_i \ll \pounds$ by (1). Let $(a_1 \vee a_2) \vee c = 1$ for some $c \in \pounds$. Then $a_2 \ll \pounds$ gives $a_1 \vee c = 1$; hence $c = 1$, as $a_1 \ll \pounds$. Now $\bigvee_{i=1}^n a_i \ll \pounds$ is obtained by induction.

(3) Let $a_1 \vee c = 1$ for some element c of $\pounds$. Since $a_1 \leq a_2$, $a_2 = a_2 \wedge (a_1 \vee c) = a_1 \vee (a_2 \wedge c)$ by modularity condition. Now, $a_1 \ll [0, a_2]$ gives $a_2 \wedge c = 1$; so $c = 1$. Thus $a_1 \ll \pounds$.

(4) If $1 \neq c \in \pounds$, then $c \vee a_2 \neq 1$ gives $a_2 \vee c \vee a_1 = c \vee a_1 \neq 1$, as needed. □

Lemma 2.4. *Let a, b be elements of a modular lattice $\pounds$ such that b is a direct join of 1 with $a \leq b$. Then $a \ll \pounds$ if and only if $a \ll [0, b]$.*

Proof. If $a \ll [0, b]$, then $a \ll \pounds$ by Lemma 2.3 (3). Conversely, assume that $a \ll \pounds$ and $1 = b \vee c$ with $b \wedge c = 0$. Let $a \vee d = b$ for some $d \in [0, b]$. Then $1 = b \vee c = a \vee (c \vee d)$ gives $c \vee d = 1$ since $a \ll \pounds$; hence $b = b \wedge (c \vee d) = d \vee (b \wedge c) = d$ by modularity condition. Thus $a \ll [0, b]$. □

Proposition 2.5. *Let $\pounds$ be a lattice. Then the following hold:*

(1) Let x be a nontrivial element of $\pounds$. Then $x \ll \pounds$ if and only if x is a universal vertex of the graph $\mathbb{CG}(\pounds)$;

(2) If $|\mathcal{V}\ (\mathbb{CG})| = n$ for some positive integer n, then x is a nontrivial small element of $\pounds$ if and only if $\deg(x) = n - 1$;

(3) If $a_1, \cdots, a_n$ are universal vertices of the graph $\mathbb{CG}(\pounds)$, then $\bigwedge_{i=1}^n a_i$ and $\bigvee_{i=1}^n a_i$ are universal vertex of $\mathbb{CG}(\pounds)$;

(4) If $a < b < 1$ and b is a direct join of 1, then a is a universal vertex of the graph $\mathbb{CG}(\pounds)$ if and only if a is a universal vertex of the graph $\mathbb{CG}([0, b])$.

Proof. (1) Let y be any vertex of $\mathbb{CG}(\pounds)$. If $x \vee y = 1$, then x is small gives $y = 1$ which is impossible. Hence x is a universal vertex. The other implication is clear.

(2) It is clear by (1).

(3) This follows from (1) and Lemma 2.3 (2).

(4) this follows from (1) and Lemma 2.4. □

Proposition 2.6. *Let $\pounds$ be a lattice with the graph $\mathbb{CG}(\pounds)$. Then $\pounds$ is a hollow lattice if and only if $\mathbb{CG}(\pounds)$ is a complete graph.*

Proof. Assume that $\pounds$ is a hollow lattice and let x, y be distinct vertices of $\mathbb{CG}(\pounds)$. Then $x \ll \pounds$ and $y \ll \pounds$ gives $x \vee y \neq 1$. Hence, $\mathbb{CG}(\pounds)$ is a complete graph. Conversely, let x be a nontrivial element of L. Since $\mathbb{CG}(\pounds)$ is complete, x adjacent to every other vertex of $\mathbb{CG}(\pounds)$; so $x \vee y \neq 1$ for every element $y \neq 1$ of $\pounds$. It follows that $x \ll \pounds$. Therefore, L is a hollow lattice. □

Example 2.7. Let $\pounds = \{0, a, b, c, d, e, 1\}$ be a lattice with the relations $0 < e < a < b < c < 1$, $0 < e < a < d < c < 1$, $b \wedge d = a$ and $b \vee d = c$. An inspection will show that the co-identity join graph of the lattice $\pounds$ is a complete graph by Proposition 2.6 since $\pounds$ is hollow.

Proposition 2.8. *If every nonzero non-small element in $\pounds$ is an atom, then $\mathbb{CG}(\pounds)$ is a complete graph.*

Proof. Let $\pounds$ be a lattice in which every nonzero non-small element is an atom. Let x, y be two distinct vertices of $\mathbb{CG}(\pounds)$. If one of x and y is a small element, then $x \backsim y$ is an edge in $\mathbb{CG}(\pounds)$ by Proposition 2.5 (1). Suppose that neither x nor y is a small element. Then by assumption, x and y are atoms. Hence, $0 < x < x \vee y$ and so $x \vee y$ is not an atom and so it is a small element in $\pounds$ (so $x \vee y \neq 1$). This implies that x and y are adjacent. Therefore, $\mathbb{CG}(\pounds)$ is a complete graph. □

Proposition 2.9. *If $\pounds$ is an Artinian lattice and contains a unique atom element, then $\mathbb{CG}(\pounds)$ is a connected graph.*

Proof. By assumption, $\pounds$ has at least one atom element. Also, $\pounds$ is Artinian gives if $0 < b$ is an element of $\pounds$, then there is an atom a of $\pounds$ such that $a \leq b$. Therefore, if $\pounds$ possesses a unique atom element, say a, then $a \leq x$ for every nonzero element x of $\pounds$. Suppose that x and y are two distinct vertices of $\mathbb{CG}(\pounds)$. Then $a \leq x$ and $a \leq y$; so $x \vee a = x \neq 1$ and $y \vee a = y \neq 1$. Then there exists a path $x \backsim a \backsim y$ of length 2; hence $\mathbb{CG}(\pounds)$ is connected. □

Theorem 2.10. *Let $\pounds$ be a lattice with the graph $\mathbb{CG}(\pounds)$. If $\Delta(\mathbb{CG}(\pounds)) = n < \infty$ and $\delta(\mathbb{CG}(\pounds)) = \delta \geq 1$, then the following hold:*

(1) $l(\pounds) \leq n + 1$;

(2) If a is a nontrivial element of $\pounds$, then $[0, a]$ has finitely many elements;

(3) If x is a nontrivial element of $\pounds$, then there exist an atom a and a coatom c such that $a < x < c$.

Proof. (1) Assume to the contrary, that $\pounds$ is not Noetherian. Suppose that $a_1 < a_2 < \cdots$ is an infinite strictly increasing sequence of non-trivial elements of $\pounds$. Since $\delta \geq 1$, there is an element b of $\pounds$ such that $a_{n+1} \vee b \neq 1$. Thus $a_i \vee b \leq a_{n+1} \vee b$ gives $a_i \vee b \neq 1$ for each $1 \leq i \leq n+1$; hence $\deg(b) \geq n+1$ which is impossible. Therefore, $\pounds$ is Noetherian. Now, we assume that $c_1 > c_2 > \cdots$ is an infinite strictly decreasing sequence of non-trivial elements of $\pounds$. Since $\delta \geq 1$, there is an element d of $\pounds$ such that $d \vee c_1 \neq 1$ which implies that $d \vee c_i \neq 1$ for each $i \geq 1$ and hence $\deg(d) = \infty$, a contradiction. Hence $l(\pounds) \leq n+1$.

(2) Since $\delta \geq 1$, there is an element b of $\pounds$ such that $a \vee b \neq 1$. Then for every element z of $[0, a]$, $z \vee b \neq 1$ and since $n < \infty$, the number of elements of $[0, a]$ is finite.

(3) By (1), $\pounds$ is Noetherian and Artinian lattice. Let x be a nontrivial element of $\pounds$. As $\pounds$ is Noetherian, it possesses a coatom element c such that $x < c$ and as $\pounds$ is Artinian, it possesses an atom element a such that $a < x$, as required. □

Corollary 2.11. *Let $\pounds$ be an Artinian lattice such that it contains a unique atom element and $\Delta(\mathbb{CG}(\pounds)) < \infty$. Then $\pounds$ is a Noetherian lattice.*

Proof. By Assumption, Proposition 2.9 gives $\mathbb{CG}(\pounds)$ is a connected graph; so $\delta(\mathbb{CG}(\pounds)) \geq 1$. Now the assertion follows from Theorem 2.10 (1). □

We are now in a position to show a finer relationship between atom elements of $\pounds$ and the connectivity of $\mathbb{CG}(\pounds)$.

Theorem 2.12. *Let $\pounds$ be a modular lattice. Then the graph $\mathbb{CG}(\pounds)$ is not connected if and only if $1 = a_1 \oplus a_2$ for some atoms a_1 and a_2.*

Proof. Suppose that $\mathbb{CG}(\pounds)$ is not connected and let V_1 and V_2 be two components of $\mathbb{CG}(\pounds)$. Let $a_1 \in V_1$ and $a_2 \in V_2$. Since a_1 and a_2 are not adjacent, then $a_1 \vee a_2 = 1$. If $a_1 \wedge a_2 \neq 0$, then $a_1 \vee (a_1 \wedge a_2) = a_1 \neq 1$ and $a_2 \vee (a_1 \wedge a_2) = a_2 \neq 1$ implies that there exists a path $a_1 \backsim a_1 \wedge a_2 \backsim a_2$ with starting vertex a_1 and end vertex a_2 which is impossible. Thus $a_1 \wedge a_2 = 0$ and so $a_1 \oplus a_2 = 1$. Now we show that a_1 and a_2 are atoms. Let b be an element of $\pounds$ such that $0 < b \leq a_1$. Then $b \vee a_1 = a_1 \neq 1$ gives b and a_1 are adjacent vertices which implies that $b \in V_1$. Hence b and a_2 are not adjacent vertices and so $b \vee a_2 = 1$. By assumption,

$a_1 = a_1 \wedge (b \vee a_2) = b \vee (a_2 \wedge a_1) = b$. Thus a_1 is an atom. Simiarly, a_2 is an atom.

Conversely, let $1 = a_1 \oplus a_2$, where a_1, a_2 are atoms. Let $b \in \mathcal{V}(\mathbb{CG}(\pounds))$. We claim that either $b = a_1$ or $b = a_2$. It is easy to see that either $b \wedge a_i = 0$ or $a_i \leq b$ $(i = 1, 2)$. If $a_1 \leq b$ and $a_2 \leq b$, then $b = 1$ which is impossible. If $a_1 \wedge b = 0$ and $a_2 \wedge b = 0$, then $b = b \wedge (a_1 \vee a_2) = 0$, a contradiction. Without loss of generality, let $a_1 \leq b$ and $b \wedge a_2 = 0$. Then $b = b \wedge (a_1 \vee a_2) = a_1 \vee (a_2 \wedge b) = a_1$ which implies that $\mathcal{V}(\mathbb{CG}(L)) = \{a_1, a_2\}$ with $a_1 \vee a_2 = 1$. Thus the graph $\mathbb{CG}(\pounds)$ is not connected. □

The following corollary we determine the conditions under which the graph $\mathbb{CG}(\pounds)$ is connected (indeed, it is a direct consequence of Theorem 2.12).

Corollary 2.13. *Let £ be a modular lattice. Then $\mathbb{CG}(\pounds)$ is connected if and only if either 1 is not semisimple or $1 = \oplus_{i=1}^{n} a_i$, where $n \geq 3$ and for each i, a_i is an atom element of £.*

Corollary 2.14. *Assume that £ is a modular lattice and let $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. Then the following hold:*

(1) If $\mathbb{CG}(\pounds)$ is not connected, then $\mathbb{CG}(\pounds)$ is a null graph;

(2) If $\mathbb{CG}(\pounds)$ has at least an edge, then $\mathbb{CG}(\pounds)$ is a connected graph.

Proof. (1) Suppose that $\mathbb{CG}(\pounds)$ is not connected. Then Theorem 2.12 gives $1 = a_{\oplus} a_2$ for some atoms a_1 and a_2. By an argument like that in the proof of Theorem 2.2, for each two distinct vertices b and c, we have $b \vee c = 1$; hence there is no edge between two distinct vertices b and c of the graph $\mathbb{CG}(\pounds)$. Therefore $\mathbb{CG}(\pounds)$ is a null graph.

(2) It is clear by (1). □

Example 2.15. (1) Let $D = \{a, b\}$. Then $\pounds(D) = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in \pounds(D)$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). It can be easily seen that $\mathcal{A}(\pounds) = \{\{a\}, \{b\}\}$ and $1 = \{a\} \oplus \{b\}$. Thus we observe that the co-identity join graph of the lattice $\pounds(D)$ is not connected by Theorem 2.12.

Example 2.16. The collection of ideals of Z, the ring of integers, form a lattice under set inclusion which we shall denote by $\pounds(Z)$ with respect to the following definitions: $mZ \vee nZ = (m, n)Z$ and $mZ \wedge nZ = [m, n]Z$ for all ideals mZ and nZ of Z, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $\pounds(Z)$ is a distributive complete lattice with least element the zero ideal and the greatest element Z [6]. If mZ is a nontrivial element of $\pounds(Z)$, then $0 < 2mZ < mZ$ gives mZ is not an atom element of $\pounds(Z)$; so $\mathcal{A}(\pounds) = \emptyset$. It can be easily seen that $\mathcal{CA}(\pounds) = \{pZ : p \text{ is a prime number}\}$.

Moreover, if mZ and nZ are distinct nontrivial elements of $\pounds(Z)$, then $nZ \backsim mnZ \backsim mZ$ gives the co-identity join graph of the lattice $\pounds(Z)$ is connected.

Lemma 2.17. *Let $\pounds$ be a modular lattice. If $1 = c \oplus a$ for some coatom c, then a is an atom.*

Proof. Assume to the contrary, that a is not an atom. Then $0 < b < a$ for some $b \in \pounds$; so $c \wedge b = 0$. Then $c < c \vee b$ gives $c \vee b = 1$. Hence, $a = a \wedge (c \vee b) = b \vee (c \wedge a) = b$ by modularity condition which is a contradiction. Thus a is an atom. □

Proposition 2.18. *Let $\pounds$ be a modular lattice. If $\mathbb{CG}(\pounds)$ is a connected graph, then the following conditions hold:*

(1) If c_1 and c_2 are distinct coatoms of $\pounds$, then $c_1 \wedge c_2 \neq 0$ and there is an edge between them;

(2) If a_1 and a_2 are distinct atoms of $\pounds$, then $a_1 \vee a_2 \neq 1$ and there is an edge between them.

Proof. (1) Clearly, $c_1 \wedge c_2 \neq 1$ and $c_1 \vee c_2 = 1$. Assume to contrary, that $c_1 \wedge c_2 = 0$. Then by Lemma 2.17, $1 = c_1 \oplus c_2$ gives c_1 and c_2 are atoms. Now, by Theorem 2.12, $\mathbb{CG}(\pounds)$ is not connected which is impossible. Thus $c_1 \wedge c_2 \neq 0$ and there is a path to form $c_1 \backsim c_1 \wedge c_2 \backsim c_2$ between them.

(2) Let a_1 and a_2 be two atoms of $\pounds$ such that $a_1 \vee a_2 = 1$. If $a_1 \wedge a_2 = 0$, then $1 = a_1 \oplus a_2$ gives $\mathbb{CG}(\pounds)$ is not connected, a contradiction. So $a_1 \wedge a_2 \neq 0$. Since $0 < a_1 \wedge a_2 \leq a_i$ $(i = 1, 2)$, we have $a_1 = a_1 \wedge a_2 = a_2$ which is impossible. Thus $a_1 \vee a_2 \neq 1$ and there is an edge between them. □

Proposition 2.19. *Let $\pounds$ be a lattice with the graph $\mathbb{CG}(\pounds)$. If $\pounds$ has no coatom or no atom elements, then the graph $\mathbb{CG}(\pounds)$ is infinite.*

Proof. If $\pounds$ has no coatom element, $0 < 1$ gives there exists an element a_0 of $\pounds$ such that $0 < a_0 < 1$, and a_0 is not coatom; so there exists an element a_1 of $\pounds$ such that $0 < a_0 < a_1 < 1$. It follows that there exists a chain $0 < a_0 < a_1 < \cdots < 1$, and for $i < k$, $a_i \vee a_k = a_k$. Thus $\pounds$ contains an infinite strictly increasing sequence of elements which implies that $\mathbb{CG}(\pounds)$ is infinite. If $\pounds$ has no atom element, $1 > 0$ gives there exists an element b_0 of $\pounds$ such that $1 > b_0 > 0$, and b_0 is not atom; so there exists an element b_1 of $\pounds$ such that $1 > b_0 > b_1 > 0$. consequently, there exists a chain $1 > b_0 > b_1 > \cdots > 0$, and for $i < k$, $b_i \vee b_k = b_i$. Thus $\pounds$ contains an infinite strictly decreasing sequence of elements which implies that $\mathbb{CG}(\pounds)$ is infinite. □

Theorem 2.20. *Let $\pounds$ be a modular lattice. If $\mathbb{CG}(\pounds)$ is a connected graph, then* $\mathrm{diam}(\mathbb{CG}(\pounds)) \leq 3$.

Proof. Let x and y be two distinct vertices of the graph $\mathbb{CG}(\pounds)$. If $x \vee y \neq 1$, then $\mathrm{d}(x, y) = 1$. Suppose that $x \vee y = 1$. If $x \wedge y \neq 0$, then there is a path $x \backsim x \wedge y \backsim y$ of length 2; so $\mathrm{d}(x, y) = 2$. If $x \wedge y = 0$, then $1 = x \oplus y$, and since $\mathbb{CG}(\pounds)$ is connected, by Proposition 2.18 (1), at least one of x and y should be non-coatom, say y. It follows that there exists an element z of $\pounds$ such that $y < z < 1$; so $y \vee z = z \neq 1$. By modularity condition, $z = z \wedge (x \vee y) = y \vee (x \wedge z)$. If $x \wedge z = 0$, then $y = z$, a contradiction. So we may assume that $x \wedge z \neq 0$. Then there is a path $x \backsim x \wedge z \backsim z \backsim y$ of length 3; so $\mathrm{d}(x, y) = 3$. Thus $\mathrm{diam}(\mathbb{CG}(\pounds)) \leq 3$. □

A lattice $\pounds$ is called $\pounds$-domain if $a \wedge b = 0$, then either $a = 0$ or $b = 0$.

Corollary 2.21. *If $\pounds$ is a $\pounds$-domain, then $\mathbb{CG}(\pounds)$ is a connected graph with* $\mathrm{diam}(\mathbb{CG}(\pounds)) = 2$.

Proof. Let x and y be two distinct vertices of the graph $\mathbb{CG}(\pounds)$. If $x \vee y \neq 1$, then $\mathrm{d}(x, y) = 1$. Suppose that $x \vee y = 1$. Since $\pounds$ is a $\pounds$-domain, $x \wedge y \neq 0$, and there is a path $x \backsim x \wedge y \backsim y$ of length 2; so $\mathrm{d}(x, y) = 2$, as needed. □

Theorem 2.22. *Let $\pounds$ be a lattice, and $\mathbb{CG}(\pounds)$ a graph, which contains a cycle. Then* $\mathrm{gr}(\mathbb{CG}(\pounds)) = 3$.

Proof. Assume to the contrary, that $\mathrm{gr}(\mathbb{CG}(\pounds)) \geq 4$. In this case, we claim that for two distinct vertices x and y of the graph $\mathbb{CG}(\pounds)$ with $x \vee y \neq 1$, we have either $x \leq y$ or $y \leq x$. Let a and b be distinct vertices of $\mathbb{CG}(\pounds)$ with $a \vee b \neq 1$ such that $a \nleq b$ and $b \nleq a$; so $a < a \vee b$ and $b < a \vee b$. Then $\mathbb{CG}(\pounds)$ has a cycle to form $a \backsim a \vee b \backsim b \backsim a$ of length 3 which is impossible. Thus the claim holds. Now, since $\mathrm{gr}(\mathbb{CG}(\pounds)) \geq 4$, $\mathbb{CG}(\pounds)$ contains a path of length 3 to form $x \backsim y \backsim z \backsim u$. Since every two distinct vertices in this path are comparable and every chain of nontrivial elements of length 2 induces a cycle of length 3 in $\mathbb{CG}(\pounds)$, the only two possible case are $x \leq y$, $z \leq y$ or $y \leq x$, $y \leq z$, $u \leq z$. The fist case implies that $x \vee y = y \neq 1$, $y \vee z = y \neq 1$, $x \vee z \leq y \neq 1$; so $x \backsim y \backsim z \backsim x$ is a cycle of length 3 in $\mathbb{CG}(\pounds)$, a contradiction. In the second case, we have $x \vee y = x \neq 1$, $y \vee z = z \neq 1$, $y \vee u \leq z \neq 1$ and $z \vee u \leq z \neq 1$; so $y \backsim z \backsim u \backsim y$ is a cycle of length 3 in $\mathbb{CG}(\pounds)$ which is impossible. Hence, $\mathrm{gr}(\mathbb{CG}(\pounds)) = 3$. □

Theorem 2.23. *If $\pounds$ is a Noetherian lattice, then $\mathbb{CG}(\pounds)$ is a complete graph if and only if $\pounds$ contains a unique coatom element.*

Proof. By assumption, $\pounds$ has at least one coatom element. Also, $\pounds$ is Noetherian gives if $a < 1$ is an element of $\pounds$, then there is a coatom c of $\pounds$ such that $a \leq c$. Hence, if $\pounds$ possesses a unique coatom element, say

c, then $x \leq c$ for every element $x < 1$ of $\mathcal{L}$. Suppose that x and y are two distinct vertices of $\mathbb{CG}(\mathcal{L})$. Then $x \leq c$ and $y \leq c$; so $x \vee y \leq c \neq 1$. Consequently, $\mathbb{CG}(\mathcal{L})$ is complete.

Conversely, assume that $\mathbb{CG}(\mathcal{L})$ is complete. Let c_1 and c_2 be two distinct coatoms of $\mathcal{L}$. By assumption, $c_1 \vee c_2 \neq 1$. Since $c_1 < c_1 \vee c_2 < 1$ and $c_2 < c_1 \vee c_2 < 1$, we have $c_1 = c_1 \vee c_2 = c_2$ which is impossible. Therefore, $\mathcal{L}$ contains a unique coatom element. □

3. On vertices of finite degree

The purpose of this section is to characterize lattices $\mathcal{L}$ such that some vertices of the co-identity join graph $\mathbb{CG}(\mathcal{L})$ are of finite degree and to determine elements of degree one. Let us begin the following theorem:

Theorem 3.1. *Assume that $\mathcal{L}$ is a modular lattice with the graph $\mathbb{CG}(\mathcal{L})$ and let e be an atom element of $\mathcal{L}$ such that $\deg(e) < \infty$. If $\mathbb{CG}(\mathcal{L})$ is a connected graph, then $|\mathcal{A}(\mathcal{L})| < \infty$.*

Proof. Since $\mathbb{CG}(\mathcal{L})$ is a connected graph, Proposition 2.18 gives $a \vee e \neq 1$ for every $a \in \mathcal{A}(\mathcal{L})$; hence the number of atom elements of $\mathcal{L}$ is finite, as $\deg(e) < \infty$. This completes the proof. □

Lemma 3.2. *Let c be an element of a lattice $\mathcal{L}$ with the graph $\mathbb{CG}(\mathcal{L})$. Then the following hold:*

If $\deg(c) = 0$, then c is an atom of $\mathcal{L}$;

(2) If $\deg(c) = 1$, then either c is an atom or a coatom element of $\mathcal{L}$;

(3) If $\deg(c) = 1$ with $c \vee d \neq 1$ for some $d \in \mathcal{L}$, then d is an atom.

Proof. (1) Assume to the contrary, that $0 < e < c$ for some element e of $\mathcal{L}$. Then $c \vee e = c \neq 1$ gives $\deg(c) \neq 0$, a contradiction. Thus c is an atom.

(2) Let $\deg(c) = 1$. Then there exists only a vertex b of $\mathbb{CG}(\mathcal{L})$ such that $0 \neq b \vee c \neq 1$. Since $(b \vee c) \vee c = b \vee c \neq 1$ and $(b \vee c) \vee b \neq 1$ and since $\deg(c) = 1$, $b \vee c = c$ or $b \vee c = b$. This shows that $c \leq b$ or $b \leq c$. If $c \leq b$, then we claim that c is an atom element of $\mathcal{L}$. Suppose that $0 < a < c$ for some nontrivial element a of $\mathcal{L}$. Then $a \vee c = c \neq 1$ gives $\deg(c) \geq 2$ which is impossible. If $b \leq c$, then we show that c is a coatom element. suppose that $c < d < 1$ for some nontrivial element d of $\mathcal{L}$. Then $c \vee d = d \neq 1$ gives $\deg(c) \geq 2$, a contradicion. This completes the proof.

(3) Suppose, on the contrary, $0 < e < d$ for some $e \in \mathcal{L}$. Then $e \vee c \neq 1$ gives $\deg(c) \geq 2$ which is impossible. Thus d is an atom. □

Proposition 3.3. *Let c be a coatom element of a lattice $\mathcal{L}$. If the graph $\mathbb{CG}(\mathcal{L})$ is triangle-free, then either c is an atom or $\deg(c) = 1$.*

Proof. At first, we claim that $\deg(c) < 2$. Assume to the contrary, that $\deg(c) \geq 2$. Then there are at least two vertices a and b such that $a \vee c \neq 1$ and $b \vee c \neq 1$. Then $c \leq b \vee c < 1$ and $c \leq a \vee c < 1$ gives $a \vee c = c = b \vee c$; so $a \vee b \leq c \neq 1$. This implies that a and b are two adjacent vertices of $\mathbb{CG}(\pounds)$. Thus c, a, b would form a triangle, a contradiction. Hence either $\deg(c) = 0$ or $\deg(c) = 1$. If $\deg(c) = 0$, then c is an atom element by Lemma 3.2, as needed. □

Lemma 3.4. *Let a be an atom of a modular lattice $\pounds$. If c is a nontrivial element of $\pounds$ with $a \vee c = 1$, then c is a coatom element of $\pounds$.*

Proof. Let b be an element of $\pounds$ such that $0 < c \leq b < 1$. Then $b \vee c = b$ and $a \wedge b \leq a$. If $a \leq b$, then $1 = a \vee c \leq b \vee c = b$ gives $b = 1$ which is impossible. Thus $a \wedge b = 0$. Then $b = b \wedge (a \vee c) = c \vee (a \wedge b) = c$ by modularity condition. Hence c is a coatom element of $\pounds$. □

Lemma 3.5. *Assume that $\pounds$ is a distributive lattice and let a be an atom element of $\pounds$. Then the following hold:*

(1) There is at most one element b of $\pounds$ such that a is not adjacent to b.

(2) There is at most one coatom element c of $\pounds$ such that a is not adjacent to c.

Proof. (1) Let c_1 and c_2 be elements of $\pounds$ such that $c_1 \vee a = 1 = c_2 \vee a$; hence c_1, c_2 are coatom elements of $\pounds$ by Lemma 3.4 (so $c_1 \vee c_2 = 1$). If $a \wedge c_1 \neq 0$, then $a \leq c_1$ gives $c = 1$, a contradiction. Thus $c_1 \wedge a = 0$. Similarly, $c_2 \wedge a = 0$. Then $a = a \wedge (c_1 \vee c_2) = (a \wedge c_1) \vee (a \wedge c_2) = 0$ which is impossible, as needed.

(2) This is a consequence of (1). □

Theorem 3.6. *Let $\pounds$ be a distributive lattice. Then $\mathbb{CG}(\pounds)$ is finite if and only if there exists an atom element a such that $\deg(a) < \infty$.*

Proof. By Lemma 3.5, $\deg(a) = |\mathcal{V}(\mathbb{CG}(\pounds))| - 1$ or $\deg(a) = |\mathcal{V}(\mathbb{CG}(\pounds))| = 2$. Therefore, $\mathbb{CG}(\pounds)$ is finite if and only if $\deg(a)$ is finite. □

Lemma 3.7. *Let $\pounds$ be a distributive lattice and c be a coatom element with degree 1. Then there is at most one element a of $\pounds$ such that c is not adjacent to a.*

Proof. Let a_1 and a_2 be two distinct vertices of $\mathbb{CG}(\pounds)$ such that $a_1 \vee c = 1 = a_2 \vee c$. We claim that either $a_1 \wedge c = 0$ or $a_2 \wedge c = 0$. Assume to the contrary, that $a_1 \wedge c \neq 0$ and $a_2 \wedge c \neq 0$. Then $(a_1 \wedge c) \vee c = c = (a_2 \wedge c) \vee c$ gives $a_2 \wedge c = a_2 \wedge c$ since $\deg(c) = 1$. So $a_1 = a_1 \wedge (a_2 \vee c) = (a_1 \wedge a_2) \vee (a_1 \wedge c) = (a_1 \wedge a_2) \vee (a_2 \wedge c) = a_2 \wedge (a_1 \vee c) = a_2$, a contradiction. Thus we can assume that $a_1 \wedge c = 0$ and $a_2 \wedge c \neq 0$ (so $a_1 \oplus c = 1$). It follows from Lemma 2.17 that a_1 is an atom element and $(a_2 \wedge c) \vee c = c \neq 1$;

hence $a_2 \wedge c$ is an atom element by Lemm 2.20 (3). Then $a_1 \wedge (a_2 \wedge c) = 0$ gives $a_1 = a_2 \wedge c \leq c$, a contradiction. This completes the proof. □

Theorem 3.8. *Let £ be a complete distributive lattice with the connected graph $\mathbb{CG}(£)$. Then the following statements are equivalent:*

(1) $\mathbb{CG}(£)$ contains a vertex with degree 1;

(2) $\mathcal{V}(\mathbb{CG}(£)) = \{a, c_1\}$ or $\mathcal{V}(\mathbb{CG}(£)) = \{a, c_1, c_2\}$, where $a \in \mathcal{A}(£)$ and $c_1, c_2 \in \mathcal{CA}(£)$.

Proof. (1) ⇒ (2) Let c be a vertex of $\mathbb{CG}(£)$ with degree 1. Then c is either a coatom or an atom by Lemma 3.2 (2). We split the proof into two cases.

Case 1: c is a coatom. At first we show that $|\mathcal{CA}(£)| \leq 2$. Suppose, on the contrary, c, c_1, c_2 are distinct coatom elements of £. So $c \wedge c_1 \neq 0$ and $c \wedge c_2 \neq 0$ by Proposition 2.18 (1) and $c \vee c_1 = 1 = c \vee c_2$. Since $\deg(c) = 1$, $c \wedge c_1 = c \wedge c_2$. Then $c_1 = c_1 \wedge (c \vee c_2) = (c_1 \wedge c_2) \vee (c_1 \wedge c) = (c_1 \wedge c_2) \vee (c_2 \wedge c) = c_2$ which is a contradiction. Thus $|\mathcal{CA}(£)| \leq 2$. Let $\mathcal{CA}(£) = \{c, c_1\}$ (so $c \vee c_1 = 1$ and $c \wedge c_1 \neq 0$). Then $(c \wedge c_1) \vee c \neq 1$ and Lemma 3.2 (2) gives $c \wedge c_1$ is an atom element of £. Let e be a vertex of $\mathbb{CG}(£)$. If $e \vee c \neq 1$, then $e = c \wedge c_1$, as $\deg(c) = 1$. If $e \vee c = 1$, then $e = c_1$ by Lemma 3.7. Therefore, $\mathcal{V}(\mathbb{CG}(£)) = \{c, c_1, c \wedge c_1\}$. So we may assume that $\mathcal{CA}(£) = \{c\}$. Since $\deg(c) = 1$, $c \vee a \neq 1$ for some atom element a by Lemma 3.2 (3); hence $\mathcal{V}(\mathbb{CG}(£)) = \{a, c\}$.

Case 2: c is not a coatom. At first we claim that $|\mathcal{CA}(£)| = 1$. If $|\mathcal{CA}(£)| \neq 1$, then by Lemma 3.5 (2), $|\mathcal{CA}(£)| = 2$. Let $\mathcal{CA}(£) = \{c_1, c_2\}$ and $c \leq c_1$; so $c \vee c_2 = 1$ and $c_1 \wedge c_2 \neq 0$, as $\deg(c) = 1$ and Proposition 2.18 (1). Then $c \vee c_1 \neq 1$ and $c \vee (c_1 \wedge c_2) \neq 1$ gives $c_1 = c_1 \wedge c_2$ which is a contradiction. Thus $|\mathcal{CA}(£)| = 1$. Let $\mathcal{CA}(£) = \{c'\}$. By Lemma 2.1, $c \leq c'$ (so $c \vee c' \neq 1$) and c is an atom by Lemma 3.2 (3). We claim that $|\mathcal{A}(£)| = 1$. If $c, c_1 \in \mathcal{A}(£)$, then $c \vee c_1 \neq 1$ by Proposition 2.18 (2) which is impossible since $\deg(c) = 1$ (so $\mathcal{A}(£) = \{c\}$). Suppose that e is a nontrivial element of £. If $c \leq e$, then $e = c$ or $e = c'$ since $\deg(c) = 1$; hence $\mathcal{V}(\mathbb{CG}(£)) = \{c, c'\}$. If $c \not\leq e$ (i.e. $c \wedge e = 0$), then $\mathcal{A}(£) = \{c\}$ implies $0 < f < e$ for some element f of £. Then $c < c \vee f \leq c'$ gives $c \vee f = c'$. It follows that $e = e \wedge c' = e \wedge (c \vee f) = f \vee (c \wedge e) = f$, a contradiction. Therefore, $c \leq e$ and $\mathcal{V}(\mathbb{CG}(£)) = \{c, c'\}$.

(2) ⇒ (1) It can be easily checked that if $\mathcal{V}(\mathbb{CG}(£)) = \{a, c_1\}$ or $\mathcal{V}(\mathbb{CG}(£)) = \{a, c_1, c_2\}$, then $\mathbb{CG}(£)$ has a vertex with degree 1. □

4. Clique number, chromatic number and domination number

In this section, we will investigate clique number, chromatic number and domination number of $\mathbb{CG}(£)$. We also study the condition under which the chromatic number of $\mathbb{CG}(£)$ is finite.

Proposition 4.1. *If $\omega(\mathbb{CG}(\pounds))$ is finite, then the following conditions hold:*

(1) $\pounds$ is Noetherian and Artinian;

(2) $\omega(\mathbb{CG}(\pounds)) = 1$ if and only if $|\mathcal{V}(\mathbb{CG}(\pounds))| = 1$ or $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$ and $1 = a_1 \oplus a_2$ for some atoms a_1 and a_2.

(3) If $\omega(\mathbb{CG}(\pounds)) > 1$, $\mathcal{A}(\pounds) < \infty$.

Proof. (1) Let $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ be an infinite strictly increasing sequence of elements of $\pounds$. For $i < j$, $x_i \vee x_j = x_j \neq 1$, so similarly for infinite strictly decreasing sequence of elements of $\pounds$. Therefore, any infinite strictly increasing or decreasing sequence of elements of $\pounds$ induces a clique in $\mathbb{CG}(\pounds)$ which is a contradiction since $\omega(\mathbb{CG}(\pounds))$ is finite. This implies that for infinite strictly (increasing and decreasing) sequence of elements of $\pounds$, $x_m = x_{m+i}$ for $i = 1, 2, 3, \cdots$. Hence, M should be Noetherian and Artinian.

(2) Suppose that $\omega(\mathbb{CG}(\pounds)) = 1$ and $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. This implies that $\mathbb{CG}(\pounds)$ is not connected. Hence, by Theorem 2.12, $a_1 \oplus a_2 = 1$ for some atoms a_1 and a_2. Conversely, it is clear by Theorem 2.12.

(3) Since $\omega(\mathbb{CG}(\pounds)) > 1$, by Part (2), 1 is not a direct join of two atom elements. Then, by Theorem 2.12, $\mathbb{CG}(\pounds)$ is connected. Therefore, by Proposition 2.18 (2), if a_1 and a_2 are distinct atoms of $\pounds$, then $a_1 \vee a_2 \neq 1$. Suppose that $\mathbb{A}(\pounds)$ is a subgraph of $\mathbb{CG}(\pounds)$ with the vertex set $\mathcal{A}(\pounds)$. Then $\mathbb{A}(\pounds)$ is a clique in $\pounds$, and $|\mathcal{A}(\pounds)| = \omega(\mathbb{A}(\pounds)) \leq \omega(\mathbb{CG}(\pounds))$, as needed. □

Theorem 4.2. *Let $\pounds$ be a modular lattice. If $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$, then the following conditions are equivalent:*

(1) $\mathbb{CG}(\pounds)$ is a star graph;

(2) $\mathbb{CG}(\pounds)$ is a tree;

(3) $\chi(\mathbb{CG}(\pounds)) = 2$;

(4) $\pounds$ has a unique atom element c such that every nontrivial element x with $c < x$ is a coatom element of $\pounds$ and $l(\pounds) = 3$.

Implications (1) ⇒ (2) ⇒ (3) are trivial by definitions.

Proof. (3) ⇒ (4). If $\chi(\mathbb{CG}(\pounds)) = 2$, then $\mathbb{CG}(\pounds)$ is not a null graph and so $\mathbb{CG}(\pounds)$ is connected by Corollary 2.14 (1). By [5, Theorem 10.3 (1)], $\omega(\mathbb{CG}(\pounds)) \leq \chi(\mathbb{CG}(\pounds))$; so $\omega(\mathbb{CG}(\pounds))$ is finite which implies that $l(\pounds)$ is finite by Proposition 3.1. Then $\pounds$ is Artinian gives $\pounds$ contains an atome element c. We claim that c is unique. Assume to the contrary, that c_1 and c_2 are two distinct atoms of $\pounds$. Then by Proposition 2.18 (2), $c_1 \vee c_2 \neq 1$. It follows that $c_1 \backsim c_1 \vee c_2 \backsim c_2 \backsim c_1$ is a cycle of length 3 in $\mathbb{CG}(\pounds)$ which contradicts $\chi(\mathbb{CG}(\pounds)) = 2$. Thus c is a unique atom element of $\pounds$. Suppose that $c < y$ for every nontrivial element y of $\pounds$. Let x be an element of $\pounds$ such that $c < x < y < 1$. Then $c \backsim x \backsim y \backsim c$

is a cycle of length 3 which is impossible. Hence, x is a coatom element with $c < x$. Finally, $0 < c < x < 1$ is a composition chain of $\pounds$ with length 3, as needed.

(4) $\Rightarrow$ (1). Suppose that $l(\pounds) = 3$ and $\pounds$ has a unique atom element c such that every nontrivial element x_i ($i \in J$) of $\pounds$ with $c < x_i$ is a coatom element of $\pounds$. Then $0 < c < x_i < 1$ for all $i \in J$ are composition series of $\pounds$ with length 3 such that $x_i \vee c = x_i \neq 1$ and $x_i \vee x_k = 1$ for $i \neq k$; hence $\mathbb{CG}(\pounds)$ is a star graph. □

Corollary 4.3. *Let $\pounds$ be a modular lattice. If $\mathbb{CG}(\pounds)$ is a forest, then each component of $\mathbb{CG}(\pounds)$ is either K_1 or a star graph.*

Proof. Notice that if $\mathbb{CG}(\pounds)$ is a forest, then each component of $\mathbb{CG}(\pounds)$ is a tree. If $|\mathcal{V}(\mathbb{CG}(\pounds))| = 1$, then $\mathbb{CG}(\pounds) \cong K_1$. So we may assume that $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. Then $\mathbb{CG}(\pounds)$ is a star graph by Theorem 4.2, as needed. □

Corollary 4.4. *Assume that $\pounds$ is a modular lattice and let $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. If $\mathbb{CG}(\pounds)$ is not connected, then $\mathbb{CG}(\pounds)$ is a forest and $\chi(\mathbb{CG}(\pounds)) = \omega(\mathbb{CG}(\pounds)) = 1$.*

Proof. By Corollary 2.14 (1), $\mathbb{CG}(\pounds)$ is a null graph. Thus $\mathbb{CG}(\pounds)$ has no cycle and hence it is forest and $\chi(\mathbb{CG}(\pounds)) = \omega(\mathbb{CG}(\pounds)) = 1$. □

Corollary 4.5. *Let $\pounds$ be a lattice with the graph $\mathbb{CG}(\pounds)$. If $\Delta(\mathbb{CG}(\pounds)) = n < \infty$, $\delta(\mathbb{CG}(\pounds)) = \delta \geq 1$ and 1 is except direct join of two atom elements, Then $l(\pounds) < \infty$, $\omega(\mathbb{CG}(\pounds)) < \infty$ and $\chi(\mathbb{CG}(\pounds)) < \infty$.*

Proof. It is clear that $l(\pounds) \leq \omega(\mathbb{CG}(\pounds)) + 1$. By assumption, Theorem 2.12 gives $\mathbb{CG}(\pounds)$ is a connected graph. It follows from [5, Theorem 10.3 (1)] that $\omega(\mathbb{CG}(\pounds)) \leq \chi(\mathbb{CG}(\pounds)) \leq n + 1$ and since $n < \infty$, we have $l(\pounds) < \infty$, $\omega(\mathbb{CG}(\pounds)) < \infty$ and $\chi(\mathbb{CG}(\pounds)) < \infty$. □

Corollary 4.6. *Let $\pounds$ be a lattice such that $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. If $\pounds$ is Noetherian which contains a unique coatom element or it is hollow, then $\chi(\mathbb{CG}(\pounds)) = \omega(\mathbb{CG}(\pounds))$ but not forest.*

Proof. By Theorem 2.23 and Proposition 2.6, $\mathbb{CG}(\pounds)$ is a complete graph. $\mathbb{CG}(\pounds)$ is not forest, as $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. Notice that all complete graphs are their own maximal cliques and $\omega(\mathbb{CG}(\pounds)) \leq \chi(\mathbb{CG}(\pounds))$. If $\omega(\mathbb{CG}(\pounds)) = \infty$, then we are done. If $\mathbb{CG}(\pounds) \cong K_n$ for some positive intiger n, then $\chi(\mathbb{CG}(\pounds)) = \omega(\mathbb{CG}(\pounds)) = n$, as required. □

Theorem 4.7. *Assume that $\pounds$ is a modular lattice with the graph $\mathbb{CG}(\pounds)$ and let e be an atom element of $\pounds$ such that $\deg(e) < \infty$. If $\mathbb{CG}(\pounds)$ is a connected graph, then $\chi(\mathbb{CG}(\pounds)) < \infty$.*

Proof. Let $\{a_i\}_{i\in\Lambda}$ be the family of nontrivial elements of $\pounds$ which are not adjacent to e; so $a_i \vee e = 1$ for all $i \in \Lambda$. Then Lemma 3.4 gives a_i is a coatom element of $\pounds$ for all $i \in \Lambda$. Since $a_i \vee a_k = 1$, for $i \neq k$, distinct vertices a_i and a_k are not two adjacent vertices of $\mathbb{CG}(\pounds)$. Hence, one can color all $\{a_i\}_{i\in\Lambda}$ by a color, and other vertices, which are a finite number of adjacent vertices e, by a new color to obtain a proper vertex coloring of $\mathbb{CG}(\pounds)$. Thus, $\chi(\mathbb{CG}(\pounds)) < \infty$. □

Theorem 4.8. *Let $\pounds$ be a complete lattice such that $\mathcal{V}(\mathbb{CG}(\pounds))$ is an infinite set and $\omega(\mathbb{CG}(\pounds)) < \infty$. Then the following hold:*
(1) $\mathbb{CA}(\pounds)$ is an infinite set;
(2) $\chi(\mathbb{CG}(\pounds)) < \infty$;
(3) $\alpha(\mathbb{CG}(\pounds)) = \infty$.

Proof. (1) Notice that if a is a vertex of the graph $\mathbb{CG}(\pounds)$, then there exists a coatom c such that $a \leq c$ by Lmma 2.1. Assume to the contrary, that the number of coatom elements of $\pounds$ is finite, say $c_1, c_2, \cdots, c_n$. For each $1 \leq i \leq n$, set $\Omega_i = \{x \in \pounds : x \leq c_i\}$. An inspection will show that there is an element $k \in \{1, \cdots, n\}$ such that the set Ω_k is infinite, as $\mathcal{V}(\mathbb{CG}(\pounds))$ is infinite; hence $\mathbb{CG}(\pounds)$ has an infinite clique which contradicts the fact that $\omega(\mathbb{CG}(\pounds)) < \infty$.

(2) If $\omega(\mathbb{CG}(\pounds)) = 1$, we are done. So we may assume that $\omega(\mathbb{CG}(\pounds)) > 1$. If c_1 and c_2 are two distinct coatom elements, then $c_1 \not\leq c_1 \vee c_2$ gives $c_1 \vee c_2 = 1$; so they are not two adjacent vertices of $\mathbb{CG}(\pounds)$. Now, by Part (1), one can color all coatom elements by a color, and other vertices, which are finite number, by a new color, to obtain a proper vertex coloring of $\mathbb{CG}(\pounds)$. Hence, $\chi(\mathbb{CG}(\pounds)) < \infty$.

(3) Since each two distinct elements of $\mathcal{CA}(\pounds)$ are not adjacent vertices of $\mathbb{CG}(\pounds)$, then $\mathcal{CA}(\pounds)$ is an independent set of the graph $\mathbb{CG}(\pounds)$. By part (1), $|\mathcal{CA}(\pounds)| = \infty$; hence $\alpha(\mathbb{CG}(\pounds)) = \infty$. □

Lemma 4.9. *If a is an atom of a distributive lattice $\pounds$ such that $a \leq b \vee c$ for some $b, c \in \pounds$, then either $a \leq b$ or $a \leq c$.*

Proof. Assume to the contrary, that $a \not\leq b$ and $a \not\leq c$. Then $a \wedge b = 0 = a \wedge c$ gives $a = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = 0$, a contradiction, as needed. □

Theorem 4.10. *Let $\pounds$ be a distributive lattice with the connected graph $\mathbb{CG}(\pounds)$. Then the following statements are equivalent:*
(1) $\deg(a) < \infty$ for some atom element a of $\pounds$;
(2) $\mathbb{CG}(\pounds)$ is a finite graph;
(3) $\omega(\mathbb{CG}(\pounds)) < \infty$.

Proof. (1) $\Leftrightarrow$ (2) is trivial by Theorem 3.6.

The implication (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (2) At first we show that if a_1, a_2 and a_3 are distinct atom elements of $\pounds$, then $a_1 \vee a_2 \neq a_1 \vee a_3$. Suppose on the contrary, $a_1 \vee a_2 = a_1 \vee a_3$. Then $a_2 \leq a_1 \vee a_3$ gives $a_2 \leq a_1$ or $a_2 \leq a_3$ by Lemma 4.9 which implies that $a_1 = a_2 = a_3$, a contradiction. Notice that $a \vee b \neq 1$ for two distinct atom elements a and b of $\pounds$ by Proposition 2.18 (2). Now we claim that $\mathcal{A}(\pounds) < \infty$. Assume to the contrary, let $\{a_i\}_{i\in\Omega}$ be an infinite set of atom elements of $\pounds$. Clearly, $a_i \vee a_j \neq a_i \vee a_k$ for $i, j, k \in \Omega$. Hence for atom element a_i of $\pounds$ we have the infinite complete subgraph $\{a_i \vee a_j\}_{j\in\Omega}$ which is a contradiction. Therefore, $\mathcal{A}(\pounds) < \infty$. Since $\omega(\mathbb{CG}(\pounds))$ is finite, for each vertex x of the graph $\mathbb{CG}(\pounds)$, $[0, x]$ contains an atom element. Now if $\mathbb{CG}(\pounds)$ is infinite, then there are infinite intervals $[0, x]$ which contain common atom element, a contradiction. Thus $\mathbb{CG}(\pounds)$ is a finite graph. □

We close this section with study the dominating sets of $\mathbb{CG}(\pounds)$. Using Proposition 2.5 (1), one can see the following remark:

Remark 4.11. Let $\pounds$ be a lattice such that $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$. Then:

(1) If $D \subseteq \mathcal{V}(\mathbb{CG}(\pounds))$ and D contains at least a non-zero small element, then D is a dominating set in $\mathbb{CG}(\pounds)$;

(2) If $\mathbb{CG}(\pounds)$ has at least a non-zero small element, then for each non-zero small element x, $D = \{x\}$ is a minimal dominating set in $\mathbb{CG}(\pounds)$; so $\gamma(\mathbb{CG}(\pounds)) = 1$.

Corollary 4.12. *Let $\pounds$ be a complete lattice with the connected graph $\mathbb{CG}(\pounds)$. If $\mathrm{rad}(\pounds) \neq 0$, then $D = \{\mathrm{rad}(\pounds)\}$ is a minimal dominating set in $\mathbb{CG}(\pounds)$; so $\gamma(\mathbb{CG}(\pounds)) = 1$.*

Proof. This is a direct consequence of Proposition 2.2 and Proposition 2.5 (1). □

Corollary 4.13. *If $\pounds$ is a Noetherian lattice such that $\pounds$ contains a unique coatom element, then every subset of the vertex set of the graph $\mathbb{CG}(\pounds)$ is a dominating set. In particular, $\gamma(\mathbb{CG}(\pounds)) = 1$.*

Proof. By Theorem 2.23, $\mathbb{CG}(\pounds)$ is a complete graph; so every subset of the vertex set of the graph $\mathbb{CG}(\pounds)$ is a dominating set; so for each non-zero element a, $D = \{a\}$ is a minimal dominating set in $\mathbb{CG}(\pounds)$ which implies that $\gamma(\mathbb{CG}(\pounds)) = 1$. □

Corollary 4.14. *Assume that $\pounds$ is a hollow lattice such that $|\mathcal{V}(\mathbb{CG}(\pounds))| \geq 2$ and let $\mathrm{rad}(\pounds)$ be a nontrivial element of $\pounds$. If $D \subseteq \mathcal{V}(\mathbb{CG}(\pounds))$, then D is a dominating set. In particular, $\gamma(\mathbb{CG}(\pounds)) = 1$.*

Proof. By Proposition 2.6, $\mathbb{CG}(\pounds)$ is a complete graph; so every subset of the vertex set of the graph $\mathbb{CG}(\pounds)$ is a dominating set and $\gamma(\mathbb{CG}(\pounds)) = 1$.

□

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