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Strongly (σ,τ)-e-Reversible Rings

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Abstract. In this paper, we introduce the definition of $(\sigma,\tau)-e$-reversible ring. Precisely, take R is a e-reversible ring such that σ and τ are automorphism mappings of R, we named R is $(\sigma,\tau)-e$-reversible ring if $\sigma(x)\tau(y)=0$ leads to $\tau(y)\sigma(x)e=0$, where $x,y\in R$. In fact, we seek and characterize the various properties of (σ,τ)-e-reversible rings.
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1. Introduction

The modern definition of abstract ring appeared in 1914 whiles the investigation of reversible rings, which represent the generalization the reduced ring. It has meaningful in the Ring Theory. In fact, Ring Theory has undergone a revolution in recent years, with the development of what is now known as e-reversible rings. Indeed, there are several authors have interested in this algebraic area.

Zhaiming Peng et al. [1] have studied the classical right quotient ring Q with the right Ore ring R. They find out R is strongly reflexive

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if and only if Q is strongly reflexive. Furthermore, they have studied that R is strongly M-reflexive when M form a unique product monoid ($u.p.$-monoid) and R is a reduced ring. Based on the result that, every reflexive ring form a central reflexive ring. Uday Shankar Chakraborty [2] presented the adequate identities for a central reflexive ring to become a reflexive one. Also, he proved that any central reversible and hence any central symmetric ring is central reflexive, furthermore, the converse collaborations are erroneous. Additionally to that, Uday studied that if R is quasi-Armendariz and central reflexive then the polynomial ring $R[x]$ is central reflexive. Moreover, he posted a generalization of α-reflexive rings via introduced the new concept which is the central α-reflexive rings. Inspired by the result that, if a ring R has a strong right (resp., left) reversible endomorphism α which acts of R then R is named the concept of a strong right (resp., left) α-reversible. Over and above, if R acts as a both strong left and right α-reversible the a ring R is called strong α-reversible.

More precisely, an endomorphism α of a ring R is called strong right (resp., left) reversible if whenever $a\alpha(b) = 0$ (resp., $\alpha(a)b = 0$) for $a, b \in R, ba = 0$. Furthermore, a ring R is called strong right (resp., left) α-reversible if there exists a strong right (resp., left) reversible endomorphism α of R, and the ring R is called strong α-reversible if R is both strong left and right α-reversible.

Muhittin Bašer and Tai Keun Kwak [3] investigated about the strong α-reversible rings with their connected properties together with extensions. M. Bašer, C. Y. Hong and T. K. Kwak [4] posted the result that indicate to that, every reduced and right α-reversible ring form a α-skew Armendariz, consequently, the reduced and the strong right α-reversible rings are α-skew Armendariz.

Moreover, P. M. Cohn [5] reveals via has [Proposition 1.2], the fact that, every semiprime and strong right α-reversible ring form α-rigid. Recently, Avanish K. Chaturvedi et al. [6] supplied some examples concerning the concept of e-reversible and non e-reversible rings. Furthermore, they characterized the strongly e-reversible rings and studied diversity properties with extensions of strongly e-reversible rings. It is provide that if D is a division ring with suppose that R equal to $D\langle x, y\rangle$ is the free D-algebra in two noncommutating variables x and y, then R is an (strongly) 1_R-reversible ring [[6], Lemma 1.7].

The principal aim of this article is to continue this line of investigation and introduce the structure of different properties of (σ, τ)-e-reversible rings.

2. Preliminaries

During this work, all rings are associative and noncommutative with the center $Z(R)$ and the identity unless otherwise cases stated. Where $x, y \in R$ if $xy = 0$ implies $yx = 0$, then a ring R is named reversible. Assume R is a ring equipped with the element e in R wherever e plays the role of an idempotent element in R.Hence e-reversible ring R is named if $xy = 0$ leads to the relation $yxe = 0$. Additionally, R is a strongly (resp. right) e-reversible ring we say that about R if $xy = 0$ which yields the identity $yex = 0$, hold for all $x, y \in R$. Moreover, if $xy = 0$ implies $yex = 0$, where $x, y \in R$. Then a ring R is named a weakly (resp. left) e-reversible ring.
The symbols $Z(R)$ denote to the center, $N(R)$ refer to the set which all elements are nilpotent while the set which has all idempotent elements in a ring R denoted via $E(R)$. Moreover, suppose $M_n(R)$, $T_n(R)$ (resp. $L_n(R)$) with $D_n(R)$ symbolize to $n \times n$ ring of matrices. For the upper triangular matrices (resp. lower triangular matrices) and diagonal matrices which defined over the ring. Then the symbol $E_{ij} \in M_n(R)$ characterize the matrix who has $(i, j)^{th}$ entry 1_R (which plays the role of the identity element in R) while 0_R (characterize the zero of a ring R). In order that enrich the formation about this area, there are several results in literature indicate to the global structure of a reversible ring which cover e-reversible ring. Hence, for more information, we refer readers to (see, e.g., [7]) for all undefined terms and notions.

3. Main result

The following definition is backbone and very crucial for developing the outcome of this work.

Definition 3.1. Assume R is e -reversible ring, σ and τ are automorphism mappings of R we named:

- (i) If $\sigma(x)\tau(y) = 0$ deduce $\tau(y)\sigma(x)e = 0$, for all $x, y \in R$ then R is a $(\sigma, \tau) - e$-ring.
- (ii) R is a strongly right $(\sigma, \tau) - e$-reversible ring if $\sigma(x)\tau(y) = 0$ implies $\tau(y)e\sigma(x) = 0$, where $x, y \in R$.
- (iii) R is a weakly left $(\sigma, \tau) - e$-reversible ring if $\sigma(x)\tau(y) = 0$ leads to $e\tau(y)\sigma(x) = 0$, where $x, y \in R$.

Example 3.2. (i) Set $\mathbb{Z}$ is the ring of integers. Consider $R = \{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x, y \in \mathbb{Z}\}$ a ring over a field $\mathbb{F}$. Clearly that $\sigma, \tau \colon R \to R$ are the mappings on R which defined by
$\sigma(t) = \sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\tau(s) = \tau(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}) = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$

for all $t, s \in R, x, y, g, h \in \mathbb{Z}$. Now we check the branches of the above definition.

(i) $\sigma(t)\tau(s) = \sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix})\tau(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Also,

$\tau(s)\sigma(t)E_{11} = \tau(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix})\sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Undoubted, R is a $(\sigma, \tau) - e$-ring.

(ii) $\tau(s)E_{11}\sigma(t) = \tau(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Obviously, R is a strongly right $(\sigma, \tau) - e$-ring.

Likewise, for the last branch

(iii) $E_{11}\tau(s)\sigma(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\tau(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix})\sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Of course, R is a weakly left $(\sigma, \tau) - e$-ring.

Theorem 3.3. *Suppose a ring R contains e as an idempotent element such that $e \in E(R)$ and the mappings σ and τ are automorphisms of R. For any $r_1, r_2, r_3, \cdots, r_{n-1} \in R$ the idempotent X of the ring $T_n(R)$ of upper triangular $n \times n$ matrices over R define via*

$$X = \begin{pmatrix} e & er_1 & er_2 & \cdots & er_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $T_n(R)$ becomes X-reversible ring if and only if R acts as $(\sigma, \tau) - e$-reversible ring.

Proof. Take $a, b \in R$ such that $\sigma(a)\tau(b) = 0$. Due to R has the two automorphism mappings σ and τ. In this situation $\sigma, \tau \colon R \to R$ are 1-1 and onto. $(\sigma(R) = R, \tau(R) = R)$. In particular, by reason of σ and τ are automorphisms of R, we use $\sigma(a) = x, \tau(b) = y$ in the above relation, we deduce

$xy = 0$ for all $v, y \in R$. Now take $A = xE_{11}$, $B = yE_{11} \in T_n(R)$. Obviously, $AB = 0$. Consequently, we find that $BAX = 0$, based on the fact $T_n(R)$ is X-reversible. Therefore, we conclude that $yxe = 0$. This result yields $\tau(b)\sigma(a)e = 0$. This means that R is $(\sigma, \tau) - e$-reversible ring.

Contrariwise, suppose $A = [\sigma(a_{ij})]$, $B = [\tau(b_{ij})] \in T_n(R)$ with strict condition that $AB = 0$, without doubt we observe $\sigma(a_{ij})\tau(b_{ij}) = 0$ for every $1 \leqslant i \leqslant n$. According to the fact that R is $(\sigma, \tau) - e$-reversible ring, we arrive to

$\tau(b_{ij})\sigma(a_{ij})e = 0$, $1 \leqslant i \leqslant n$. Hence, we find that

$$BAX = \begin{pmatrix} \tau(b_{ij})\sigma(a_{ij})e & \tau(b_{ij})\sigma(a_{ij})er_1 & \tau(b_{ij})\sigma(a_{ij})er_2 & \cdots & \tau(b_{ij})\sigma(a_{ij})er_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Without doubt, we obtain $T_n(R)$ is X-reversible ring. This step represents completes the proof. □

Likewise, one find the proof of the following results

Corollary 3.4. *Let R be a ring containing an idempotent $e \in E(R)$, σ and τ be automorphism mappings of R. For any $r_1, r_2, r_3, \cdots, r_{n-1} \in R$. Then the idempotent element X which belong to the ring $L_n(R)$ of the lower triangular $n \times n$ matrices on R define via*

$$X = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e & er_1 & er_2 & \cdots & er_{n-1} \end{pmatrix}.$$

Then R is strongly $(\sigma, \tau) - e$-reversible ring (resp. weakly $(\sigma, \tau) - e$-reversible ring) if and only if $L_n(R)$ is X-reversible ring.

Theorem 3.5. *Let R be a ring has an idempotent $e \in E(R)$, σ and τ be automorphism mappings of R. For any $r_1, r_2, r_3, \cdots, r_{n-1} \in R$ which represent to the idempotent X of the ring $D_n(R)$ of diagonal (anti diagonal) $n \times n$ matrices on a ring R become*

$$X = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & er_1 & 0 & \cdots & 0 \\ 0 & 0 & er_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & er_{n-1} \end{pmatrix} \text{(resp. } X = \begin{pmatrix} 0 & 0 & 0 & \cdots & e \\ 0 & 0 & \cdots & er_1 & 0 \\ 0 & 0 & er_2 & \cdots & \vdots \\ \vdots & & \vdots & \cdots & 0 \\ er_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix}\text{)}$$

Consequently R is $(\sigma, \tau) - e$-reversible ring if and only if $D_n(R)$ acts as X-reversible ring.

Corollary 3.6. *Let R be a ring containing an idempotent $e \in E(R)$, σ and τ be automorphism mappings of R. For any $r_1, r_2, r_3, \cdots, r_{n-1} \in R$ there exists the idempotent X which acts on the ring $D_n(R)$ of diagonal (resp. anti diagonal) $n \times n$ matrices over R define via*

$$X=\begin{pmatrix} e & 0 & 0 & \cdots & 0\\ 0 & er_1 & 0 & \cdots & 0\\ 0 & 0 & er_2 & \cdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \cdots & er_{n-1}\end{pmatrix} \textit{(resp. } X=\begin{pmatrix} 0 & 0 & 0 & \cdots & e\\ 0 & 0 & \cdots & er_1 & 0\\ 0 & 0 & er_2 & \cdots & \vdots\\ \vdots & & \vdots & \cdots & 0\\ er_{n-1} & 0 & 0 & \cdots & 0\end{pmatrix}\textit{)}$$

Then R becomes strongly $(\sigma,\tau)-e$-reversible ring (resp. weakly $(\sigma,\tau)-e$-reversible ring) if and only if $D_n(R)$ acts as X-reversible ring.

In the reference [8], we can note an idempotent e of the ring R is called a right (resp. left) semicentral if $ea = eae$ (resp. $ae = eae$) where $a \in R$.
In the following theorem, one can observe the relations are equivalent.

Theorem 3.7. *For $e \in E(R)$ and a ring R:*

(i) R acts as an $(\sigma,\tau)-e$-reversible ring.
(ii) The term eRe forms a $(\sigma,\tau)-e$-reversible ring with e is left semicentral.

Proof. $(i)\Rightarrow(ii)$ Suppose R is an $(\sigma,\tau)-e$-reversible ring. When $x \in R$, we deduce
$e(1-e)\sigma(x) = 0$. It follows that
$(1-e)xe = (1-e)xe^2 = 0$, this situation yields the identity $xe = exe$. Undoubtedly, e is left semicentral. In this stage of the proof, we have to observe that eRe is a $(\sigma,\tau)-e$-reversible ring. Suppose $a, b \in eRe$ such that $ab = 0$. Based on the fact that eRe form a subring of R with R is an $(\sigma,\tau)-e$-reversible ring. That means, we arrive to $\tau(b)\sigma(a)e = 0$. Moreover, $ae = a$ implies that $\tau(b)\sigma(a) = 0$. Consequently, eRe is a $(\sigma,\tau)-e$-reversible ring.
$(ii)\Rightarrow(i)$ Let e be left semicentral and eRe be a $(\sigma,\tau)-e$-reversible ring. Now, assume $a, b \in R$ such that $\sigma(a)\tau(b) = 0$. Consequently, we deduce $\sigma(a)\tau(b)e = e\sigma(a)e\tau(b)e = 0$ also we obtain $\tau(b)\sigma(a)e = e\tau(b)e\sigma(a)e = 0$. Thus, R is an $(\sigma,\tau)-e$-reversible ring. This step represents completes the proof.

□

In the following theorem the identities are equivalent.

Theorem 3.8. *For the element e in $E(R)$ of a ring R:*

(i) R is a strongly $(\sigma,\tau)-e$-reversible ring (resp. weakly $(\sigma,\tau)-e$-reversible ring),
(ii) e in $Z(R)$ with the term eRe is a $(\sigma,\tau)-e$-reversible ring.

Proof. $(i)\Rightarrow(ii)$ Suppose R is a strongly $(\sigma,\tau)-e$-reversible ring. For any $a \in R$, $\sigma(a)(1-e)e = 0$. Consequently, $ea(1-e) = e^2a(1-e) = 0$.

Hence, we conclude that $ea = eae$. By reason R is a strongly $(\sigma, \tau) - e$-reversible ring, we deduce $(\sigma, \tau) - e$-reversible. According to Theorem 3.7, we arrive to e plays the role of the left semicentral which yields $e \in Z(R)$. Applying Theorem 3.7, gives us eRe is a $(\sigma, \tau) - e$-reversible ring.
$(ii) \Rightarrow (i)$ assume eRe is a $(\sigma, \tau) - e$-reversible ring and $e \in Z(R)$. Consequently $ae = aee = ae$. Obviously, e is left semicentral. Employing Theorem 3.7, we see that R is $(\sigma, \tau) - e$-reversible ring. According to the fact that e is a central element, then R is a strongly $(\sigma, \tau) - e$-reversible ring (resp. weakly $(\sigma, \tau) - e$-reversible ring). □

The authors in reference [9] contributed, R is a ring named as a left e-reflexive if $aRe = 0$ implies to $eRa = 0$ where $a \in R$. Hence, from this definition we obtain

Definition 3.9. Let σ and τ be automorphism mappings of R a ring. For any arbitrary $a \in R$ such that $\sigma(a)R\tau(e) = 0$ yields $\tau(e)R\sigma(a) = 0$, R is named to be left $(\sigma, \tau) - e$-reflexive.

Example 3.10. Consider $R = \{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x, y \in \mathbb{Z}\}$ is a ring over a field $\mathbb{F}$ and $\mathbb{Z}$ is a ring of integers. Suppose $\sigma, \tau\colon R \to R$ are the mappings of R defined by
$\sigma(t) = \sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\tau(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
where $t, s \in R, x, y, g, h \in \mathbb{Z}$.

Now we check the the aforementioned definition.

$\sigma(t)R\tau(e) = \sigma(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Moreover,
$\sigma(t)R\tau(s) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Certainly, R is a left $(\sigma, \tau) - e$-reflexive.

In the following proposition the conditions are equivalent.

Proposition 3.11. *For e in $E(R)$, where R is a ring:*

- (i) *R acts as strongly $(\sigma, \tau) - e$-reversible ring (resp. weakly $(\sigma, \tau) - e$-reversible ring),*
- (ii) *R becomes left $(\sigma, \tau) - e$-reflexive and $(\sigma, \tau) - e$-reversible.*

Proof. $(i) \Rightarrow (ii)$ Suppose R acts as a strongly $(\sigma, \tau) - e$-reversible. Based on Theorem 3.8, we deduce that R is $(\sigma, \tau) - e$-reversible and e

is central. Assume that $a \in R$ such that $aRe = 0$. Follows from that $ae = 0$. By reason of e is central of R, we arrive to $eRa = aRae = 0$. Consequently, R is left $(\sigma, \tau) - e$-reflexive.
$(ii) \Rightarrow (i)$ It is appropriate to show that a ring R is strongly $(\sigma, \tau) - e$-reversible ring (resp. weakly $(\sigma, \tau) - e$-reversible ring). Hence, , assume that a ring R is $(\sigma, \tau) - e$-reversible with left $(\sigma, \tau) - e$-reflexive. Based on R is a $(\sigma, \tau) - e$-reversible ring, applying Theorem3.7 gives the element e which acts as a left semicentral. Obviously, $(1 - e)Re = 0$ deduces $eR(1 - e) = 0$ as R is left $(\sigma, \tau) - e$-reflexive. Employing Theorems 3.5 and 3.7, e is $(\sigma, \tau) - e$-central and consequently R is strongly $(\sigma, \tau) - e$-reversible. □

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