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A Population biological model with a singular nonlinearity and Caffarelli-Kohn-Nirenberg exponents

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Abstract. We consider the existence of positive solutions of singular nonlinear semipositone problem of the form

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) & \\ \qquad = |x|^{-(\alpha+1)p+\beta}(au^{p-1} - bu^r - f(u) - \frac{c}{u^\gamma}), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $1 < p < N$, $0 \leq \alpha < \frac{N-p}{p}$,$r > p-1$,$\gamma \in (0,1)$, a, b, c, β are positive parameters, and $f : [0, +\infty) \to \mathbb{R}$ is a continuous function . This model arises in the studies of population biology of one species with u representing the concentration of the species. We obtain our results via the method of sub and supersolutions.

Keywords: Population biology, Singular weights, infinite semipositone systems, Sub and supersolutions method, Caffarelli-Kohn-Nirenberg exponents.

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1. Introduction

We study the existence of positive solutions to the singular infinite semipositone problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) \\ \qquad = |x|^{-(\alpha+1)p+\beta}(au^{p-1} - bu^r - f(u) - \frac{c}{u^\gamma}), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p < N$, $0 \leq \alpha < \frac{N-p}{p}$,$r > p-1$, $\gamma \in (0,1)$, a, b, c, β are positive parameters , and $f : [0, +\infty) \to \mathbb{R}$ is a continuous function . We make the following assumptions:
(H_1) There exist $A > 0$ and $l > p-1$ such that $f(s) \leq As^l$, for all $s \geq 0$.
(H_2) There exist a constant $S > 0$ such that $au^{p-1} - bu^r \leq f(u) + S$ for all $u > 0$.
Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u)$, were motivated by the following Caffarelli, Kohn and Nirenberg's inequality (see [1, 2]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [3, 4]). More recently, reaction-diffusion models have been used to describe spatiotemporal phenomena in disciplines other than ecology, such as physics, chemistry, and biology (see[5, 6, 7]). In addition, most ecological systems have some form of predation or harvesting of the population, for example, hunting or fishing is often used as an effective means of wildlife management. This model describes the dynamics of the fish population with predation. In such cases u denotes the population density and the term $\frac{c}{u^\gamma}$

corresponds to predation. So, the study of positive solutions of (1.1) has more practical meanings. In [13], the authors have studied the equation $-\Delta_p u = au^{p-1} - bu^r - f(u) - \frac{c}{u^\gamma}$ Here we focus on extending the study ([13]). In fact this paper is motivated, in part, by the mathematical difficulty posed by the degenerate quasilinear elliptic operator compared to the Laplacian operator .This extension is nontrivial and requires more careful analysis of the nonlinearity. Our approach is based on the method of sub-super- solutions, ([11, 12]).

2. Main result

In this paper, we denote $W_0^{1,p}(\Omega, |x|^{-\alpha p})$ the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = (\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx)^{\frac{1}{p}}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi) = \lambda|x|^{-(\alpha+1)p+\beta}|\phi|^{p-2}\phi, & x\in\Omega, \\ \phi = 0, & x\in\partial\Omega. \end{cases} \tag{2.1}$$

Let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2.1) such that $\phi_{1,p}(x) > 0$ in Ω and $\|\phi_{1,p}\|_\infty = 1$ (see [8, 9]). It can be shown that $\frac{\partial\phi_{1,p}}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. We will also consider the unique solution $\zeta_p(x) \in W_0^{1,p}(\Omega, |x|^{-\alpha p})$ for the problem

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) = |x|^{-(\alpha+1)p+\beta}, & x\in\Omega, \\ u = 0, & x\in\partial\Omega, \end{cases} \tag{2.2}$$

to discuss our existence. It is known that $\zeta_p(x) > 0$ in Ω and $\frac{\partial\zeta_p}{\partial n} < 0$ on $\partial\Omega$ ([8]).

Now we give the definition of weak solution and sub-supersolution of (1.1) . A nonnegative function ψ is called a sub-solution of (1.1) if it satisfy $\psi \geq 0$ on $\partial\Omega$ and

$$\begin{aligned} &\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx \\ &\leq \int_\Omega |x|^{-(\alpha+1)p+\beta}(a\psi^{p-1} - b\psi^r - f(\psi) - \frac{c}{\psi^\gamma})w dx, \end{aligned} \tag{2.3}$$

and a nonnegative function φ is called a super-solution of (1.1) if it satisfy $\varphi \geq 0$ on $\partial\Omega$ and

$$\begin{aligned} &\int_\Omega |x|^{-\alpha p}|\nabla\varphi|^{p-2}\nabla\varphi\cdot\nabla w dx \\ &\geq \int_\Omega |x|^{-(\alpha+1)p+\beta}(a\varphi^{p-1} - b\varphi^r - f(\varphi) - \frac{c}{\varphi^\gamma})w dx. \end{aligned} \tag{2.4}$$

for all $w \in W = \{w \in C_0^\infty(\Omega) \mid w \geq 0, x \in \Omega\}$. Then the following result holds:

Lemma 2.1. *Suppose that there exist sub and super-solutions ψ and φ respectively of (1.1) such that $\psi \leq \varphi$. Then (1.1) has a solution u such that $\psi \leq u \leq \varphi$. ([8])*

We are now ready to give our existence result.

Theorem 2.2. *Let $(H_1) - (H_2)$ hold. If $a > (\frac{p}{p-1+\gamma})^{p-1}\lambda_{1,p}$, then there exists positive constant $c_0 > 0$ such that if $0 < c < c_0$, then the problem (1.1) has a positive solution.*

Proof. We start the construction of a positive subsolution for (1.1). To get a positive subsolution, we can apply an anti-maximum principle (

[10]), from which we know that there exist a $\delta_1 > 0$ and a solution z_λ of

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla z|^{p-2}\nabla z) = |x|^{-(\alpha+1)p+\beta}(\lambda z^{p-1} - 1), & x \in \Omega, \\ z = 0, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

for $\lambda \in (\lambda_{1,p}, \lambda_{1,p} + \delta_1)$. Fix $\hat{\lambda} \in (\lambda_{1,p}, \min\{(\frac{p-1+\gamma}{p})^{p-1}a, \lambda_{1,p} + \delta_1\})$. It is well known that $z_{\hat{\lambda}} > 0$ in Ω and $\frac{\partial z_{\hat{\lambda}}}{\partial n} < 0$ on $\partial\Omega$, where n is the outer unit normal to Ω. Hence there exist positive constants ϵ, δ, σ such that

$$|x|^{-\alpha p}|\nabla z_{\hat{\lambda}}|^p \geq \epsilon \quad x \in \bar{\Omega}_\delta \tag{2.6}$$

$$z_{\hat{\lambda}} \geq \sigma \quad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\delta \tag{2.7}$$

where $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$.

Choose $\eta_1, \eta_2 > 0$ such that $\eta_1 \leq \min |x|^{-(\alpha+1)p+\beta}$, and $\eta_2 \geq \max |x|^{-(\alpha+1)p+\beta}$ in $\bar{\Omega}_\delta$. We construct a subsolution ψ of (1.1) using $z_{\hat{\lambda}}$. Define $\psi = M z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}$, where

$$M := \min\Bigg\{ \Bigg(\frac{(\frac{p}{p-1+\gamma})^{p-1}}{2b\|z_{\hat{\lambda}}\|_\infty^{\frac{rp-(p-1)(\gamma-1)}{p-1+\gamma}}} \Bigg)^{\frac{1}{r-p+1}}, \Bigg(\frac{a - (\frac{p}{p-1+\gamma})^{p-1}\hat{\lambda}}{3b\|z_{\hat{\lambda}}\|_\infty^{\frac{p(r-p+1)}{p-1+\gamma}}} \Bigg)^{\frac{1}{r-p+1}},$$

$$\Bigg(\frac{(\frac{p}{p-1+\gamma})^{p-1}}{2A\|z_{\hat{\lambda}}\|_\infty^{\frac{lp-(p-1)(\gamma-1)}{p-1+\gamma}}} \Bigg)^{\frac{1}{l-p+1}}, \Bigg(\frac{a - (\frac{p}{p-1+\gamma})^{p-1}\hat{\lambda}}{3A\|z_{\hat{\lambda}}\|_\infty^{\frac{p(l-p+1)}{p-1+\gamma}}} \Bigg)^{\frac{1}{l-p+1}} \Bigg\}$$

Let $w \in W$. Then a calculation shows that

$$\nabla\psi = M\Big(\frac{p}{p-1+\gamma}\Big) z_{\hat{\lambda}}^{\frac{1-\gamma}{p-1+\gamma}} \nabla z_{\hat{\lambda}}$$

$$\begin{aligned} & \int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\nabla w dx \\ = \;& M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1} \int_\Omega |x|^{-\alpha p} z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}} |\nabla z_{\hat{\lambda}}|^{p-2}\nabla z_{\hat{\lambda}}\nabla w dx \\ = \;& M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1} \int_\Omega |x|^{-\alpha p}|\nabla z_{\hat{\lambda}}|^{p-2}\nabla z_{\hat{\lambda}}\Big[\nabla\big(z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}} w\big) \\ & -\nabla z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}} w\Big] dx \end{aligned}$$

$$
\begin{aligned}
&= M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\int_\Omega\Big[|x|^{-(\alpha+1)p+\beta}z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}(\hat\lambda z_{\hat\lambda}^{p-1}-1)\\
&\quad -|x|^{-\alpha p}\frac{(1-\gamma)(p-1)}{p-1+\gamma}\frac{|\nabla z_{\hat\lambda}|^p}{z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}}\Big]wdx\\
&= \int_\Omega\Big[|x|^{-(\alpha+1)p+\beta}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat\lambda z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}}\\
&\quad -|x|^{-(\alpha+1)p+\beta}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\\
&\quad -|x|^{-\alpha p}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\frac{(1-\gamma)(p-1)}{p-1+\gamma}\frac{|\nabla z_{\hat\lambda}|^p}{z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}}\Big]wdx,
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&\int_\Omega |x|^{-(\alpha+1)p+\beta}\Big[a\psi^{p-1}-b\psi^r-f(\psi)-\frac{c}{\psi^\gamma}\Big]wdx\\
&= \int_\Omega\Big[|x|^{-(\alpha+1)p+\beta}aM^{p-1}z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}}-|x|^{-(\alpha+1)p+\beta}bM^r z_{\hat\lambda}^{\frac{pr}{p-1+\gamma}}\\
&\quad -|x|^{-(\alpha+1)p+\beta}f\Big(Mz_{\hat\lambda}^{\frac{p}{p-1+\gamma}}\Big)\\
&\quad -|x|^{-(\alpha+1)p+\beta}\frac{c}{M^\gamma z_{\hat\lambda}^{\frac{\gamma r}{p-1+\gamma}}}\Big]wdx.
\end{aligned}
\tag{2.9}
$$

Let $c_0 = M^{p-1+\gamma}\min\Big\{\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\Big(\big(\frac{(p-1)(1-\gamma)}{p-1+\gamma}\big)\Big)\frac{\epsilon}{\eta_2}, \frac{1}{3}\sigma^p\Big(a-\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat\lambda\Big)\Big\}$.

Let $x\in\bar\Omega_\delta$, $c\le c_0$. Since $\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat\lambda<a$, we have

$$
|x|^{-(\alpha+1)p+\beta}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat\lambda z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}} < |x|^{-(\alpha+1)p+\beta}a\Big(Mz_{\hat\lambda}^{\frac{p}{p-1+\gamma}}\Big)^{p-1}. \tag{2.10}
$$

From the choice of M, we have

$$
\begin{aligned}
&\frac{1}{2}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\\
&\ge bM^{r-p+1}\|z_{\hat\lambda}\|_\infty^{\frac{rp-(p-1)(\gamma-1)}{p-1+\gamma}},
\end{aligned}
\tag{2.11}
$$

$$\frac{1}{2}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1} \geq AM^{l-p+1}\|z_{\hat{\lambda}}\|_{\infty}^{\frac{lp-(p-1)(\gamma-1)}{p-1+\gamma}}, \tag{2.12}$$

and by (2.11), (2.12) and (H_1), we know that

$$\begin{aligned}
&-\frac{1}{2}|x|^{-(\alpha+1)p+\beta}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\\
&\leq -|x|^{-(\alpha+1)p+\beta}b\Big(Mz_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\Big)^{r}, &(2.13)\\
&-\frac{1}{2}|x|^{-(\alpha+1)p+\beta}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\\
&\leq -|x|^{-(\alpha+1)p+\beta}A\Big(Mz_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\Big)^{l}\\
&\leq -|x|^{-(\alpha+1)p+\beta}f\Big(Mz_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\Big). &(2.14)
\end{aligned}$$

Since $|x|^{-\alpha p}|\nabla z_{\hat{\lambda}}|^p \geq \epsilon$ in $\bar{\Omega}_\delta$, from the choice of c_0 we have

$$\begin{aligned}
&-|x|^{-\alpha p}M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\Big(\frac{(p-1)(1-\gamma)}{p-1+\gamma}\Big)\frac{|\nabla z_{\hat{\lambda}}|^p}{z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}}\\
&\leq -|x|^{-(\alpha+1)p+\beta}\frac{c}{\Big(Mz_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\Big)^{\gamma}}. &(2.15)
\end{aligned}$$

Hence by using (2.10), (2.13), (2.14) and (2.15) we have

$$\begin{aligned}
\int_{\bar{\Omega}_\delta}|x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\nabla w dx &\leq \int_{\Omega}\Big[|x|^{-(\alpha+1)p+\beta}aM^{p-1}z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}}\\
&-|x|^{-(\alpha+1)p+\beta}bM^{r}z_{\hat{\lambda}}^{\frac{pr}{p-1+\gamma}} - |x|^{-(\alpha+1)p+\beta}f\Big(Mz_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\Big)\\
&-|x|^{-(\alpha+1)p+\beta}\frac{c}{\Big(Mz_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\Big)^{\gamma}}\Big]wdx\\
&= \int_{\bar{\Omega}_\delta}|x|^{-(\alpha+1)p+\beta}\Big[a\psi^{p-1} - b\psi^{r} - f(\psi) - \frac{c}{\psi^{\gamma}}\Big]wdx. \qquad (2.16)
\end{aligned}$$

On the other hand on $\Omega_0 = \Omega\setminus\bar{\Omega}_\delta$, we have $z_{\hat{\lambda}} \geq \sigma$, and from the definition of c_0, for $c \leq c_0$ we have

$$\frac{c}{M^{\gamma}} \leq \frac{1}{3}M^{p-1}z_{\hat{\lambda}}^{p}\Big(a - \Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat{\lambda}\Big), \tag{2.17}$$

and also from the choice of M we have

$$bM^{r-p+1}z_{\hat{\lambda}}^{\frac{p(r-p+1)}{p-1+\gamma}} \leq \frac{1}{3}\Big(a - \Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat{\lambda}\Big), \tag{2.18}$$

$$AM^{l-p+1}z_{\hat{\lambda}}^{\frac{p(l-p+1)}{p-1+\gamma}} \leq \frac{1}{3}\Big(a - \Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat{\lambda}\Big). \tag{2.19}$$

By combining (2.17), (2.18) and (2.19) we have

$$\begin{aligned}
&\int_{\Omega_0} |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\nabla w dx\\
&= \int_{\Omega_0} \Big[|x|^{-(\alpha+1)p+\beta} M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat{\lambda} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}}\\
&- |x|^{-(\alpha+1)p+\beta} M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1} z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\\
&- |x|^{-\alpha p} M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\frac{(1-\gamma)(p-1)}{(p-1+\gamma)}\frac{|\nabla z_{\hat{\lambda}}|^p}{z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}}\Big] w dx\\
&\leq \int_{\Omega_0} |x|^{-(\alpha+1)p+\beta} M^{p-1}\Big(\frac{p}{p-1+\gamma}\Big)^{p-1}\hat{\lambda} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}} w dx\\
&= \int_{\Omega_0} \Big[\frac{|x|^{-(\alpha+1)p+\beta}}{z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}}\sum_{k=1}^{3}\Big(\frac{1}{3}M^{p-1}(\frac{p}{p-1+\gamma})^{p-1}\hat{\lambda} z_{\hat{\lambda}}^{p}\Big)\Big] w dx\\
&\leq \int_{\Omega_0} \Big[\frac{|x|^{-(\alpha+1)p+\beta}}{z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}}\Big\{(\frac{1}{3}M^{p-1} z_{\hat{\lambda}}^{p} a - \frac{c}{M^{\gamma}})\\
&+ M^{p-1} z_{\hat{\lambda}}^{p}(\frac{1}{3}a - bM^{r-p+1} z_{\hat{\lambda}}^{\frac{p(r-p+1)}{p-1+\gamma}})\\
&+ M^{p-1} z_{\hat{\lambda}}^{p}(\frac{1}{3}a - AM^{l-p+1} z_{\hat{\lambda}}^{\frac{p(l-p+1)}{p-1+\gamma}})\Big\}\Big] w dx\\
&\leq \int_{\Omega_0} \Big[|x|^{-(\alpha+1)p+\beta}\Big(aM^{p-1} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}} - bM^{r} z_{\hat{\lambda}}^{\frac{rp}{p-1+\gamma}}\\
&- AM^{l} z_{\hat{\lambda}}^{\frac{pl}{p-1+\gamma}} - \frac{c}{M^{\gamma}} z_{\hat{\lambda}}^{-\frac{\gamma p}{p-1+\gamma}}\Big)\Big] w dx\\
&\leq \int_{\Omega_0} \Big[|x|^{-(\alpha+1)p+\beta}\Big(a(M z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}})^{p-1} - b(M z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}})^{r}\\
&- f(M z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}) - \frac{c}{(M z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}})^{\gamma}}\Big)\Big] w dx\\
&= \int_{\Omega_0} |x|^{-(\alpha+1)p+\beta}(a\psi^{p-1} - b\psi^{r} - f(\psi) - \frac{c}{\psi^{\gamma}}) w dx. \qquad (2.20)
\end{aligned}$$

By using (2.16) and (2.20) we see that ψ is a sub-solution of (1.1). Next, we construct a supersolution φ of (1.1) such that $\varphi \geq \psi$. By (H_2) and $r > p-1$ we can choose a S^* such that $au^{p-1} - bu^r - f(u) - \frac{c}{u^{\gamma}} \leq S^*$ for all $u > 0$. Let $\varphi = (S^*)^{\frac{1}{p-1}}\zeta(x)$, where $\zeta(x)$ is the unique positive solution of (2.2). We shall verify that φ is a super solution of (1.1). To this end, let $w \in W$. Then we have

$$\int_\Omega |x|^{-\alpha p}|\nabla\varphi|^{p-2}\nabla\varphi\cdot\nabla w dx$$
$$= S^*\int_\Omega |x|^{-(\alpha+1)p+\beta} w dx$$
$$\geq \int_\Omega |x|^{-(\alpha+1)p+\beta}(a\varphi^{p-1} - b\varphi^r - f(\varphi) - \frac{c}{\varphi^\gamma}) w dx.$$

Then, φ is a supersolution of (1.1). Finally, we can choose $S^* \gg 1$ such that $\varphi \geq \psi$ in Ω. This completes the proof of Theorem 2.2. □

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