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# The Meir-Keeler fixed point theorem in incomplete 2-normed spaces

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Abstract. In this paper, we give a new fixed point theorem for the Meir-Keeler mappings definded on incomplete orthogonal 2-Banach spaces. This can be 2-Banach version of the Meir-Keeler fixed point theorem.

Keywords: Meir-Keeler mapping; Orthogonal 2-Banach spaces; Fixed point.

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## 1. Introduction

Throughout this article, $\mathbb{N}$ stands for the set of all positive integers, and $\mathbb{R}_+ = [0, \infty)$. Fixed point theory for contractive and non-expansive mapping defined in Banach spaces has been extensively developed since the mid 1960s; see for example [3, 11, 15]. The Banach contraction principle [2] is the most wildely applied fixed point result in many branches of mathematics and considered as the main source of metric fixed point theory.

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**Theorem 1.1.** [2] *Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be a mapping such that there exists a nonnegative number* $k < 1$ *and for all* $x, y \in X$

$$d(Tx, Ty) \leq kd(x, y).$$

*Then,* $T$ *has a unique fixed point in* $X$*.*

In 1969, Meir and Keeler [17] gave an interesting generalization the Banach contraction principle as the following; see also [19].

**Theorem 1.2.** *If a mapping* $T : X \to X$ *where* $(X, d)$ *is a complete metric space, and for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for* $x, y \in X$ *,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon.$$

*Then,* $T$ *has a unique fixed point.*

In 1965, Gahler [9, 10] initiated the notion of 2-normed spaces. There was a strong interest to study the fixed point property in 2-Banach spaces after the first paper [14] was published in 1977. Many problems in metric and Banach space can be reformulated in 2-Banach spaces. The fixed point property in 2-Banach spaces has been defined and investigate by many authors. For more details on 2-Banach spaces, the reader may consult the book [8].

In order to generalized Banach contraction principle Eshaghi Gordji et al. [7] introduced the notion of orthogonal set in 2017. Fixed point properties in orthogonal spaces have been studied by many authors [1, 5, 6, 7, 18].

Our aim in this paper is to prepare a fixed point theorem for the Meir-Keeler mappings on incomplete orthogonal 2-Banach spaces. In other words, 2-Banach version of the Meir-Keeler fixed point theorem is introduced. Our results are extremely new in 2-banach space.

## 2. Preliminaries

We begin this section by recalling the defnition of an orthogonal set. This property is central in our considerations; see [1, 7].

**Definition 2.1.** Let $X$ be a non-empty set and $\perp$ be a binary relation defined on $X \times X$. Then $(X, \perp)$ is said to be an orthogonal set (abbreviated as $O$- set) if there exists $x_0 \in X$ such that

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

The element $x_0$ is called an orthogonal element. An orthogonal set may have more than one orthogonal element.

**Example 2.2.** Let $X$ be a norm linear space. For any two element $x, y \in X$, we define the notion $\perp$ by $x \perp y$ iff $\|x\| \leq \|x + \lambda y\|$ for all

$\lambda \in \mathbb{R}$. It is easy to see that $x \perp 0$ and $0 \perp x$ for all $x \in X$. Then $(X, \perp)$ is an $O$- set.

**Definition 2.3.** Let $(X, \perp)$ be an $O$- set. A sequence $(x_n)$ in $X$ is said to be an orthogonal sequence (abbreviated as $O$- sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

**Definition 2.4.** Let $(X, \perp)$ be an $O$- set. A sequence $(x_n)$ in $X$ is said to be a strongly orthogonal sequence (abbreviated as $SO$- sequence) if

$$(\forall n, k \in \mathbb{N}, x_k \perp x_{n+k}) \text{ or } (\forall n, k \in \mathbb{N}, x_{n+k} \perp x_k).$$

From the definition, it is obvious that each $SO$-sequence is an $O$-sequence, but the converse is not true; see [1, 7]. The following example shows that the converse is not true.

**Example 2.5.** Let us set $X = \mathbb{R}_+$, and we define a binary relation $\perp$ on $X$ by

$$x \perp y \text{ if and only if } xy = \min\{x, y\}.$$

It is easy to see that $x \perp 0$ for all $x \in X$. Hence $(X, \perp)$ is an $O$- set. Define the sequence $(x_n)$ in $X$ as follows:

$$x_n = \begin{cases} 0 & \text{if } n = 2k, \\ 2 & \text{if } n = 2k+1 \end{cases}$$

such that $k \in \mathbb{N}$. Clearly, for all $n \in \mathbb{N}$, $x_n \perp x_{n+1}$. But $x_{2n+1}$ is not orthogonal to $x_{4n+1}$. This shows that, $(x_n)$ is an $O$-sequence but not an $SO$-sequence.

**Definition 2.6.** Let $\|.\|$ be a norm on linear space $X$, and $(X, \perp)$ be an $O$- set.The triplet $(X, \perp, \|.\|)$ is called an orthogonal norm space.

**Definition 2.7.** Let $(X, \perp, \|.\|)$ be an orthogonal norm space. $X$ is called strongly orthogonal complete (abbreviated as $SO$-complete) if every Cauchy $SO$-sequence is convergent.

Obviously, every complete norm space (Banach space) is $SO$-complete. The following example shows that the converse is not true.

**Example 2.8.** Let $C_b(\mathbb{R}))$ denotes the set of all real-valued bounded continuous functions on $\mathbb{R}$. The collection of real valued functions on $\mathbb{R}$ that have compact support is denoted by $C_c(\mathbb{R})$. In other words, $f \in C_b(\mathbb{R})$ belongs to $C_c(\mathbb{R})$ if and only if the set $\{x \in X : f(x) \neq 0\}$ has compact closure. We note that the space $C_c(\mathbb{R})$ with the norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is not complete.

Let $X = C_c(\mathbb{R})$. For any two elements $f, g \in X$, we define the orthogonal relation $\perp$ as follows:

$$f \perp g \text{ if and only if } f(t)g(t) \geq \max\{f(t), g(t)\} \text{ for all } t \in \mathbb{R}.$$

We show that $X$ is $SO$-complete. Let $\{f_n\}$ be a cauchy $SO$-sequence in $X$. The completeness of $C_b(\mathbb{R})$ with respect to the supremum norm, yields the existence of the limit. So there exists $f \in C_b(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$. Since for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $f_n(t) \geq 1$ it could be concluded that, $f \in C_c(\mathbb{R})$.

**Definition 2.9.** Let $C$ be a non-empty subset an orthogonal norm space $(X, \perp, \|.\|)$. A mapping $T : C \to C$ is said to be strongly orthogonal $\|.\|$-continuous (abbreviated as $SO$- $\|.\|$-continuous) in $x \in C$ if for any $SO$-sequence $(x_n) \subset C$ which $\|.\|$-convergent to $x$, then $(T(x_n))$ be $\|.\|$-convergent to $Tx$.

It can be easily derived that every continuous mapping is $SO$-continuous. By the following example, we show that the converse is not true.

**Example 2.10.** Let $X = \mathbb{R}$ equipped with the usual metric. That is, for $x, y \in X$, $d(x, y) = |x - y|$. We define a binary relation $\perp$ on $X$ by

$$x \perp y \text{ if and only if } xy = \min\{x, y\}.$$

Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in Q, \\ x & \text{if } x \in Q^c \end{cases}$$

It is easy to see that $f$ is not continuous in 0. Let $(x_n)$ be an $SO$-sequence in $X$ which converges to 0. According to the definition of relation$\perp$, we have $x_n = 0$, for all $n \in \mathbb{N}$. This implies that $f(x_n) = 1 \to 1 = f(0)$. Thus, f is $SO$-continuous in 0.

## 3. Meir-Keeler mapping in orthogonal 2-Banach spaces

We state a brief recollection of basic concepts and facts in 2-norm spaces [8].

**Definition 3.1.** Let $X$ be a real vector space of dimension greater than 1. A function $\|.,.\| : X \times X \to [0, \infty)$ is called a 2-norm on $X$ if the following properteis hold:

(1) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent,
(2) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(3) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for all $\alpha \in \mathbb{R}$ and $x, y \in X$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

The pair $(X, \|.,.\|)$ is called a vector 2-norm space.

**Definition 3.2.** Let $(X, \|.,.\|)$ be a vector 2-norm space.

(1) A sequence $(x_n)$ in $(X, \|.,.\|)$ is said to be convergent to $x \in X$ if $\|x_n - x, y\| \to 0$ as $n \to \infty$, for all $y \in X$.

(2) The sequence $(x_n)$ in $(X, \|.,.\|)$ is said to be Cauchy if $\|x_n - x_m, y\| \to 0$ as $n, m \to \infty$, for all $x, y \in X$.
(3) A vector 2-norm space $(X, \|.,.\|)$ is said to be 2-Banach space if any Cauchy sequence in $(X, \|.,.\|)$ is a convergent sequence.

**Definition 3.3.** Let $(X, \|.,.\|)$ be a 2-norm space and $\perp$ be an orthogonal relation on $X$. The triplet $(X, \perp, \|.,.\|)$ is said an orthogonal 2-norm space.

**Definition 3.4.** Let $(X, \perp, \|.,.\|)$ be a orthogonal 2-Banach space.
(1) A subset $B \subset X$ is said to be $SO$-$\|.,.\|$-closed if for any $SO$-sequence $(x_n) \subset B$ which $\|.,.\|$-convergent to $x$, then $x \in B$.
(2) $(X, \perp, \|.,.\|)$ is said to be strongly orthogonal $\|.,.\|$-complete (abbreviated as $SO$- $\|.,.\|$-complete) if every $\|.,.\|$-Cauchy $SO$-sequence is $\|.,.\|$-convergent in $X$.
(3) Let $C$ be a non-empty subset of $X$. A mapping $T : C \to C$ is said to be $\perp$-preserving if for each $x, y \in K$ such that $x \perp y$, then $Tx \perp Ty$.

Now, we give the 2-Banach space version of the Meir-Keeler mapping introduced in [17].

**Definition 3.5.** Let $K$ be a non-empty $SO$-$\|.,.\|$-closed subset of orthogonal 2-Banach space $(X, \perp, \|.,.\|)$, and $c, l \in (0, \infty)$ with $c > l$. A mapping $T : K \to K$ satisfies the Meir-Keeler condition whenever for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \|l(x - y), z\| < \varepsilon + \delta(\varepsilon) \Rightarrow \|c(Tx - Ty), z\| < \varepsilon,$$

for all $x, y, z \in K$ and $x \neq y$, $x \perp y$.

Now we are ready to state our main result. In fact, we give the 2-Banach version of the Meir-Keeler fixed point theorem for incomplete orthogonal sets.

**Theorem 3.6.** *Let $(X, \perp, \|.,.\|)$ be an $SO$- $\|.,.\|$-complete orthogonal 2-Banach space (not necessarily $\|.,.\|$-complete). Let $K$ be a nonempty, $SO$-$\|.,.\|$-closed subset of $X$. Assume that $T : K \to K$ 5s a Meir-Keeler's type contraction, $\|.,.\|$-preserving, and $SO$- $\|.,.\|$-continuous. Then $T$ has a unique fixed point $x \in K$ such that for all $z \in K$ the sequence $(T^n(z))$ is $\|.,.\|$-convergent to $x$.*

*Proof.* Let $x_0$ is an orthogonal element in $X$, we have

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

This implies that $x_0 \perp T(x_0)$ or $T(x_0) \perp x_0$. Let us consider the sequence $(x_n)$ where $x_n = T^n(x_0)$ for all $n \in \mathbb{N}$. It is obvious that

$$(\forall n \in \mathbb{N}, x_0 \perp x_n) \text{ or } (\forall n \in \mathbb{N}, x_n \perp x_0).$$

Since $T$ is a $\perp$-preserving map, we have either

$$(\forall n, k \in \mathbb{N}, x_k = T^k(x_0) \perp x_{n+k} = T^k(x_n))$$

or

$$(\forall n, k \in \mathbb{N}, x_{n+k} = T^k(x_n)) \perp x_k = T^k(x_0).$$

This implies that $(x_n)$ is an $SO$-sequence.
Now, we verify that for all $z \in X$, $\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = 0$.
If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the result follows. Now, let for all $n \in \mathbb{N}$, $x_n \neq x_{n+1}$. By using the Meir-Keeler condition, for all $n \in \mathbb{N}$ and $z \in X$, we have

$$\|l(x_{n+1} - x_n), z\| \leq \|l(x_n - x_{n-1}), z\|.$$

Since the sequence $(\|l(x_{n+1} - x_n), z\|)$ is strictly decreasing in $\mathbb{R}$, put

$$r := \lim_{n\to\infty} \|l(x_{n+1} - x_n), z\|.$$

We claim that $r = 0$. Suppose that $r > 0$. By the Meir-Keeler condition, we have $\delta(r) > 0$ such that $x \neq y$, $x \perp y$ and $r \leq \|l(x - y), z\| < r + \delta(r)$, then $\|c(Tx - Ty), z\| < r$. On the other hand, $\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = r$, then there exists $t_0 \in \mathbb{N}$ such that $r \leq \|l(x_{t_0} - x_{t_0-1}), z\| < r + \delta(r)$, then $\|c(Tx_{t_0} - Tx_{t_0-1}), z\| < r$. So $\|c(x_{t_0} - x_{t_0-1}), z\| < r$ and this is a contradiction. Therefore it must be $r = 0$ and we obtain

$$\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = 0.$$

Next we prove that $(x_n)$ is $\|.,.\|$-cauchy in $X$. Let $\varepsilon > 0$ be given. Choose $\xi > 0$ with $\Im\xi < \varepsilon$. By using the Meir-Keeler condition, there exists $\delta(\xi) > 0$ such that $x \neq y$, $x \perp y$ and $\xi \leq \|l(x - y), z\| < \xi + \delta(\xi)$. Therefore

$$\|c(Tx - Ty), z\| < \xi.$$

Put $\delta' = \min\ \{1, \xi, \delta(\xi)\}$. So we have $x \neq y$, $x \perp y$ and $\xi < \|l(x - y), z\| < \xi + \delta'$. Therefore $\|c(Tx - Ty), z\| < \xi$.
Since $\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = 0$, there exists $j_0 \in N$ such that for all $n \geq j_0$, we have $\|l(x_{n+1} - x_n), z\| < \frac{\delta'}{\delta}$.
Let

$$\sum = \left\{ k \in N : k \geq j_0 \text{ and } \|l(x_k - x_{j_0}), z\| < \xi + \frac{\delta'}{2} \right\}$$

clearly, $\sum \neq \emptyset$. we must next prove that $m \in \sum$ such that $m + 1 \in \sum$. Let $m \in \sum$. So $\|l(x_m - x_{j_0}), z\| < \xi + \frac{\delta'}{2}$.
If $m = j_0$, then $m + 1 \in \sum$, by ..... Let $m > j_0$. Now, we consider two cases:

Case 1. Let $\xi \le \|l(x_m - x_{j_0}), z\| < \xi + \frac{\delta'}{2}$.
Since $x_m$ and $x_{j_0}$ are $\perp$-comparable. We have

$$\|c(Tx_m - Tx_{j_0}, z)\| < \xi.$$

From the definition of 2-norm, we get

$$\begin{aligned}\|c(x_{m+1} - x_{j_0}, z)\| &= \|c(Tx_m - x_{j_0}, z)\| \\ &= \|c(Tx_m - Tx_{j_0} + Tx_{j_0} - x_{j_0}), z)\| \\ &\le \|c(Tx_m - Tx_{j_0}), z\| + \|c(Tx_{j_0} - x_{j_0}), z\| \\ &< \xi + \frac{\delta'}{8} \\ &< \xi + \frac{\delta'}{2}\end{aligned}$$

which means that $m + 1 \in \Sigma$.

Case 2. $\|c(x_m - x_{j_0}), z\| < \xi$. We then have

$$\begin{aligned}\|c(x_{m+1} - x_{j_0}), z)\| &= \|cx_{m+1} - cx_{j_0} + cx_m - cx_m, z\| \\ &\le \|c(x_{m+1} - x_m), z\| + \|c(x_m - x_{j_0}), z\| \\ &< \xi + \frac{\delta'}{8} \\ &< \xi + \frac{\delta'}{2}\end{aligned}$$

So $m + 1 \in \Sigma$.

Therefore, we get for all $k \ge j_0$

$$\|l(x_k - x_{j_0}), z)\| < \xi + \frac{\delta'}{2}.$$

For all $m, n \in \mathbb{N}$ with $m \ge n \ge j_0$, we obtain

$$\begin{aligned}\|l(x_m - x_n), z)\| &\le \|l(x_m - x_{j_0}), z\| + \|l(x_n - x_{j_0}), z)\| \\ &< 2\xi + \delta' \\ &\le 3\xi \\ &< \varepsilon.\end{aligned}$$

Therefore, $(x_n)$ is $\|.,.\|$-Cauchy So-sequence in $X$. Since $X$ is an $SO$-$\|.,.\|$-complete orthogonal 2-Banach space and $B$ is an $SO$- $\|.,.\|$-closed subset of $X$, then there exists $x \in B$ such that for all $z \in X$ ,

$$\lim_{n\to\infty} \|c(x_n - x), z\| = 0.$$

On the other hand, $T$ is $SO$- $\|.,.\|$-continuous mapping. Therefore $\lim_{n\to\infty}\|c(Tx_n - Tx), z\| = 0$. So for all $\varepsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|c(x_{n_0+1} - x), z\| < \frac{\varepsilon}{2} \text{ and } \|c(Tx_{n_0} - Tx), z\| < \frac{\varepsilon}{2}.$$

Now, we have

$$\|c(Tx - x), z)\| \leq \|c(Tx - Tx_{n_0}), z\| + \|c(Tx_{n_0} - x), z)\| < \varepsilon$$

Therefore $\|c(Tx - x), z)\| = 0$, for all $z \in X$, which implies that $Tx = x$. Since $x_0$ is an orthogonal element in $X$, one of the following holds:

$$x_0 \perp x \text{ and } x_o \perp y$$

or

$$x \perp x_0 \text{ and } y \perp x_0$$

fot all $y \in B$. Since $T$ is $\perp$-preserving, we have $T^n x_o \perp x$ and $T^n x_o \perp T^n y$ or $x \perp T^n x_o$ and $T^n y \perp T^n x_o$ for all $n \in \mathbb{N}$.
Using Meir-Keeler contraction, we conclude that the sequence $\{\|l(T^n x - T^n x_o), z\|\}$ is strictly decreasing for all $z \in X$. According to the scheme used in step 1, it could be obtained that $\lim_{n\to\infty}\|c(T^n x - T^n x_o), z\| = 0$. For all $n \in \mathbb{N}$, we have

$$\|c(T^n y - x), z\| \leq \|c(T^n y - T^n x_0), z\| + \|c(T^n x_0 - x), z\|.$$

Thus, $\lim_{n\to\infty}\|c(T^n y - x), z\| = 0$. Now we show that the set of the fixed points of $T$ is a singletone set. Assume that $t\ inB$ is a fixed point of $T$. Then $T^n(t) = t$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty}\|T^n(t) - t, z\| = 0$ for all $z \in X$ and also we know that $\lim_{n\to\infty}\|T^n(t) - x, z\| = 0$ for all $z \in X$. Therefore the uniqueness of limit implies that $t = x$ and so the proof is complete. □

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