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## The Meir-Keeler fixed point theorem in incomplete 2-normed spaces

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**ABSTRACT.** In this paper, we give a new fixed point theorem for the Meir-Keeler mappings defined on incomplete orthogonal 2-Banach spaces. This can be 2-Banach version of the Meir-Keeler fixed point theorem.

**Keywords:** Meir-Keeler mapping; Orthogonal 2-Banach spaces; Fixed point.

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### 1. INTRODUCTION

Throughout this article,  $\mathbb{N}$  stands for the set of all positive integers, and  $\mathbb{R}_+ = [0, \infty)$ . Fixed point theory for contractive and non-expansive mapping defined in Banach spaces has been extensively developed since the mid 1960s; see for example [3, 11, 15]. The Banach contraction principle [2] is the most widely applied fixed point result in many branches of mathematics and considered as the main source of metric fixed point theory.

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**Theorem 1.1.** [2] *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow X$  be a mapping such that there exists a nonnegative number  $k < 1$  and for all  $x, y \in X$*

$$d(Tx, Ty) \leq kd(x, y).$$

*Then,  $T$  has a unique fixed point in  $X$ .*

In 1969, Meir and Keeler [17] gave an interesting generalization the Banach contraction principle as the following; see also [19].

**Theorem 1.2.** *If a mapping  $T : X \rightarrow X$  where  $(X, d)$  is a complete metric space, and for every  $\varepsilon > 0$  there exists  $\delta(\varepsilon) > 0$  such that for  $x, y \in X$ ,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon.$$

*Then,  $T$  has a unique fixed point.*

In 1965, Gähler [9, 10] initiated the notion of 2-normed spaces. There was a strong interest to study the fixed point property in 2-Banach spaces after the first paper [14] was published in 1977. Many problems in metric and Banach space can be reformulated in 2-Banach spaces. The fixed point property in 2-Banach spaces has been defined and investigate by many authors. For more details on 2-Banach spaces, the reader may consult the book [8].

In order to generalized Banach contraction principle Eshaghi Gordji et al. [7] introduced the notion of orthogonal set in 2017. Fixed point properties in orthogonal spaces have been studied by many authors [1, 5, 6, 7, 18].

Our aim in this paper is to prepare a fixed point theorem for the Meir-Keeler mappings on incomplete orthogonal 2-Banach spaces. In other words, 2-Banach version of the Meir-Keeler fixed point theorem is introduced. Our results are extremely new in 2-banach space.

## 2. PRELIMINARIES

We begin this section by recalling the definition of an orthogonal set. This property is central in our considerations; see [1, 7].

**Definition 2.1.** Let  $X$  be a non-empty set and  $\perp$  be a binary relation defined on  $X \times X$ . Then  $(X, \perp)$  is said to be an orthogonal set (abbreviated as  $O$ - set) if there exists  $x_0 \in X$  such that

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

The element  $x_0$  is called an orthogonal element. An orthogonal set may have more than one orthogonal element.

**Example 2.2.** Let  $X$  be a norm linear space. For any two element  $x, y \in X$ , we define the notion  $\perp$  by  $x \perp y$  iff  $\|x\| \leq \|x + \lambda y\|$  for all

$\lambda \in \mathbb{R}$ . It is easy to see that  $x \perp 0$  and  $0 \perp x$  for all  $x \in X$ . Then  $(X, \perp)$  is an  $O$ -set.

**Definition 2.3.** Let  $(X, \perp)$  be an  $O$ -set. A sequence  $(x_n)$  in  $X$  is said to be an orthogonal sequence (abbreviated as  $O$ -sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

**Definition 2.4.** Let  $(X, \perp)$  be an  $O$ -set. A sequence  $(x_n)$  in  $X$  is said to be a strongly orthogonal sequence (abbreviated as  $SO$ -sequence) if

$$(\forall n, k \in \mathbb{N}, x_k \perp x_{n+k}) \text{ or } (\forall n, k \in \mathbb{N}, x_{n+k} \perp x_k).$$

From the definition, it is obvious that each  $SO$ -sequence is an  $O$ -sequence, but the converse is not true; see [1, 7]. The following example shows that the converse is not true.

**Example 2.5.** Let us set  $X = \mathbb{R}_+$ , and we define a binary relation  $\perp$  on  $X$  by

$$x \perp y \text{ if and only if } xy = \min\{x, y\}.$$

It is easy to see that  $x \perp 0$  for all  $x \in X$ . Hence  $(X, \perp)$  is an  $O$ -set. Define the sequence  $(x_n)$  in  $X$  as follows:

$$x_n = \begin{cases} 0 & \text{if } n = 2k, \\ 2 & \text{if } n = 2k + 1 \end{cases}$$

such that  $k \in \mathbb{N}$ . Clearly, for all  $n \in \mathbb{N}$ ,  $x_n \perp x_{n+1}$ . But  $x_{2n+1}$  is not orthogonal to  $x_{4n+1}$ . This shows that,  $(x_n)$  is an  $O$ -sequence but not an  $SO$ -sequence.

**Definition 2.6.** Let  $\|\cdot\|$  be a norm on linear space  $X$ , and  $(X, \perp)$  be an  $O$ -set. The triplet  $(X, \perp, \|\cdot\|)$  is called an orthogonal norm space.

**Definition 2.7.** Let  $(X, \perp, \|\cdot\|)$  be an orthogonal norm space.  $X$  is called strongly orthogonal complete (abbreviated as  $SO$ -complete) if every Cauchy  $SO$ -sequence is convergent.

Obviously, every complete norm space (Banach space) is  $SO$ -complete. The following example shows that the converse is not true.

**Example 2.8.** Let  $C_b(\mathbb{R})$  denotes the set of all real-valued bounded continuous functions on  $\mathbb{R}$ . The collection of real valued functions on  $\mathbb{R}$  that have compact support is denoted by  $C_c(\mathbb{R})$ . In other words,  $f \in C_b(\mathbb{R})$  belongs to  $C_c(\mathbb{R})$  if and only if the set  $\{x \in X : f(x) \neq 0\}$  has compact closure. We note that the space  $C_c(\mathbb{R})$  with the norm  $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$  is not complete.

Let  $X = C_c(\mathbb{R})$ . For any two elements  $f, g \in X$ , we define the orthogonal relation  $\perp$  as follows:

$f \perp g$  if and only if  $f(t)g(t) \geq \max \{f(t), g(t)\}$  for all  $t \in \mathbb{R}$ .

We show that  $X$  is  $SO$ -complete. Let  $\{f_n\}$  be a cauchy  $SO$ -sequence in  $X$ . The completeness of  $C_b(\mathbb{R})$  with respect to the supremum norm, yields the existence of the limit. So there exists  $f \in C_b(\mathbb{R})$  such that  $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$ . Since for all  $n \in \mathbb{N}$  and  $t \in \mathbb{R}$ ,  $f_n(t) \geq 1$  it could be concluded that,  $f \in C_c(\mathbb{R})$ .

**Definition 2.9.** Let  $C$  be a non-empty subset an orthogonal norm space  $(X, \perp, \|\cdot\|)$ . A mapping  $T : C \rightarrow C$  is said to be strongly orthogonal  $\|\cdot\|$ -continuous (abbreviated as  $SO$ -  $\|\cdot\|$ -continuous) in  $x \in C$  if for any  $SO$ -sequence  $(x_n) \subset C$  which  $\|\cdot\|$ -convergent to  $x$ , then  $(T(x_n))$  be  $\|\cdot\|$ -convergent to  $Tx$ .

It can be easily derived that every continuous mapping is  $SO$ -continuous. By the following example, we show that the converse is not true.

**Example 2.10.** Let  $X = \mathbb{R}$  equipped with the usual metric. That is, for  $x, y \in X$ ,  $d(x, y) = |x - y|$ . We define a binary relation  $\perp$  on  $X$  by

$$x \perp y \text{ if and only if } xy = \min \{x, y\}.$$

Define  $f : X \rightarrow X$  by

$$f(x) = \begin{cases} 1 & \text{if } x \in Q, \\ x & \text{if } x \in Q^c \end{cases}$$

It is easy to see that  $f$  is not continuous in 0. Let  $(x_n)$  be an  $SO$ -sequence in  $X$  which converges to 0. According to the definition of relation  $\perp$ , we have  $x_n = 0$ , for all  $n \in \mathbb{N}$ . This implies that  $f(x_n) = 1 \rightarrow 1 = f(0)$ . Thus,  $f$  is  $SO$ -continuous in 0.

### 3. MEIR-KEELER MAPPING IN ORTHOGONAL 2-BANACH SPACES

We state a brief recollection of basic concepts and facts in 2-norm spaces [8].

**Definition 3.1.** Let  $X$  be a real vector space of dimension greater than 1. A function  $\|\cdot, \cdot\| : X \times X \rightarrow [0, \infty)$  is called a 2-norm on  $X$  if the following properteis hold:

- (1)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly dependent,
- (2)  $\|x, y\| = \|y, x\|$  for all  $x, y \in X$ ,
- (3)  $\|\alpha x, y\| = |\alpha| \|x, y\|$  for all  $\alpha \in \mathbb{R}$  and  $x, y \in X$ ,
- (4)  $\|x, y + z\| \leq \|x, y\| + \|x, z\|$  for all  $x, y, z \in X$ .

The pair  $(X, \|\cdot, \cdot\|)$  is called a vector 2-norm space.

**Definition 3.2.** Let  $(X, \|\cdot, \cdot\|)$  be a vector 2-norm space.

- (1) A sequence  $(x_n)$  in  $(X, \|\cdot, \cdot\|)$  is said to be convergent to  $x \in X$  if  $\|x_n - x, y\| \rightarrow 0$  as  $n \rightarrow \infty$ , for all  $y \in X$ .

- (2) The sequence  $(x_n)$  in  $(X, \|\cdot, \cdot\|)$  is said to be Cauchy if  $\|x_n - x_m, y\| \rightarrow 0$  as  $n, m \rightarrow \infty$ , for all  $x, y \in X$ .
- (3) A vector 2-norm space  $(X, \|\cdot, \cdot\|)$  is said to be 2-Banach space if any Cauchy sequence in  $(X, \|\cdot, \cdot\|)$  is a convergent sequence.

**Definition 3.3.** Let  $(X, \|\cdot, \cdot\|)$  be a 2-norm space and  $\perp$  be an orthogonal relation on  $X$ . The triplet  $(X, \perp, \|\cdot, \cdot\|)$  is said an orthogonal 2-norm space.

**Definition 3.4.** Let  $(X, \perp, \|\cdot, \cdot\|)$  be a orthogonal 2-Banach space.

- (1) A subset  $B \subset X$  is said to be  $SO$ - $\|\cdot, \cdot\|$ -closed if for any  $SO$ -sequence  $(x_n) \subset B$  which  $\|\cdot, \cdot\|$ -convergent to  $x$ , then  $x \in B$ .
- (2)  $(X, \perp, \|\cdot, \cdot\|)$  is said to be strongly orthogonal  $\|\cdot, \cdot\|$ -complete (abbreviated as  $SO$ - $\|\cdot, \cdot\|$ -complete) if every  $\|\cdot, \cdot\|$ -Cauchy  $SO$ -sequence is  $\|\cdot, \cdot\|$ -convergent in  $X$ .
- (3) Let  $C$  be a non-empty subset of  $X$ . A mapping  $T : C \rightarrow C$  is said to be  $\perp$ -preserving if for each  $x, y \in K$  such that  $x \perp y$ , then  $Tx \perp Ty$ .

Now, we give the 2-Banach space version of the Meir-Keeler mapping introduced in [17].

**Definition 3.5.** Let  $K$  be a non-empty  $SO$ - $\|\cdot, \cdot\|$ -closed subset of orthogonal 2-Banach space  $(X, \perp, \|\cdot, \cdot\|)$ , and  $c, l \in (0, \infty)$  with  $c > l$ . A mapping  $T : K \rightarrow K$  satisfies the Meir-Keeler condition whenever for every  $\varepsilon > 0$  there exists  $\delta(\varepsilon) > 0$  such that

$$\varepsilon \leq \|l(x - y), z\| < \varepsilon + \delta(\varepsilon) \Rightarrow \|c(Tx - Ty), z\| < \varepsilon,$$

for all  $x, y, z \in K$  and  $x \neq y, x \perp y$ .

Now we are ready to state our main result. In fact, we give the 2-Banach version of the Meir-Keeler fixed point theorem for incomplete orthogonal sets.

**Theorem 3.6.** *Let  $(X, \perp, \|\cdot, \cdot\|)$  be an  $SO$ - $\|\cdot, \cdot\|$ -complete orthogonal 2-Banach space (not necessarily  $\|\cdot, \cdot\|$ -complete). Let  $K$  be a nonempty,  $SO$ - $\|\cdot, \cdot\|$ -closed subset of  $X$ . Assume that  $T : K \rightarrow K$  is a Meir-Keeler's type contraction,  $\|\cdot, \cdot\|$ -preserving, and  $SO$ - $\|\cdot, \cdot\|$ -continuous. Then  $T$  has a unique fixed point  $x \in K$  such that for all  $z \in K$  the sequence  $(T^n(z))$  is  $\|\cdot, \cdot\|$ -convergent to  $x$ .*

*Proof.* Let  $x_0$  is an orthogonal element in  $X$ , we have

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

This implies that  $x_0 \perp T(x_0)$  or  $T(x_0) \perp x_0$ . Let us consider the sequence  $(x_n)$  where  $x_n = T^n(x_0)$  for all  $n \in \mathbb{N}$ . It is obvious that

$$(\forall n \in \mathbb{N}, x_0 \perp x_n) \text{ or } (\forall n \in \mathbb{N}, x_n \perp x_0).$$

Since  $T$  is a  $\perp$ -preserving map, we have either

$$(\forall n, k \in \mathbb{N}, x_k = T^k(x_0) \perp x_{n+k} = T^k(x_n))$$

or

$$(\forall n, k \in \mathbb{N}, x_{n+k} = T^k(x_n) \perp x_k = T^k(x_0)).$$

This implies that  $(x_n)$  is an  $SO$ -sequence.

Now, we verify that for all  $z \in X$ ,  $\lim_{n \rightarrow \infty} \|l(x_{n+1} - x_n), z\| = 0$ .

If there exists  $n_0 \in \mathbb{N}$  such that  $x_{n_0} = x_{n_0+1}$ , then the result follows.

Now, let for all  $n \in \mathbb{N}$ ,  $x_n \neq x_{n+1}$ . By using the Meir-Keeler condition, for all  $n \in \mathbb{N}$  and  $z \in X$ , we have

$$\|l(x_{n+1} - x_n), z\| \leq \|l(x_n - x_{n-1}), z\|.$$

Since the sequence  $(\|l(x_{n+1} - x_n), z\|)$  is strictly decreasing in  $\mathbb{R}$ , put

$$r := \lim_{n \rightarrow \infty} \|l(x_{n+1} - x_n), z\|.$$

We claim that  $r = 0$ . Suppose that  $r > 0$ . By the Meir-Keeler condition, we have  $\delta(r) > 0$  such that  $x \neq y$ ,  $x \perp y$  and  $r \leq \|l(x - y), z\| < r + \delta(r)$ , then  $\|c(Tx - Ty), z\| < r$ . On the other hand,  $\lim_{n \rightarrow \infty} \|l(x_{n+1} - x_n), z\| = r$ , then there exists  $t_0 \in \mathbb{N}$  such that  $r \leq \|l(x_{t_0} - x_{t_0-1}), z\| < r + \delta(r)$ , then  $\|c(Tx_{t_0} - Tx_{t_0-1}), z\| < r$ . So  $\|c(x_{t_0} - x_{t_0-1}), z\| < r$  and this is a contradiction. Therefore it must be  $r = 0$  and we obtain

$$\lim_{n \rightarrow \infty} \|l(x_{n+1} - x_n), z\| = 0.$$

Next we prove that  $(x_n)$  is  $\|\cdot, \cdot\|$ -cauchy in  $X$ . Let  $\varepsilon > 0$  be given. Choose  $\xi > 0$  with  $\Im \xi < \varepsilon$ . By using the Meir-Keeler condition, there exists  $\delta(\xi) > 0$  such that  $x \neq y$ ,  $x \perp y$  and  $\xi \leq \|l(x - y), z\| < \xi + \delta(\xi)$ . Therefore

$$\|c(Tx - Ty), z\| < \xi.$$

Put  $\delta' = \min \{1, \xi, \delta(\xi)\}$ . So we have  $x \neq y$ ,  $x \perp y$  and  $\xi < \|l(x - y), z\| < \xi + \delta'$ . Therefore  $\|c(Tx - Ty), z\| < \xi$ .

Since  $\lim_{n \rightarrow \infty} \|l(x_{n+1} - x_n), z\| = 0$ , there exists  $j_0 \in \mathbb{N}$  such that for all  $n \geq j_0$ , we have  $\|l(x_{n+1} - x_n), z\| < \frac{\delta'}{8}$ .

Let

$$\Sigma = \left\{ k \in \mathbb{N} : k \geq j_0 \text{ and } \|l(x_k - x_{j_0}), z\| < \xi + \frac{\delta'}{2} \right\}$$

clearly,  $\Sigma \neq \emptyset$ . we must next prove that  $m \in \Sigma$  such that  $m + 1 \in \Sigma$ .

Let  $m \in \Sigma$ . So  $\|l(x_m - x_{j_0}), z\| < \xi + \frac{\delta'}{2}$ .

If  $m = j_0$ , then  $m + 1 \in \Sigma$ , by ..... Let  $m > j_0$ . Now, we consider two cases:

Case 1. Let  $\xi \leq \|l(x_m - x_{j_0}), z\| < \xi + \frac{\delta'}{2}$ .  
Since  $x_m$  and  $x_{j_0}$  are  $\perp$ -comparable. We have

$$\|c(Tx_m - Tx_{j_0}, z)\| < \xi.$$

From the definition of 2-norm, we get

$$\begin{aligned} \|c(x_{m+1} - x_{j_0}, z)\| &= \|c(Tx_m - x_{j_0}, z)\| \\ &= \|c(Tx_m - Tx_{j_0} + Tx_{j_0} - x_{j_0}), z)\| \\ &\leq \|c(Tx_m - Tx_{j_0}), z\| + \|c(Tx_{j_0} - x_{j_0}), z\| \\ &< \xi + \frac{\delta'}{8} \\ &< \xi + \frac{\delta'}{2} \end{aligned}$$

which means that  $m + 1 \in \Sigma$ .

Case 2.  $\|c(x_m - x_{j_0}), z\| < \xi$ . We then have

$$\begin{aligned} \|c(x_{m+1} - x_{j_0}, z)\| &= \|cx_{m+1} - cx_{j_0} + cx_m - cx_m, z\| \\ &\leq \|c(x_{m+1} - x_m), z\| + \|c(x_m - x_{j_0}), z\| \\ &< \xi + \frac{\delta'}{8} \\ &< \xi + \frac{\delta'}{2} \end{aligned}$$

So  $m + 1 \in \Sigma$ .

Therefore, we get for all  $k \geq j_0$

$$\|l(x_k - x_{j_0}), z\| < \xi + \frac{\delta'}{2}.$$

For all  $m, n \in \mathbb{N}$  with  $m \geq n \geq j_0$ , we obtain

$$\begin{aligned} \|l(x_m - x_n), z\| &\leq \|l(x_m - x_{j_0}), z\| + \|l(x_n - x_{j_0}), z\| \\ &< 2\xi + \delta' \\ &\leq 3\xi \\ &< \varepsilon. \end{aligned}$$

Therefore,  $(x_n)$  is  $\|\cdot, \cdot\|$ -Cauchy So-sequence in  $X$ . Since  $X$  is an  $SO$ - $\|\cdot, \cdot\|$ -complete orthogonal 2-Banach space and  $B$  is an  $SO$ - $\|\cdot, \cdot\|$ -closed subset of  $X$ , then there exists  $x \in B$  such that for all  $z \in X$ ,

$$\lim_{n \rightarrow \infty} \|c(x_n - x), z\| = 0.$$

On the other hand,  $T$  is  $SO$ - $\|\cdot, \cdot\|$ -continuous mapping. Therefore  $\lim_{n \rightarrow \infty} \|c(Tx_n - Tx), z\| = 0$ . So for all  $\varepsilon \geq 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\|c(x_{n_0+1} - x), z\| < \frac{\varepsilon}{2} \text{ and } \|c(Tx_{n_0} - Tx), z\| < \frac{\varepsilon}{2}.$$

Now, we have

$$\|c(Tx - x), z\| \leq \|c(Tx - Tx_{n_0}), z\| + \|c(Tx_{n_0} - x), z\| < \varepsilon$$

Therefore  $\|c(Tx - x), z\| = 0$ , for all  $z \in X$ , which implies that  $Tx = x$ . Since  $x_0$  is an orthogonal element in  $X$ , one of the following holds:

$$x_0 \perp x \text{ and } x_0 \perp y$$

or

$$x \perp x_0 \text{ and } y \perp x_0$$

for all  $y \in B$ . Since  $T$  is  $\perp$ -preserving, we have  $T^n x_0 \perp x$  and  $T^n x_0 \perp T^n y$  or  $x \perp T^n x_0$  and  $T^n y \perp T^n x_0$  for all  $n \in \mathbb{N}$ .

Using Meir-Keeler contraction, we conclude that the sequence  $\{\|l(T^n x - T^n x_0), z\|\}$  is strictly decreasing for all  $z \in X$ . According to the scheme used in step 1, it could be obtained that  $\lim_{n \rightarrow \infty} \|c(T^n x - T^n x_0), z\| = 0$ . For all  $n \in \mathbb{N}$ , we have

$$\|c(T^n y - x), z\| \leq \|c(T^n y - T^n x_0), z\| + \|c(T^n x_0 - x), z\|.$$

Thus,  $\lim_{n \rightarrow \infty} \|c(T^n y - x), z\| = 0$ . Now we show that the set of the fixed points of  $T$  is a singleton set. Assume that  $t \in B$  is a fixed point of  $T$ . Then  $T^n(t) = t$  for all  $n \in \mathbb{N}$ . So  $\lim_{n \rightarrow \infty} \|T^n(t) - t, z\| = 0$  for all  $z \in X$  and also we know that  $\lim_{n \rightarrow \infty} \|T^n(t) - x, z\| = 0$  for all  $z \in X$ . Therefore the uniqueness of limit implies that  $t = x$  and so the proof is complete.  $\square$

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