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The Meir-Keeler fixed point theorem in incomplete 2-normed spaces

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Abstract. In this paper, we give a new fixed point theorem for the Meir-Keeler mappings definded on incomplete orthogonal 2-Banach spaces. This can be 2-Banach version of the Meir-Keeler fixed point theorem.

Keywords: Meir-Keeler mapping; Orthogonal 2-Banach spaces; Fixed point.

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1. Introduction

Throughout this article, $\mathbb{N}$ stands for the set of all positive integers, and $\mathbb{R}_+ = [0, \infty)$. Fixed point theory for contractive and non-expansive mapping defined in Banach spaces has been extensively developed since the mid 1960s; see for example [3, 11, 15]. The Banach contraction principle [2] is the most wildely applied fixed point result in many branches of mathematics and considered as the main source of metric fixed point theory.

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Theorem 1.1. [2] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists a nonnegative number $k < 1$ and for all $x, y \in X$*

$$d(Tx, Ty) \leq kd(x, y).$$

Then, T has a unique fixed point in X.

In 1969, Meir and Keeler [17] gave an interesting generalization the Banach contraction principle as the following; see also [19].

Theorem 1.2. *If a mapping $T : X \to X$ where (X, d) is a complete metric space, and for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for $x, y \in X$,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon.$$

Then, T has a unique fixed point.

In 1965, Gahler [9, 10] initiated the notion of 2-normed spaces. There was a strong interest to study the fixed point property in 2-Banach spaces after the first paper [14] was published in 1977. Many problems in metric and Banach space can be reformulated in 2-Banach spaces. The fixed point property in 2-Banach spaces has been defined and investigate by many authors. For more details on 2-Banach spaces, the reader may consult the book [8].

In order to generalized Banach contraction principle Eshaghi Gordji et al. [7] introduced the notion of orthogonal set in 2017. Fixed point properties in orthogonal spaces have been studied by many authors [1, 5, 6, 7, 18].

Our aim in this paper is to prepare a fixed point theorem for the Meir-Keeler mappings on incomplete orthogonal 2-Banach spaces. In other words, 2-Banach version of the Meir-Keeler fixed point theorem is introduced. Our results are extremely new in 2-banach space.

2. Preliminaries

We begin this section by recalling the defnition of an orthogonal set. This property is central in our considerations; see [1, 7].

Definition 2.1. Let X be a non-empty set and $\perp$ be a binary relation defined on $X \times X$. Then $(X, \perp)$ is said to be an orthogonal set (abbreviated as O- set) if there exists $x_0 \in X$ such that

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

The element x_0 is called an orthogonal element. An orthogonal set may have more than one orthogonal element.

Example 2.2. Let X be a norm linear space. For any two element $x, y \in X$, we define the notion $\perp$ by $x \perp y$ iff $\|x\| \leq \|x + \lambda y\|$ for all

$\lambda \in \mathbb{R}$. It is easy to see that $x \perp 0$ and $0 \perp x$ for all $x \in X$. Then $(X, \perp)$ is an O- set.

Definition 2.3. Let $(X, \perp)$ be an O- set. A sequence (x_n) in X is said to be an orthogonal sequence (abbreviated as O- sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

Definition 2.4. Let $(X, \perp)$ be an O- set. A sequence (x_n) in X is said to be a strongly orthogonal sequence (abbreviated as SO- sequence) if

$$(\forall n, k \in \mathbb{N}, x_k \perp x_{n+k}) \text{ or } (\forall n, k \in \mathbb{N}, x_{n+k} \perp x_k).$$

From the definition, it is obvious that each SO-sequence is an O-sequence, but the converse is not true; see [1, 7]. The following example shows that the converse is not true.

Example 2.5. Let us set $X = \mathbb{R}_+$, and we define a binary relation $\perp$ on X by

$$x \perp y \text{ if and only if } xy = \min\{x, y\}.$$

It is easy to see that $x \perp 0$ for all $x \in X$. Hence $(X, \perp)$ is an O- set. Define the sequence (x_n) in X as follows:

$$x_n = \begin{cases} 0 & \text{if } n = 2k, \\ 2 & \text{if } n = 2k+1 \end{cases}$$

such that $k \in \mathbb{N}$. Clearly, for all $n \in \mathbb{N}$, $x_n \perp x_{n+1}$. But x_{2n+1} is not orthogonal to x_{4n+1}. This shows that, (x_n) is an O-sequence but not an SO-sequence.

Definition 2.6. Let $\|.\|$ be a norm on linear space X, and $(X, \perp)$ be an O- set.The triplet $(X, \perp, \|.\|)$ is called an orthogonal norm space.

Definition 2.7. Let $(X, \perp, \|.\|)$ be an orthogonal norm space. X is called strongly orthogonal complete (abbreviated as SO-complete) if every Cauchy SO-sequence is convergent.

Obviously, every complete norm space (Banach space) is SO-complete. The following example shows that the converse is not true.

Example 2.8. Let $C_b(\mathbb{R}))$ denotes the set of all real-valued bounded continuous functions on $\mathbb{R}$. The collection of real valued functions on $\mathbb{R}$ that have compact support is denoted by $C_c(\mathbb{R})$. In other words, $f \in C_b(\mathbb{R})$ belongs to $C_c(\mathbb{R})$ if and only if the set $\{x \in X : f(x) \neq 0\}$ has compact closure. We note that the space $C_c(\mathbb{R})$ with the norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is not complete.

Let $X = C_c(\mathbb{R})$. For any two elements $f, g \in X$, we define the orthogonal relation $\perp$ as follows:

$$f \perp g \text{ if and only if } f(t)g(t) \geq \max\{f(t), g(t)\} \text{ for all } t \in \mathbb{R}.$$

We show that X is SO-complete. Let $\{f_n\}$ be a cauchy SO-sequence in X. The completeness of $C_b(\mathbb{R})$ with respect to the supremum norm, yields the existence of the limit. So there exists $f \in C_b(\mathbb{R})$ such that $\lim\limits_{n\to\infty} \|f_n - f\| = 0$. Since for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $f_n(t) \geq 1$ it could be concluded that, $f \in C_c(\mathbb{R})$.

Definition 2.9. Let C be a non-empty subset an orthogonal norm space $(X, \perp, \|.\|)$. A mapping $T : C \to C$ is said to be strongly orthogonal $\|.\|$-continuous (abbreviated as SO- $\|.\|$-continuous) in $x \in C$ if for any SO-sequence $(x_n) \subset C$ which $\|.\|$-convergent to x, then $(T(x_n))$ be $\|.\|$-convergent to Tx.

It can be easily derived that every continuous mapping is SO-continuous. By the following example, we show that the converse is not true.

Example 2.10. Let $X = \mathbb{R}$ equipped with the usual metric. That is, for $x, y \in X$, $d(x, y) = |x - y|$. We define a binary relation $\perp$ on X by

$$x \perp y \text{ if and only if } xy = \min\{x, y\}.$$

Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in Q, \\ x & \text{if } x \in Q^c \end{cases}$$

It is easy to see that f is not continuous in 0. Let (x_n) be an SO-sequence in X which converges to 0. According to the definition of relation$\perp$, we have $x_n = 0$, for all $n \in \mathbb{N}$. This implies that $f(x_n) = 1 \to 1 = f(0)$. Thus, f is SO-continuous in 0.

3. Meir-Keeler mapping in orthogonal 2-Banach spaces

We state a brief recollection of basic concepts and facts in 2-norm spaces [8].

Definition 3.1. Let X be a real vector space of dimension greater than 1. A function $\|.,.\| : X \times X \to [0, \infty)$ is called a 2-norm on X if the following properteis hold:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(3) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for all $\alpha \in \mathbb{R}$ and $x, y \in X$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

The pair $(X, \|.,.\|)$ is called a vector 2-norm space.

Definition 3.2. Let $(X, \|.,.\|)$ be a vector 2-norm space.

(1) A sequence (x_n) in $(X, \|.,.\|)$ is said to be convergent to $x \in X$ if $\|x_n - x, y\| \to 0$ as $n \to \infty$, for all $y \in X$.

(2) The sequence (x_n) in $(X, \|., .\|)$ is said to be Cauchy if $\|x_n - x_m, y\| \to 0$ as $n, m \to \infty$, for all $x, y \in X$.
(3) A vector 2-norm space $(X, \|., .\|)$ is said to be 2-Banach space if any Cauchy sequence in $(X, \|., .\|)$ is a convergent sequence.

Definition 3.3. Let $(X, \|., .\|)$ be a 2-norm space and $\perp$ be an orthogonal relation on X. The triplet $(X, \perp, \|., .\|)$ is said an orthogonal 2-norm space.

Definition 3.4. Let $(X, \perp, \|., .\|)$ be a orthogonal 2-Banach space.
(1) A subset $B \subset X$ is said to be SO-$\|., .\|$-closed if for any SO-sequence $(x_n) \subset B$ which $\|., .\|$-convergent to x, then $x \in B$.
(2) $(X, \perp, \|., .\|)$ is said to be strongly orthogonal $\|., .\|$-complete (abbreviated as SO- $\|., .\|$-complete) if every $\|., .\|$-Cauchy SO-sequence is $\|., .\|$-convergent in X.
(3) Let C be a non-empty subset of X. A mapping $T : C \to C$ is said to be $\perp$-preserving if for each $x, y \in K$ such that $x \perp y$, then $Tx \perp Ty$.

Now, we give the 2-Banach space version of the Meir-Keeler mapping introduced in [17].

Definition 3.5. Let K be a non-empty SO-$\|., .\|$-closed subset of orthogonal 2-Banach space $(X, \perp, \|., .\|)$, and $c, l \in (0, \infty)$ with $c > l$. A mapping $T : K \to K$ satisfies the Meir-Keeler condition whenever for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \|l(x - y), z\| < \varepsilon + \delta(\varepsilon) \Rightarrow \|c(Tx - Ty), z\| < \varepsilon,$$

for all $x, y, z \in K$ and $x \neq y$, $x \perp y$.

Now we are ready to state our main result. In fact, we give the 2-Banach version of the Meir-Keeler fixed point theorem for incomplete orthogonal sets.

Theorem 3.6. *Let $(X, \perp, \|., .\|)$ be an SO- $\|., .\|$-complete orthogonal 2-Banach space (not necessarily $\|., .\|$-complete). Let K be a nonempty, SO-$\|., .\|$-closed subset of X. Assume that $T : K \to K$ 5s a Meir-Keeler's type contraction, $\|., .\|$-preserving, and SO- $\|., .\|$-continuous. Then T has a unique fixed point $x \in K$ such that for all $z \in K$ the sequence $(T^n(z))$ is $\|., .\|$-convergent to x.*

Proof. Let x_0 is an orthogonal element in X, we have

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

This implies that $x_0 \perp T(x_0)$ or $T(x_0) \perp x_0$. Let us consider the sequence (x_n) where $x_n = T^n(x_0)$ for all $n \in \mathbb{N}$. It is obvious that

$$(\forall n \in \mathbb{N}, x_0 \perp x_n) \text{ or } (\forall n \in \mathbb{N}, x_n \perp x_0).$$

Since T is a $\perp$-preserving map, we have either

$$(\forall n, k \in \mathbb{N}, x_k = T^k(x_0) \perp x_{n+k} = T^k(x_n))$$

or

$$(\forall n, k \in \mathbb{N}, x_{n+k} = T^k(x_n)) \perp x_k = T^k(x_0).$$

This implies that (x_n) is an SO-sequence.
Now, we verify that for all $z \in X$, $\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = 0$.
If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the result follows. Now, let for all $n \in \mathbb{N}$, $x_n \neq x_{n+1}$. By using the Meir-Keeler condition, for all $n \in \mathbb{N}$ and $z \in X$, we have

$$\|l(x_{n+1} - x_n), z\| \leq \|l(x_n - x_{n-1}), z\|.$$

Since the sequence $(\|l(x_{n+1} - x_n), z\|)$ is strictly decreasing in $\mathbb{R}$, put

$$r := \lim_{n\to\infty} \|l(x_{n+1} - x_n), z\|.$$

We claim that $r = 0$. Suppose that $r > 0$. By the Meir-Keeler condition, we have $\delta(r) > 0$ such that $x \neq y$, $x \perp y$ and $r \leq \|l(x - y), z\| < r + \delta(r)$, then $\|c(Tx - Ty), z\| < r$. On the other hand, $\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = r$, then there exists $t_0 \in \mathbb{N}$ such that $r \leq \|l(x_{t_0} - x_{t_0-1}), z\| < r + \delta(r)$, then $\|c(Tx_{t_0} - Tx_{t_0-1}), z\| < r$. So $\|c(x_{t_0} - x_{t_0-1}), z\| < r$ and this is a contradiction. Therefore it must be $r = 0$ and we obtain

$$\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = 0.$$

Next we prove that (x_n) is $\|.,.\|$-cauchy in X. Let $\varepsilon > 0$ be given. Choose $\xi > 0$ with $\Im\xi < \varepsilon$. By using the Meir-Keeler condition, there exists $\delta(\xi) > 0$ such that $x \neq y$, $x \perp y$ and $\xi \leq \|l(x - y), z\| < \xi + \delta(\xi)$. Therefore

$$\|c(Tx - Ty), z\| < \xi.$$

Put $\delta' = \min\ \{1, \xi, \delta(\xi)\}$. So we have $x \neq y$, $x \perp y$ and $\xi < \|l(x - y), z\| < \xi + \delta'$. Therefore $\|c(Tx - Ty), z\| < \xi$.
Since $\lim_{n\to\infty} \|l(x_{n+1} - x_n), z\| = 0$, there exists $j_0 \in N$ such that for all $n \geq j_0$, we have $\|l(x_{n+1} - x_n), z\| < \frac{\delta'}{\delta}$.
Let

$$\sum = \left\{ k \in N : k \geq j_0 \text{ and } \|l(x_k - x_{j_0}), z\| < \xi + \frac{\delta'}{2} \right\}$$

clearly, $\sum \neq \emptyset$. we must next prove that $m \in \sum$ such that $m + 1 \in \sum$.
Let $m \in \sum$. So $\|l(x_m - x_{j_0}), z\| < \xi + \frac{\delta'}{2}$.
If $m = j_0$, then $m + 1 \in \sum$, by ..... Let $m > j_0$. Now, we consider two cases:

Case 1. Let $\xi \leq \|l(x_m - x_{j_0}), z\| < \xi + \frac{\delta'}{2}$.
Since x_m and x_{j_0} are $\perp$-comparable. We have

$$\|c(Tx_m - Tx_{j_0}, z)\| < \xi.$$

From the definition of 2-norm, we get

$$\begin{aligned}
\|c(x_{m+1} - x_{j_0}, z)\| &= \|c(Tx_m - x_{j_0}, z)\| \\
&= \|c(Tx_m - Tx_{j_0} + Tx_{j_0} - x_{j_0}), z)\| \\
&\leq \|c(Tx_m - Tx_{j_0}), z\| + \|c(Tx_{j_0} - x_{j_0}), z\| \\
&< \xi + \frac{\delta'}{8} \\
&< \xi + \frac{\delta'}{2}
\end{aligned}$$

which means that $m + 1 \in \Sigma$.

Case 2. $\|c(x_m - x_{j_0}), z\| < \xi$. We then have

$$\begin{aligned}
\|c(x_{m+1} - x_{j_0}), z)\| &= \|cx_{m+1} - cx_{j_0} + cx_m - cx_m, z\| \\
&\leq \|c(x_{m+1} - x_m), z\| + \|c(x_m - x_{j_0}), z\| \\
&< \xi + \frac{\delta'}{8} \\
&< \xi + \frac{\delta'}{2}
\end{aligned}$$

So $m + 1 \in \Sigma$.

Therefore, we get for all $k \geq j_0$

$$\|l(x_k - x_{j_0}), z)\| < \xi + \frac{\delta'}{2}.$$

For all $m, n \in \mathbb{N}$ with $m \geq n \geq j_0$, we obtain

$$\begin{aligned}
\|l(x_m - x_n), z)\| &\leq \|l(x_m - x_{j_0}), z\| + \|l(x_n - x_{j_0}), z)\| \\
&< 2\xi + \delta' \\
&\leq 3\xi \\
&< \varepsilon.
\end{aligned}$$

Therefore, (x_n) is $\|.,.\|$-Cauchy So-sequence in X. Since X is an SO-$\|.,.\|$-complete orthogonal 2-Banach space and B is an SO- $\|.,.\|$-closed subset of X, then there exists $x \in B$ such that for all $z \in X$,

$$\lim_{n\to\infty} \|c(x_n - x), z\| = 0.$$

On the other hand, T is SO- $\|.,.\|$-continuous mapping. Therefore $\lim\limits_{n\to\infty}\|c(Tx_n - Tx), z\| = 0$. So for all $\varepsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|c(x_{n_0+1} - x), z\| < \tfrac{\varepsilon}{2} \text{ and } \|c(Tx_{n_0} - Tx), z\| < \tfrac{\varepsilon}{2}.$$

Now, we have

$$\|c(Tx - x), z)\| \leq \|c(Tx - Tx_{n_0}), z\| + \|c(Tx_{n_0} - x), z)\| < \varepsilon$$

Therefore $\|c(Tx - x), z)\| = 0$, for all $z \in X$, which implies that $Tx = x$. Since x_0 is an orthogonal element in X, one of the following holds:

$$x_0 \perp x \text{ and } x_o \perp y$$

or

$$x \perp x_0 \text{ and } y \perp x_0$$

fot all $y \in B$. Since T is $\perp$-preserving, we have $T^n x_o \perp x$ and $T^n x_o \perp T^n y$ or $x \perp T^n x_o$ and $T^n y \perp T^n x_o$ for all $n \in \mathbb{N}$.
Using Meir-Keeler contraction, we conclude that the sequence $\{\|l(T^n x - T^n x_o), z\|\}$ is strictly decreasing for all $z \in X$. According to the scheme used in step 1, it could be obtained that $\lim\limits_{n\to\infty}\|c(T^n x - T^n x_o), z\| = 0$. For all $n \in \mathbb{N}$, we have

$$\|c(T^n y - x), z\| \leq \|c(T^n y - T^n x_0), z\| + \|c(T^n x_0 - x), z\|.$$

Thus, $\lim\limits_{n\to\infty}\|c(T^n y - x), z\| = 0$. Now we show that the set of the fixed points of T is a singletone set. Assume that $t\ inB$ is a fixed point of T. Then $T^n(t) = t$ for all $n \in \mathbb{N}$. So $\lim\limits_{n\to\infty}\|T^n(t) - t, z\| = 0$ for all $z \in X$ and also we know that $\lim\limits_{n\to\infty}\|T^n(t) - x, z\| = 0$ for all $z \in X$. Therefore the uniqueness of limit implies that $t = x$ and so the proof is complete. □

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