Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **13**(1)(2024), 106-117 (Research paper)

On Quasi Primary Ideals and Weakly Quasi Primary Ideals

A. R. Alehafttan [1]
[1] Department of Mathematics, Jundi-Shapur University of Technology, Dezful, Iran

Abstract. Let R be a commutative ring with identity. A proper ideal Q of R is called quasi primary (weakly quasi primary) if whenever $ab \in Q$ $(0 \neq ab \in Q)$ for some $a, b \in R$, then $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. In this paper, we study quasi primary (weakly quasi primary) ideals which are generalization of prime ideals. Our study provides an analogous to the prime avoidance theorem. We determined the Noetherian rings that each ideal of them is quasi primary and the rings that each ideal of them is weakly quasi primary. Besides giving various examples and characterizations of quasi primary and weakly quasi primary and we investigate the relations between them.

Keywords: prime ideal, quasi primary ideal, weakly quasi primary ideal.

2000 Mathematics subject classification: 13A15, 13C05.

1. Introduction

Prime ideals play a central role in commutative ring theory and so this notion has been generalized and studied in several directions. The importance of some of these generalizations is same as the prime ideals, say primary ideals. In a sense they determine how far an ideal is from being prime. For instance, Hedstrom and Houston [11] defined the strongly prime ideal, that is a proper ideal P of R such that for $a, b \in F$ with

[1]Corresponding author: A.R.Alehafttan@jsu.ac.ir
Received: 27 February 2024
Revised: 02 May 2024
Accepted: 21 May 2024

$ab \in P$, either $a \in P$ or $b \in P$ where F is the quotient field of R. Anderson and Smith [1] introduced the notion of a weakly prime ideal, i.e., a proper ideal P of ring R with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. So a prime ideal is weakly prime. Bhatwadekar and Sharma [8] introduced the notion of almost prime ideal which is also a generalization of prime ideal. A proper ideal I of an integral domain D is said to be almost prime if for $a, b \in D$ with $ab \in I \setminus I^2$, then either $a \in I$ or $b \in I$, and it is clear that every weakly prime ideal is an almost prime ideal. The notion of 2-absorbing ideals were introduced and investigated by Badawi [3]. A nonzero proper ideal I of R is called a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. For more about generalizations of prime ideals see [4]-[7] and [12]-[14]. This paper is in this theme and it is devoted to study a generalization of (weakly) prime ideals so called (weakly) quasi primary ideals.

A proper ideal Q of R is said to be (weakly) quasi primary if whenever $a, b \in R$ and $ab \in Q$ $(0 \neq ab \in Q)$, then either a^n or b^n lies in Q, for some $n \in \mathbb{N}$. The concept of quasi primary ideals, was frst introduced and studied by Fuchs in [10]. In Section 2 , we study more about quasi primary ideals and show many their properties. For example we show that such prime and maximal ideals, $\frac{Q}{I}$ is a quasi primary ideal of $\frac{R}{I}$ if and only if, Q is a quasi primary ideal of R and completely determine quasi primary of ring $R_1 \times R_2$ and also Noetherian rings that every its ideals are quasi primary. In Theorem 2.20 we prove the quasi primary avoidance theorem for ideals. Also we introduce and study weakly quasi primary ideal as a genaralization of quasi primary ideal and prove that weakly quasi primary Q is a quasi primary ideal or $Q^2 = 0$, also we show that If every proper ideal of R is a weakly quasi primary ideal of R, then R has at most two maximal ideals. We note that, every prime (quasi primary ideal) ideal is a quasi primary (weakly quasi primary) ideal. However, the converse is not true. For example, $9\mathbb{Z}$ is a quasi primary ideal of R, but it is not prime. For nontrivial quasi primary ideals (weakly quasi primary) see Examples 2.6, 2.7, 2.8 and 3.5.

Throughout this paper rings are commutative with non-zero identity and if S is a subring of R, then $1_S = 1_R$.

2. Quasi Primary Ideals

Definition 2.1. Let Q be a proper ideal of a ring R. We say that Q is quasi primary if for all $a, b \in R$ such that $ab \in Q$, then either $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$.

Example 2.2. Every primary ideal is quasi primary.

Lemma 2.3. *An ideal Q of a ring R is a quasi primary ideal of R if and only if $\sqrt{Q}$ is a prime ideal of R.*

Proof. Let Q be a quasi primary ideal and $ab \in \sqrt{Q}$, then $a^n b^n \in Q$ for some positive integer n. Since Q is quasi primary, $a^{nm} b^{nm} \in Q$ for some positive integer m. That means $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. Hence $\sqrt{Q}$ is a prime ideal. Conversely; Let $\sqrt{Q}$ be a prime ideal of R and $ab \in Q$, so $ab \in \sqrt{Q}$. But $\sqrt{Q}$ is a prime ideal, thus $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$ and this implies $a^n \in Q$ or $b^n \in Q$, for some $n \in \mathbb{N}$. Therefore Q is quasi primary. □

Note 2.4. If Q is a quasi primary and $P = \sqrt{Q}$, then we say that Q is $P-$quasi primary.

Corollary 2.5. *If P is a prime ideal of a ring R, then P^n is a $P-$quasi primary ideal of A for all $n \geq 1$.*

The following examples show that quasi primary ideals are not necessarily prime, primary or power of a prime ideal.

Example 2.6. Let F be a field, and consider the residue class ring R of the ring $F[X_1, X_2, X_3]$ of polynomials over F in indeterminates X_1, X_2, X_3 given by $R = \frac{F[X_1,X_2,X_3]}{(X_1X_3 - X_2^2)}$ and $x_i = X_i + (X_1X_3 - X_2^2)$. Then $Q = (x_1^2, x_2^2, x_1x_2)$ is a non-primary quasi primary ideal of R.

Example 2.7. Let F be a field, and $R = \frac{F[X_1,X_2,X_3,X_4]}{(X_1X_2 - X_3^2)}$ and $x_i = X_i + (X_1X_2 - X_3^2)$. Then (x_3, x_4^2) is a non-primary quasi primary of R which is not also as a power of a prime ideal.

Example 2.8. Let F be a field and $R = F[X, Y]$ where X and Y are two indeterminates. Consider the ideal $Q = (X^2, XY)$ of R. Then Q is not a weakly quasi primary ideal since $0 \neq XY \in Q$ but $X \notin P$ and $Y \notin Q$. But $\sqrt{Q} = (X)$ is a prime ideal of R, so Q is a weakly quasi primary ideal of R.

Proposition 2.9. *Let R be a ring and S be a multiplicatively closed subset of R. If Q is a quasi primary ideal of R, then $S^{-1}Q$ is a quasi primary ideal of $S^{-1}R$.*

Proof. Let $a, b \in R$ and $s_1, s_2 \in S$ such that $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}Q$. Then there exist $x \in Q$ and $r \in S$ such that $\frac{a}{s_1}\frac{b}{s_2} = \frac{x}{r}$. Hence $t(abr - xs_1s_2) = 0$ for some $t \in S$. This shows that $(ta)(rb) \in Q$, so $(ta)^n \in Q$ or $(rb)^n \in Q$, for some $n \in \mathbb{N}$. If $(ta)^n \in Q$, then $(\frac{a}{s_1})^n = \frac{t^n a^n}{(ts_1)^n} \in S^{-1}Q$ and if $(rb)^n \in Q$, then $(\frac{b}{s_2})^n = \frac{r^n b^n}{(rs_2)^n} \in S^{-1}Q$. Therefore $S^{-1}Q$ is a quasi primary ideal of $S^{-1}R$. □

Proposition 2.10. *If $\varphi : S \longrightarrow R$ is a ring homomorphism and Q is a quasi primary ideal of R, then $\varphi^{-1}(Q)$ is a quasi primary ideal of S.*

Proof. If $ab \in \varphi^{-1}(Q)$, then $\varphi(a)\varphi(b) = \varphi(ab) \in Q$. Thus either $\varphi^n(a) = \varphi(a^n) \in Q$ or $\varphi^n(b) = \varphi(b^n) \in Q$, for some $n \in \mathbb{N}$, so either $a^n\varphi^{-1}(Q)$ or $b^n\varphi^{-1}(Q)$. Hence $\varphi^{-1}(Q)$ is a quasi primary ideal of S. □

Corollary 2.11. *Let S be a subring of R and Q be a quasi primary ideal of R with $S \nsubseteq Q$. Then $Q \cap S$ is a quasi primary ideal of S.*

Proposition 2.12. *Let I be an ideal of ring R. Then $\frac{Q}{I}$ is a quasi primary ideal of $\frac{R}{I}$ if and only if, Q is a quasi primary ideal of R.*

Proof. Let Q is a quasi primary ideal of R and $(a+I)(b+I) \in \frac{Q}{I}$, so $ab \in Q$, thus a^n or $b^n \in Q$ for some $n \in \mathbb{N}$ and hence $(a+I)^n \in \frac{Q}{I}$ or $(b+I)^n \in \frac{Q}{I}$. Therefore $\frac{Q}{I}$ is a quasi primary ideal of $\frac{R}{I}$. Conversely; If $\frac{Q}{I}$ is a quasi primary ideal of $\frac{R}{I}$ and $ab \in Q$, then $(a+I)(b+I) \in \frac{Q}{I}$ and hence $(a+I)^n \in \frac{Q}{I}$ or $(b+I)^n \in \frac{Q}{I}$, for some $n \in \mathbb{N}$. Thus $a^n \in Q$ or $b^n \in Q$. This yields that Q is a quasi primary ideal of R. □

Proposition 2.13. *Let R be a principal ideal domain and Q be an ideal of R. Then Q is primary ideal if and only if Q is quasi primary.*

Proof. Assume that Q be quasi primary, so $\sqrt{Q}$ is a prime ideal of R. As R is a principal ideal domain, $\sqrt{Q}$ is a maximal ideal of R. Hence Q is a primary ideal. Conversely; It is evident. □

Proposition 2.14. *Let $\varphi : R \longrightarrow S$ be an epimorphism of rings. If Q is a quasi primary ideal of R containing $ker(\varphi)$, then $\varphi(Q)$ is a quasi primary ideal of S.*

Proof. Let $a, b \in R$ and $ab \in \varphi(Q)$. Since φ is onto, there are $x, y \in S$ such that $a = \varphi(x)$ and $b = \varphi(y)$. Thus $ab = \varphi(xy) \in \varphi(Q)$. This means that there is $q \in Q$ such that $\varphi(xy) = \varphi(q)$. Thus $xy - q \in ker(\varphi) \subseteq Q$. Therefore $xy = (xy - q) + q \in Q$. But Q is a quasi primary ideal, so either $x^n \in Q$ or $y^n \in Q$, for some $n \in \mathbb{N}$ and consequently either $a^n = \varphi(x^n) \in \varphi(Q)$ or $b^n = \varphi(y^n) \in \varphi(Q)$. Hence$\varphi(Q)$ is a quasi primary ideal of S. □

Proposition 2.15. *Let R_1 and R_2 be rings, and let $R = R_1 \times R_2$. Then Q_1 (resp. Q_2) is a quasi primary ideal of R_1 (resp. R_2) if and only if $Q_1 \times R_2$ (resp. $R_1 \times Q_2$) is a quasi primary ideal of R.*

Proof. Suppose that Q_1 is a quasi primary ideal of R_1. Let $(a, b)(x, y) \in Q_1 \times R_2$. Then $ax \in Q_1$. Since Q_1 is quasi prime, either $a^n \in Q_1$ or $x^n \in Q_1$, for some $n \in \mathbb{N}$. Hence either $(a, b)^n \in Q_1 \times R_2$ or $(x, y)^n \in Q_1 \times R_2$.

Thus $Q_1 \times R_2$ is a quasi primary ideal of R. Conversely; Let $Q_1 \times R_2$ be a quasi primary ideal of R, and let $ab \in Q_1$. Then $(a,1)(b,1) \in Q_1 \times R_2$. Hence $(a,1)^n \in Q_1 \times R_2$ or $(b,1)^n \in Q_1 \times R_2$, for some $n \in \mathbb{N}$. Therefore $a^n \in Q_1$ or $b^n \in Q_1$. Thus Q_1 is a quasi primary ideal of R_1. □

Proposition 2.16. *If Q is a $P-$quasi primary ideal of ring R and $a \notin P$, then $(Q : a^n)$ is $P-$quasi primary ideal, for all $n \geq 1$.*

Proof. Let $n \geq 1$ and $x \in (Q : a^n)$. Then $xa^n \in Q$ and $a \notin P$. Since Q is a $P-$quasi primary ideal, we get $x \in \sqrt{Q} = P$. Thus $Q \subseteq (Q : a^n) \subseteq P$ and so $P = \sqrt{Q} \subseteq \sqrt{(Q : a^n)} \subseteq \sqrt{P} = P$. Hence $(Q : a^n)$ is a $P-$quasi primary ideal. □

Proposition 2.17. *If Q is irreducible and for every $a \notin Q$ there exists $n > 1$, such that $(Q : a^n) = (Q : a^{n+1})$, then Q is quasi primary.*

Proof. Let Q be irreducible and let $ab \in Q$ be such that $a \notin \sqrt{Q}$ and $(Q : a^n) = (Q : a^{n+1})$ for some $n \in \mathbb{N}$. If $b \in Q$, then there is nothing to prove. Assume that $b \notin Q$. We show that $b \in \sqrt{Q}$. Suppose to the contrary, $b \notin \sqrt{Q}$. Let $x \in (Q + Ra^n) \cap (Q + Rb^n)$. Then there are $c, d, \in Q$ and $s, t \in R$ such that $x = c + sa^n = d + tb^n$. Hence $xa = ca + sa^{n+1} = da + tb^n a \in Q$. Thus $sa^{n+1} \in Q$, and since $(Q : a^n) = (Q : a^{n+1})$, we conclude that $sa^n \in Q$. Therefore, $x = c + sa^n \in Q$. This shows that $(Q + Ra^n) \cap (Q + Rb^n) \subseteq Q$, and hence $(Q + Ra^n) \cap (Q + Rb^n) = Q$, a contradiction. Thus Q is a quasi primary ideal of R. □

It is well-known that in Noetherian rings irreducible ideals are primary, so are quasi primary too and each ideal has primary decomposition, so it is intersection of finitely many quasi primary ideals. Proposition 2.17 shows that in Noetherian rings, irreducible ideals are quasi primary and also every proper ideal can be expressed as an intersection of finitely many quasi primary ideals in another way.

Proposition 2.18. *Let $Q_1, Q_2, \ldots, Q_n, I$ be ideals of R, such that $I \subseteq \bigcup_{i=1}^{n} Q_i$ and I is not contained in the union of any $n-1$ of the ideals $Q_1, Q_2, \ldots, Q_n$ and let $I \cap \sqrt{Q_i} \nsubseteq I \cap \sqrt{Q_j}$, for every $i \neq j$. Then Q_i's can not be quasi primary.*

Proof. Suppose to the contrary, Q_k is a quasi primary ideal of R, for some $1 \leq k \leq n$. It is easy to show that $I = \bigcup_{i=1}^{n}(I \cap Q_i)$ and I is not contained in the union of any $n-1$ of the ideals $I \cap Q_1, I \cap Q_2, \ldots, I \cap Q_n$. Thus there exists an element $a_k \in I \setminus \sqrt{Q_k}$, for all $1 \leq k \leq n$. Now we claim that $I \cap (\bigcap_{i \neq k} Q_i) \subseteq (I \cap Q_k)$, because if $x \in \bigcap_{i \neq k} Q_i$ and $y \in I \setminus \bigcup_{i \neq k} Q_i$, then $x + y \in I \setminus \bigcup_{i \neq k} Q_i$. Hence $x + y \in Q_k$ and as $y \in Q_k$ we have $x = (x + y) - y \in Q_k$. Therefore $\bigcup_{i \neq k} Q_i \subseteq Q_k$ and this implies $I \cap (\bigcap_{i \neq k} Q_i) \subseteq (I \cap Q_k)$. Since $I \cap \sqrt{Q_i} \nsubseteq I \cap \sqrt{Q_j}$, for every $i \neq j$, we have $\sqrt{Q_i} \nsubseteq \sqrt{Q_j}$, for every $i \neq j$. Thus there exists $b_i \in \sqrt{Q_i} \setminus \sqrt{Q_j}$, for every $i \neq j$. Let $b = \prod_{i\neq k} b_i$. Then $b \in \sqrt{Q_i}$, for all $i \neq k$ and $b \notin \sqrt{Q_k}$. Therefore, there exist $\{m_i\}_{i\neq k} \subset \mathbb{N}$, such that $b^{m_i} \in Q_i$. Put $m = \sum_i m_i$. Then $b^m \in Q_i$, for all $i \neq k$ and $b^m \notin \sqrt{Q_k}$. Hence $b^m a_k \in (I \cap Q_i)$ for all $i \neq k$, but $b^m a_k \notin (I \cap \sqrt{Q_k})$, otherwise, assume that $b^m a_k \in (I \cap \sqrt{Q_k})$. Since Q_k is quasi primary, we have either $b^{mt} \in Q_k$ or $a^t \in Q^j$, for some $t \in \mathbb{N}$, which is impossible as neither $b \notin \sqrt{Q_k}$ nor $a_k \notin Q_k$. Therefore $b^m a_k \notin (I \cap \sqrt{Q_k})$ Ij and this contradicts the fact that $I \cap (\bigcap_{i\neq k} Q_i) \subseteq (I \cap Q_k)$. □

Proposition 2.19. *Let Q be a quasi primary ideal of ring R and let P be ideal of R containing Q. Then for each $m, n \in \mathbb{N}$, $Q^n P^m$, is a quasi primary ideal of R.*

Proof. Let $ab \in Q^n P^m$. Then $ab \in Q$ and this implies $a^t \in Q$ or $b^t \in Q$, for some $t \in \mathbb{N}$. Since $Q \subseteq P$, we get $a^{tm+tn} \in Q^n P^m$ or $b^{tm+tn} \in Q^n P^m$. Thus $Q^n P^m$ is quasi primary ideal. □

Theorem 2.20. *(Quasi primary avoidance theorem). Let $I \subseteq \bigcup_{i=1}^{n} Q_i$ and I is not contained in the union of any $n-1$ of the ideals $Q_1, Q_2, \ldots, Q_n$, where for each $1 \leq i \leq n$, Q_i is quasi primary. If $I \cap \sqrt{Q_i} \nsubseteq I \cap \sqrt{Q_j}$, then $I \subseteq Q_k$, for some $1 \leq k \leq n$.*

Proof. The claim is evident for $n \leq 2$. Assume that the claim is true for all $k \leq n$. Now, we will prove that the claim is true for$k = n + 1$. Suppose that $I \subseteq \bigcup_{i=1}^{n+1} Q_i$, where Q_j is P_j-quasi primary ideal for each $1 \leq j \leq n+1$ and $I \cap \sqrt{Q_i} \nsubseteq I \cap \sqrt{Q_j}$ for each $i \neq j$. Now, consider the set $T = \{I \cap P_1, I \cap P_2, \ldots, I \cap P_{n+1}\}$. Then T has a minimal element, say $I \cap P_m$. Then for each $t \in \{1, 2, \ldots, n+1\} \setminus \{m\}$ there exists $a_t \in (I \cap P_t) \setminus (I \cap P_m)$. Then we have $a_t^{s_t} \in I \cap Q_t \setminus P_m$, for some $s_t \in \mathbb{N}$. This implies $a_1^{s_1} \ldots a_{m-1}^{s_{m-1}} a_{m+1}^{s_{m+1}} \ldots a_{n+1}^{s_{n+1}} \in I \cap (\bigcap_{i\neq m} Q_i)$. Now, we will show that $I \subseteq \bigcup_{i\neq m} Q_i$. Suppose to the contrary. Then it can be easily seen that $I \cap (\bigcap_{i\neq m} Q_i) \subseteq Q_m$. Since $a_1^{s_1} \ldots a_{m-1}^{s_{m-1}} a_{m+1}^{s_{m+1}} \ldots a_{n+1}^{s_{n+1}} \in Q_m$ and Q_m is quasi primary ideal of R, we conclude either $a_1^{us_1} \in Q_m$ or $\ldots$ $a_{m-1}^{us_{m-1}} \in Q_m$ or $a_{m+1}^{us_{m+1}} \in Q_m$ or $\ldots$ or $a_{n+1}^{us_{n+1}} \in Q_m$. Then we deduce that either $a_1 \in Q_m$ or $\ldots$ $a_{m-1} \in Q_m$ or $a_{m+1} \in Q_m$ or $\ldots$ or $a_{n+1} \in Q_m$. which are contradictions. Thus $I \subseteq \bigcup_{i\neq m} Q_i$ so by induction hypothesis, we get $I \subseteq Q_k$ for some $k \in \{1, 2, \ldots, n+1\} \setminus \{m\}$. □

Proposition 2.21. *Let R be a ring such that $P \cap Q$ is quasi primary, where P and Q are prime ideals. Then prime ideals of R are comparable. In particular, R is local and $nil(R)$ is a prime ideal of R.*

Proof. Let P and Q be two prime ideals of R. Since $P \cap Q$ is quasi primary, we get that $\sqrt{P \cap Q} = P \cap Q$ is prime, and so $P \subseteq Q$ or $Q \subseteq P$. Thus, prime ideals of R are comparable. □

Corollary 2.22. *Let R be a ring such that every proper ideal of R is weakly quasi primary. Then prime ideals of R are comparable and R is local and $nil(R)$ is a prime ideal of R.*

Corollary 2.23. *Let R be a reduced ring such that every proper ideal of R is quasi primary. Then, R is a domain.*

Recall from [9], that a ring R is said to be an $UN-$ring if every nonunit element a of R is a product a unit and a nilpotent elements.

Proposition 2.24. *Let R be a Noetherian ring. Then, every proper ideal of R is quasi primary if and only if R is either UN-ring, or (R, M) is a local ring such that $Spec(R) = \{nil(R), M\}$ and $x^n = 0$ for each $x \in nil(R)$, for some $n \in \mathbb{N}$.*

Proof. By Corollary 2.22, R is local (with maximal ideal M) and $nil(R)$ is prime. Suppose that R is not a $UN-$ ring and let P be a non maximal prime ideal of R. Let $n \geq 1$ be an integer. We have $M^n \nsubseteq P$, otherwise $M = P$, a contradiction. Hence, consider $b_n \in M^n \setminus P$. For each $a \in P$, we have $ab_n \in M^n \cap P$ and $b_n \notin \sqrt{M^n \cap P} = P$. Hence, since $M^n \cap P$ quasi primary, we obtain that $a^{mn} \cap P \subseteq M^n$. Consequently, $a^m \in \bigcap_{n \geq 1} M^n$. By Krull's intersection theorem, we have

$$\bigcap_{n \geq 1} M^n = \{0\}$$

Thus, $a^m = 0$. Hence, $P = nil(R)$. Accordingly, $nil(R)$ is the unique non-maximal prime ideal of R and $a^m = 0$ for all $a \in nil(R)$. Conversely; If R is a $UN-$ring then every proper ideal of R is primary, and so quasi primary. Now, suppose that R is not a $UN-$ring and let Q be a proper ideal of R. Consider $a, b \in R$ such that $ab \in Q$. If $ab \in nil(R)$ then either $a \in nil(R)$ or $b \in nil(R)$. Hence, $a^n = 0$ or $b^n = 0$, for some $n \in \mathbb{N}$. Now, if $ab \notin nil(R)$ then$\sqrt{(ab)} = \sqrt{Q} = M$ and so Q is primary, and so quasi primary. □

Proposition 2.25. *Let $Q_1, Q_2, \ldots, Q_n$ be quasi primary ideals with $\sqrt{Q_i} = P$ for each $1 \leq i \leq n$. Then $Q = \bigcap_{i=1}^n Q_i$ is a $P-$quasi primary ideal of R.*

Proof. Let $Q_1, Q_2, \ldots, Q_n$ be $P-$quasi primary ideals. Then

$$\sqrt{Q} = \sqrt{\bigcap_{i=1}^n Q_i} = \bigcap_{i=1}^n \sqrt{Q_i} = P$$

Thus Q is a $P-$quasi primary ideal of R. □

3. Weakly Quasi Primary Ideals

Definition 3.1. A proper ideal Q of ring R is said to be a weakly quasi primary ideal if whenever $0 \neq ab \in Q$ for some $a, b \in R$, then either $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$.

Definition 3.2. The ideal P of ring R is weakly prime ideal, if for all $a, b \in R$ such that $0 \neq ab \in P$, then either $a \in P$ or $b \in P$.

Example 3.3. Every weakly prime ideal is weakly quasi primary.

Example 3.4. Every quasi primary ideal is weakly quasi primary.

Example 3.5. Zero ideal is a weakly quasi primary ideal of $\mathbb{Z}_{10}$, but is not a quasi primary ideal.

Proposition 3.6. *Let Q be a weakly quasi primary ideal of ring R. Then Q is a quasi primary ideal or $Q^2 = 0$.*

Proof. Suppose that Q is a weakly quasi primary ideal of R that is not quasi primary. Then there exist $a, b \in R$ such that $0 = ab \in Q$ but $a, b \notin \sqrt{Q}$. Now, we will show that $aQ = 0 = bQ$. Assume that $aQ \neq 0$. Then we have $aq \neq 0$ for some $q \in Q$. Since Q is a weakly quasi primary ideal and $0 \neq aq = a(b+q) \in Q$, we have $a^n \in Q$ or $(b+q)^n = b^n + rq \in Q$, for some $n \in \mathbb{N}$ and $r \in R$, implying $a^n \in Q$ or $b^n \in Q$, which is a contradiction. Thus $aQ = 0$ and similarly, $bQ = 0$. Now, choose $x, y \in Q$. If $xy = 0$, then we are done. Assume that $xy \neq 0$. Then we have $0 \neq xy = (x+a)(y+b) \in Q$. Since Q is a weakly quasi primary ideal, we get $(x+a)^n = x^n + as + a^n = x^n + a^n \in Q$ or $(y+b)^n = y^n + bt + b^n = y^n + b^n \in Q$, for some $n \in \mathbb{N}$ and $s, t \in Q$ (we note that $aQ = 0 = bQ$). As $x, y \in Q$, we conclude that $a^n \in Q$ or $b^n \in Q$, again a contradiction. Hence, $Q^2 = 0$. □

Corollary 3.7. *If Q is a weakly quasi primary ideal of R that is not quasi primary, then $\sqrt{Q} = \sqrt{0}$.*

Corollary 3.8. *Let R be reduced ring and $Q \neq 0$ be a weakly quasi primary ideal of R. Then Q is quasi primary ideal.*

Proposition 3.9. *Let Q be a weakly quasi primary ideal of ring R and let P be ideal of R containing Q. Then for each $m, n \in \mathbb{N}$, $Q^n P^m$, is a quasi primary ideal of R.*

Proof. Similar to the proof of Proposition 2.19. □

Corollary 3.10. *Let Q be a weakly quasi primary ideal of R. Then for each $n \in \mathbb{N}$, Q^n is a weakly quasi primary ideal.*

Proposition 3.11. *If Q is a weakly quasi primary ideal of ring R, then $\sqrt{Q}$ is weakly prime ideal if and only if, $\sqrt{Q}$ weakly quasi primary ideal.*

Proof. Suppose that $\sqrt{Q}$ is a weakly quasi primary ideal of R. Let $0 \neq ab \in \sqrt{Q}$ for some $a, b \in R$. Since$\sqrt{Q}$ is a weakly quasi primary ideal, we have $a^n \in \sqrt{Q}$ or $b^n \in \sqrt{Q}$, for some $n \in \mathbb{N}$. which implies that $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. Therefore, $\sqrt{Q}$ is a weakly prime ideal of R. The converse is clear. □

Corollary 3.12. *Let Q be a weakly quasi primary ideal and let Q be semiprime ideal of ring R. Then Q is weakly prime ideal.*

Corollary 3.13. *If R is a von Neumann regular ring. Then a proper ideal Q of R is a weakly quasi primary ideal if and only if it is a weakly prime ideal.*

Proposition 3.14. *Let $\varphi : R \longrightarrow S$ be a ring epimorphism and Q be a weakly quasi primary ideal of R containing $ker(\varphi)$. Then $\varphi(Q)$ is a weakly quasi primary ideal of S.*

Proof. Let $0 \neq ab \in \varphi(Q)$ for some $a, b \in R$. Since φ is epimorphism, we can write $a = \varphi(x)$ and $b = \varphi(y)$ for some $x, y \in R$. Then we have $ab = \varphi(x)\varphi(y) = \varphi(xy) \in \varphi(Q)$. As $ker(\varphi) \subseteq Q$ we have $0 \neq xy \in Q$. Since Q is weakly quasi primary ideal, we get either $x^n \in Q$ or $y^n \in Q$, for some $n \in \mathbb{N}$ and this yields $a^n = \varphi^n(x) = \varphi(x^n) \in \varphi(Q)$ or $b^n = \varphi^n(y) = \varphi(y^n) \in \varphi(Q)$. Hence$\varphi(Q)$ is a weakly quasi primary ideal of S. □

Corollary 3.15. *Let Q be a weakly quasi primary ideal and ideal of ring R containing an ideal I of R. Then $\frac{Q}{I}$ is a weakly quasi primary ideal of $\frac{R}{I}$.*

Proposition 3.16. *Let $\varphi : R \longrightarrow S$ be a ring monomorphism and and Q be a weakly quasi primary ideal of S. Then $\varphi^{-1}(Q)$ is a weakly quasi primary ideal of R.*

Proof. Let $0 \neq ab \in \varphi^{-1}(Q)$ for some $a, b \in R$. As φ is monic, $0 \neq \varphi(ab) = \varphi(a)\varphi(b) \in Q$. Since Q is a weakly quasi primary ideal, we get either $varphi(a^n) = \varphi^n(a) \in Q$ or $varphi(b^n) = \varphi^n(b) \in Q$, for some $n \in \mathbb{N}$ and thus $a^n \in \varphi^{-1}(Q)$ or $b^n \in \varphi^{-1}(Q)$ Hence, $\varphi^{-1}(Q)$ is a weakly quasi primary ideal of S. □

Corollary 3.17. *Let S be a subring of R and Q be a weakly quasi primary ideal of R with $S \nsubseteq Q$. Then $Q \cap S$ is a weakly quasi primary ideal of S.*

Proposition 3.18. *Let $I \subseteq Q$ be two ideals of R. If $\frac{Q}{I}$ is a weakly quasi primary ideal of $\frac{R}{I}$ and I is a weakly quasi primary ideal of R, then Q is a quasi primary ideal of R.*

Proof. Let $ab \in Q$ for some $a, b \in R$. If $ab \in I$, then we have $a^n \in I \subseteq Q$ or $b^n \in I \subseteq Q$ since I is a quasi primary ideal of R. Now, assume that $ab \notin I$. This implies that $0 \neq (a + I)(b + I) \in \frac{Q}{I}$. As $\frac{Q}{I}$ is a weakly quasi primary ideal of $\frac{R}{I}$, we get either $a^n + I = (a + I)^n \in \frac{Q}{I}$ or $b^n + I = (b + I)^n \in \frac{Q}{I}$, for some $n \in \mathbb{N}$, which implies that $a^n \in Q$ or $b^n \in Q$. Therefore, Q is a quasi primary ideal of R. □

Proposition 3.19. *Let R be a ring and S be a multiplicatively closed subset of R such that $Q \cap S = \emptyset$. If Q is a weakly quasi primary ideal of R, then $S^{-1}Q$ is a weakly quasi primary ideal of $S^{-1}R$.*

Proof. Let $0 \neq \frac{a}{s}\frac{b}{t} = \frac{ab}{st} \in S^{-1}Q$. Then there exists $u \in S$ such that $0 \neq u(ab) = (ua)b \in Q$. So $(ua)^n \in Q$ or $b^n \in Q$, for some $n \in \mathbb{N}$ and this implies $(\frac{a}{s})^n = (\frac{ua}{us})^n \in S^{-1}Q$ or $\frac{b^n}{t^n} \in S^{-1}Q$. Thus $S^{-1}Q$ is a weakly quasi primary ideal of $S^{-1}R$. □

Proposition 3.20. *Assume that Q_1 and Q_2 be ide of rings R_1 and R_2 respectively. Let $R = R_1 \times R_2$ and $Q = Q_1 \times Q_2$. Then the following are equivalent:*

(1) *Q is quasi primary ideal of R.*
(2) *Q is weakly quasi primary ideal of R.*
(3) *Q_1 is quasi primary ideal of R_1 and $Q_2 = R_2$ or Q_2 is quasi primary ideal of R_2 and $Q_1 = R_1$.*

Proof. (1) $\Longrightarrow$ (2): It is evident.
(2) $\Longrightarrow$ (3): Let Q be a weakly quasi primary ideal of R. Since $Q \neq 0$, either $Q_1 \neq 0$ or $Q_2 \neq 0$. Without loss of generality, we may assume that $Q_1 \neq 0$. Then there exists $0 \in Q_1$. Take any $b \in Q_2$. Then $0 \neq (a, 1)(1, b) \in Q$. Since Q is a weakly quasi primary of R, we conclude that $(a^n, 1) = (a, 1)^n \in Q$ or$(b^n, 1) = (b, 1)^n \in Q$, for some $n \in \mathbb{N}$, which implies $Q_1 = R_1$ or $Q_2 = R_2$ First assume that $Q_1 = R_1$. Now, we will show that Q_2 is a quasi primary ideal of R_2. Choose $xy \in Q_2$ for some $x, y \in R_2$. Then we have $(0, 0) \neq (1, x)(1, y) \in Q$. As Q is a weakly quasi primary ideal of R, we have $(1, x^m) = (1, x)^m \in Q$ or $(1, y^m) = (1, y)^m \in Q$, for some $m \in \mathbb{N}$ and hence $x^m \in Q_2$ or $y^m \in Q_2$. If $Q_2 = R_2$, then similarly one can prove that Q_1 is quasi primary ideal of R_1.
(2) $\Longrightarrow$ (3): By Proposition 2.16, Q is a quasi primary ideal of R. □

Proposition 3.21. *If every proper ideal of R is a weakly quasi primary ideal of R, then R has at most two incomparable prime ideals.*

Proof. Suppose that R has three incomparable prime ideals, say P_1, P_2 and P_3. Let $I = P_1 \cap P_2$. Then by assumption, I is a weakly quasi primary ideal of R. Also, note that $\sqrt{I} = \sqrt{P_1 \cap P_2} = P_1 \cap P_2 = I$ is a semiprime ideal of R. Then by Corollary 3.12, I is a weakly prime ideal of R. Since I is not a prime ideal of R, by [1, Theorem 1], $I^2 = 0$ which implies that $P_1^2 P_2^2 = 0 \subseteq P_3$. Then we have either $P_1 \subseteq P_3$ or $P_2 \subseteq P_3$ which completes the proof. □

Proposition 3.22. *If every proper ideal of R is a weakly quasi primary ideal of R, then R has at most two maximal ideals. In particular, either R, M is a local ring or $R = F_1 \times F_2$, where F_1, F_2 are fields.*

Proof. By Proposition 3.22, we have $|Max(R)| \leq 2$. Now, assume that R is not local. Then M_1, M_2 are the only maximal ideals of R. By assumption, M_1M_2 is a weakly quasi primary ideal of R that is not quasi primary. By Proposition 3.6, $M_1^2M_2^2 = 0$. By Chinese Remainder Theorem, we have $R \simeq \frac{R}{M_1^2} \times \frac{R}{M_2^2}$. Without loss of generality, we may assume that $R = \frac{R}{M_1^2} \times \frac{R}{M_2^2}$. Now, we will show that $\frac{R}{M_1^2}$ is a field. Take a prime ideal P of $\frac{R}{M_2^2}$. Choose a nonzero ideal I of $\frac{R}{M_1^2}$. Then $Q = I \times P$ is a weakly quasi primary ideal of R. By Proposition 3.20, $I = \frac{R}{M_1^2}$ which implies that $\frac{R}{M_1^2}$ is a field. Similarly, one can show that $\frac{R}{M_2^2}$ is a field. □

Proposition 3.23. *Suppose that R is not a local ring. Then every proper ideal of R is a weakly quasi primary ideal if and only if $R = F_1 \times F_2$, where F_1, F_2 are fields.*

Proof. If every proper ideal of R is a weakly quasi primary ideal of R, then by Proposition 3.22, $R = F_1 \times F_2$, where F_1, F_2 are fields.. The converse is clear. □

Acknowledgement

The author is deeply grateful to the referees for their meticulous reading and valuable suggestions which have definitely improved the paper.

References

[1] D. D. Anderson, E. Smith, *Weakly prime ideals*, Houston J. Math. (2003). 29:831-840.
[2] D. D. Anderson, M. Bataineh, *Generalizations of prime ideals*, Commun. Algebra. (2008). 36(2):686-696.
[3] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. 75 (2007), no. 3, 417-429.
[4] A. Badawi, U. Tekir, and E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc, 51(4), (2014),1163-1173.
[5] A. Badawi, U. Tekir, and E. Yetkin, *On weakly 2-absorbing primary ideals of commutative rings*, J. Korean Math. Soc 52(01), (2015): 97-111.
[6] A. Badawi and E. Yetkin, *On 1-absorbing primary ideals of commutative rings*, J. Algebra Appl. 19(06), (2020): 2050111.
[7] A. Badawi, U. Tekir, E. A. Ugurlu, G. Ulucak, and E. Y. Celikel, *Generalizations of 2-absorbing primaryideals of commutative rings*, Turkish Journal of Mathematics 40(3), (2016): 703-717.
[8] S. M. Bhatwadekar and P. K. Sharma, *Unique factorization and birth of almost primes*, Comm. Algebra. (2005). 33:1, 43-49.
[9] C. Calugareanu, *UN-rings*, J. Algebra Appl. 15(10), (2016), 1650182.
[10] L. Fuchs, *On quasi-primary ideals*, Acta Sci. Math. (Szeged) 11 (1947), 174-183
[11] J. R. Hedstrom, and E. G. Houston, *Pseudo-valuation domains*, Pacific J. Math. 75 (1978), no. 1, 137-147.

[12] R. Nikandish, M. J. Nikmehr and A. Yassine, *More on the 2-prime ideals of commutative rings*, Bulletin of the Korean Mathematical Society 57(01) (2020): 117-126.

[13] M. J. Nikmehr, R. Nikandish and A. Yassine, *On classical 2-prime subsemimodules*, J. Algebra Appl 22(02) (2023): 2350041.

[14] A. Yassine, M. J. Nikmehr and R. Nikandish , *On 1-absorbing prime ideals of commutative rings*, J. Algebra Appl 20(10) (2021): 2150175.