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Weakly compact weighted composition operators on pointed Lipschitz spaces

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Abstract. Let (X, d) be a pointed compact metric space with the base point x_0 and let $\text{Lip}((X, d), x_0)$ ($\text{lip}((X, d), x_0)$) denote the pointed (little) Lipschitz space on (X, d). In this paper, we prove that every weakly compact composition operator uC_φ on $\text{Lip}((X, d), x_0)$ is compact provided that $\text{lip}((X, d), x_0)$ has the uniform separation property, φ is a base point preserving Lipschitz self-map of X and $u \in \text{Lip}(X, d)$ with $u(x) \neq 0$ for all $x \in X \setminus \{x_0\}$.

Keywords: Compact operator, Lipschitz space, weakly compact operator, weighted composition operator.

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1. Introduction and Preliminaries

The symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two normed linear spaces over $\mathbb{K}$. The space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. The *adjoint operator* of $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the operator $T^* \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ which is defined by $T^*(y^*) = y^* \circ T$ whenever $y^* \in \mathcal{Y}^*$. We say that a linear operator T from $\mathcal{X}$ into $\mathcal{Y}$ is (weakly) compact if T maps bounded sets in $\mathcal{X}$ into relatively (weakly) compact sets in $\mathcal{Y}$. Clearly, every (weakly) compact operator is bounded. It is known [3, Proposition V.4.1] that if $\mathcal{X}$ is a normed

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linear space over $\mathbb{K}$, the $\pi_{\mathcal{X}}(\mathcal{B}_{\mathcal{X}})$ is $\sigma(\mathcal{X}^{**}, \mathcal{X}^{*})$ dense in $\mathcal{B}_{\mathcal{X}^{**}}$,where $\pi_{\mathcal{X}}$ is the natural embedding from $\mathcal{X}$ to $\mathcal{X}^{**}$ and $\mathcal{B}_{\mathcal{Y}}$ is the closed unit ball of the normed space $\mathcal{Y}$. Therefore, exactly as the proof of the case that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, one can show that for normed linear spaces $\mathcal{X}$ and $\mathcal{Y}$, an operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is weakly compact if and only if $T^{**}(\mathcal{X}^{**})$ is contained in $\pi_{\mathcal{Y}}(\mathcal{Y})$.(see [5, Theorem VI.4.2])

Let X be a nonempty set and A be a nonempty subset of $\mathbb{K}^X$, the set of all $\mathbb{K}$-valued functions on X. For each $u \in \mathbb{K}^X$ and every self-map φ of X, the map $f \mapsto u.(f \circ \varphi) : A \to \mathbb{K}^X$ is denoted by uC_φ on A. A map $T : A \to A$ is called a weighted composition operator on A if there exist a function $u \in \mathbb{K}^X$ and a self-map φ of X such that $T = uC_\varphi$ on A. It is clear that such a map T is a linear map over $\mathbb{K}$ if A is a linear subspace of $\mathbb{K}^X$ over $\mathbb{K}$. In the special case $u = 1_X$, the weighted composition operator $T = uC_\varphi : A \to A$ reduces to the composition operator C_φ on A.

Let (X, d) and (Y, ρ) be metric spaces. A function $\varphi : X \to Y$ is said to be a *Lipschitz mapping* from (X, d) to (Y, ρ) if

$$\mathrm{L}(\varphi) = \sup\{\frac{\rho(\varphi(x), \varphi(y))}{d(x, y)} : x, y \in X, \, x \neq y\} < \infty$$

and a *supercontractive* if

$$\lim_{d(x,y)\to 0} \frac{\rho(\varphi(x), \varphi(y))}{d(x, y)} = 0.$$

Let (X, d) be a metric space. We denote by $\mathrm{Lip}(X, d)$ the set of all $\mathbb{K}$-valued bounded functions f on X for which

$$\mathrm{L}_{(X,d)}(f) = \sup\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, \, x \neq y\} < \infty.$$

Then $\mathrm{Lip}(X, d)$ is a commutative unital Banach algebra over $\mathbb{K}$ with unit 1_X, the constant function with value 1, and with the algebra norm $\|\cdot\|_{\mathrm{Lip}(X,d)}$ defined by

$$\| f \|_{\mathrm{Lip}(X,d)} = \| f \|_X + \mathrm{L}_{(X,d)}(f) \qquad (f \in \mathrm{Lip}(X, d)),$$

where $\| f \|_X = \sup\{|f(x)| : x \in X\}$. We denote by $\mathrm{lip}(X, d)$ the set of all $f \in \mathrm{Lip}(X, d)$ for which

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x, y)} = 0.$$

Then $\mathrm{lip}(X, d)$ is a closed subalgebra of $\mathrm{Lip}(X, d)$ that contains 1_X.

Let (X, d) be a metric space and A be the Lipschitz algebra $\mathrm{Lip}(X, d)$. It is known that if $\varphi : X \to X$ is a Lipschitz mapping from (X, d) to

(X, d) and $u \in \mathrm{Lip}(X, d)$, then $uC_\varphi : A \to A$ is a bounded weighted composition operator on A. (see [1, Theorems 2.2 and 2.4])

Let B be a nonempty subset of $\mathrm{Lip}(X, d)$. We say that B *separates uniformly the points* of X if there exists a constant $C > 1$ such that for every $x, y \in X$, there exists a function $f \in B$ with $\mathrm{L}_{(X,d)}(f) \leq C$ such that $|f(x) - f(y)| = d(x, y)$. It is known that $\mathrm{Lip}(X, d)$ has the uniform separation property. Let (X, d) be a metric space. For $\alpha \in (0, 1]$, the map $d^\alpha : X \times X \to \mathbb{R}$ defined by $d^\alpha(x, y) = (d(x, y))^\alpha$, $(x, y \in X)$, is a metric on X and the induced topology by d^α on X coincides by the induced topology by d on X. It is known that $\mathrm{lip}(X, d^\alpha)$ separates uniformly the points of X whenever (X, d) is compact and $\alpha \in (0, 1)$.

A metric space (X, d) is pointed if it carries a distinguished element or a base point, say x_0. We denote by $((X, d), x_0)$ the pointed metric space (X, d) with the base point x_0. We denote by $\mathrm{Lip}((X, d), x_0)$ the set of all $\mathbb{K}$-valued Lipschitz functions f on X for which $f(x_0) = 0$. It is easy to see that $\mathrm{Lip}((X, d), x_0)$ is a maximal ideal of $\mathrm{Lip}(X, d)$ whenever (X, d) is bounded and a Banach space with the Lipschitz norm $\mathrm{L}_{(X,d)}(\cdot)$. Note that as a Banach space, $\mathrm{Lip}((X, d), x_0)$ does not depend on the base point x_0. Explicitly, if x_0 and x_1 are two different choices, then the map $f \mapsto f - f(x_1)$ takes $\mathrm{Lip}((X, d), x_0)$ linearly and isometrically onto $\mathrm{Lip}((X, d), x_1)$. We denote by $\mathrm{lip}((X, d), x_0)$ the set of all $f \in \mathrm{Lip}((X, d), x_0)$ for which f is supercontractive. It is easy to see that $\mathrm{lip}((X, d), x_0)$ is closed subset of $\mathrm{Lip}((X, d), x_0)$.

Let $((X, d), x_0)$ be a pointed compact metric space and A be the pointed Lipschitz space $\mathrm{Lip}((X, d), x_0)$. If $\varphi : X \to X$ is a Lipschitz mapping with $\varphi(x_0) = x_0$ and $u \in \mathrm{Lip}(X, d)$, then $uC_\varphi : A \to A$ is a weighted composition operator on A. (see [4, Theorem 2.1])

Let B be a nonempty subset of $\mathrm{Lip}((X, d), x_0)$. We say that B separates uniformly the points of X if there exists a constant $C > 1$ such that for every $x, y \in X$, there exists a function $f \in B$ with $\mathrm{L}_{(X,d)}(f) \leq C$ such that $|f(x) - f(y)| = d(x, y)$. It is known [10, Proposition 3.2.2] that $\mathrm{lip}((X, d^\alpha), x_0)$ has the uniform separation property where $0 < \alpha < 1$.

Golbaharan and Mahyar in [7] studied weighted composition operators on Lipschitz algebras, whenever (X, d) is a compact metric space. In [1], these operators studied between Lipschitz algebras $\mathrm{Lip}(X, d)$ and $\mathrm{Lip}(Y, \rho)$ whenever metric spaces (X, d) and (Y, ρ) are not necessarily compact. Compact composition operators between pointed Lipschitz spaces studied in [9]. A. Jiménez–Vargas in [8] studied weakly compact composition operators on pointed Lipschitz spaces $\mathrm{Lip}((X, d), x_0)$ and pointed little Lipschitz space $\mathrm{lip}((X, d), x_0)$, where $((X, d), x_0)$ is a pointed compact metric space and $\mathrm{lip}((X, d), x_0)$ has the uniform separation property. Compact weighted composition operators between pointed Lipschitz spaces were studied in [2, 4]. A. Golbaharan in [6] studied weakly compact weighted composition operators on $\mathrm{Lip}(X,d)$ where (X,d) is a compact metric space.

In this paper, we study weakly compact weighted composition operators on pointed Lipschitz spaces $\mathrm{Lip}((X,d),x_0)$, where $((X,d),x_0)$ is a pointed compact metric space. We first show that if $((X,d),x_0)$ is a pointed compact metric space, $\mathrm{lip}((X,d),x_0)$ separates uniformly the points of X, $\varphi : X \to X$ is a Lipschitz mapping with $\varphi(x_0) = x_0$, $u \in \mathrm{Lip}(X,d)$ with $u(x) \neq 0$ for all $x \in X\backslash\{x_0\}$ and weighted composition operator $uC_\varphi : \mathrm{Lip}((X,d),x_0) \to \mathrm{Lip}((X,d),x_0)$ is weakly compact, then uC_φ is compact. We next show that if (X,d) is a compact metric space, $\mathrm{lip}(X,d)$ separates uniformly the points of X, $\varphi : X \to X$ is a Lipschitz mapping, $u \in \mathrm{Lip}(X,d)$ with $u(x) \neq 0$ for all $x \in X$ and $uC_\varphi : \mathrm{Lip}(X,d) \to \mathrm{Lip}(X,d)$ is weakly compact, then uC_φ is compact.

2. Main result

To prove the main result , we need the following lemmas.

Lemma 2.1. *Let $((X,d),x_0)$ be a pointed compact metric space and A be the pointed Lipschitz space $\mathrm{Lip}((X,d),x_0)$. Let $\varphi : X \to X$ be a Lipschitz mapping with $\varphi(x_0) = x_0$ and $u \in \mathrm{Lip}(X,d)$. If composition operator $C_\varphi : A \to A$ is compact, then weighted composition operator $uC_\varphi : A \to A$ is compact.*

Proof. Let $C_\varphi : A \to A$ be a compact operator. Suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ be a bounded net in $(A, \mathrm{L}_{(X,d)}(\cdot))$ which converges to 0_X on X. By [2, Proposition 3.2], there exists a subnet $\{f_\gamma\}_{\gamma\in\Gamma}$ of $\{f_\lambda\}_{\lambda\in\Lambda}$ such that

$$\lim_{\gamma} \mathrm{L}_{(X,d)}(C_\varphi(f_\gamma)) = 0. \tag{2.1}$$

Let $\gamma \in \Gamma$. For all $x, y \in X$ we have

$$\begin{aligned}
|uC_\varphi(f_\gamma)(x) - uC_\varphi(f_\gamma)(y)| \leq& |u(x)||f_\gamma(\varphi(x)) - f_\gamma(\varphi(y))| \\
&+ |u(x) - u(y)||f_\gamma(\varphi(y))| \\
\leq& \,\|u\|_X \,\mathrm{L}_{(X,d)}(C_\varphi(f_\gamma))d(x,y) \\
&+ \mathrm{L}_{(X,d)}(u)d(x,y)|f_\gamma(\varphi(y)) - f_\gamma(\varphi(x_0))| \\
\leq& \,\|u\|_X \,\mathrm{L}_{(X,d)}(C_\varphi(f_\gamma))d(x,y) \\
&+ \mathrm{L}_{(X,d)}(u)d(x,y)\mathrm{L}_{(X,d)}(C_\varphi(f_\gamma))d(y,x_0) \\
\leq& \,\|u\|_X \,\mathrm{L}_{(X,d)}(C_\varphi(f_\gamma))d(x,y) \\
&+ \mathrm{L}_{(X,d)}(u)d(x,y)\mathrm{L}_{(X,d)}(C_\varphi(f_\gamma))\,\mathrm{diam}(X).
\end{aligned}$$

Therefore,

$$\mathrm{L}_{(X,d)}(uC_\varphi(f_\gamma)) \leq (\|u\|_X + \mathrm{L}_{(X,d)}(u)\,\mathrm{diam}(X))\mathrm{L}_{(X,d)}(C_\varphi(f_\gamma)). \tag{2.2}$$

Since (2.2) holds for all $\gamma \in \Gamma$, according to (2.1) we get

$$\lim_{\gamma} \mathrm{L}_{(X,d)}(uC_\varphi(f_\gamma) = 0.$$

Hence, uC_φ is compact by [2, Proposition 3.2]. □

Lemma 2.2. *Let $((X,d),x_0)$ be a pointed compact metric space and let B be a linear subspace of $\mathrm{Lip}((X,d),x_0)$ over $\mathbb{K}$.*

(i) For each $x \in X$, the map $e_{B,x} : B \to \mathbb{K}$ defined by

$$e_{B,x}(g) = g(x) \qquad (g \in B),$$

belongs to B^* and $\| e_{B,x} \| \leq \mathrm{diam}(X)$, where B^* is the dual space of normed space $(B, \mathrm{L}_{(X,d)}(\cdot))$.

(ii) $\| e_{B,x} - e_{B,y} \| \leq d(x,y)$ for all $x, y \in X$.

Proof. (i) Let $x \in X$. It is clear that $e_{B,x}$ is a linear functional on B. Since

$$\begin{aligned} |e_{B,x}(g)| &= |g(x)| = |g(x) - g(x_0)| \\ &\leq \mathrm{L}_{(X,d)}(g)d(x,x_0) \leq \mathrm{L}_{(X,d)}(g)\,\mathrm{diam}(X) \end{aligned}$$

for all $g \in B$, we deduce that $e_{B,x} \in B^*$ and $\| e_{B,x} \| \leq \mathrm{diam}(X)$.

(ii) Let $x, y \in X$. Then $e_{B,x} - e_{B,y} \in B^*$ and

$$|(e_{B,x} - e_{B,y})(g)| = |g(x) - g(y)| \leq \mathrm{L}_{(X,d)}(g)d(x,y)$$

for all $g \in B$. Hence, $\| e_{B,x} - e_{B,y} \| \leq d(x,y)$. □

Lemma 2.3. *Let $((X,d),x_0)$ be a pointed compact metric space and let B be a linear subspace of $\mathrm{lip}((X,d),x_0)$ over $\mathbb{K}$. Then the map $\Phi_B : B^{**} \to \mathrm{Lip}((X,d),x_0)$ defined*

$$\Phi_B(\Lambda) = \Lambda \circ E_{X,B} \qquad (\Lambda \in B^{**})$$

is a bounded linear operator and $\| \Phi_B \| \leq 1$.

Proof. Let $\Lambda \in B^{**}$. By part (ii) of Lemma 2.2, for all $x, y \in X$ we have

$$\begin{aligned} |\Phi_B(\Lambda)(x) - \Phi_B(\Lambda)(y)| &= |(\Lambda \circ E_{X,B})(x) - (\Lambda \circ E_{X,B})(y)| \\ &= |\Lambda(e_{B,x}) - \Lambda(e_{B,y})| \\ &= |\Lambda(e_{B,x} - e_{B,y})| \\ &\leq \|\Lambda\| \|e_{B,x} - e_{B,y}\| \\ &\leq \|\Lambda\| d(x,y). \end{aligned}$$

This implies that $\Phi_B(\Lambda) \in \mathrm{Lip}(X,d)$ and

$$\mathrm{L}_{(X,d)}(\Phi_B(\Lambda)) \leq \|\Lambda\|. \tag{2.3}$$

Since x_0 is the base-point of X, we have

$$e_{B,x_0}(g) = g(x_0) = 0$$

for all $g \in B$. Thus $e_{B,x_0} = 0_{B^*}$, where 0_{B^*} is the zero linear functional on B. Therefore,

$$\Phi_B(\Lambda)(x_0) = (\Lambda o E_{B,X})(x_0) = \Lambda(e_{B,x_0}) = \Lambda(0_{B^*}) = 0.$$

Hence, $\Phi_B(\Lambda) \in \mathrm{Lip}((X,d),x_0)$ and so Φ_B is well-defined. It is easy to see that Φ_B is a linear operator. Since (2.3) holds for all $\Lambda \in B^*$, we deduce that Φ_B is bounded and $\|\Phi_B\| \leq 1$. This completes the proof. □

Theorem 2.4. *Let $((X,d),x_0)$ be a pointed compact metric space , $\varphi : X \to X$ be a Lipschitz mapping with $\varphi(x_0) = x_0$ and $u \in \mathrm{Lip}(X,d)$ with $u(x) \neq 0$ for all $x \in X \backslash \{x_0\}$. Suppose that $\mathrm{lip}((X,d),x_0)$ separates uniformly the points of X. If weighted composition operator $uC_\varphi : \mathrm{Lip}((X,d),x_0) \to \mathrm{Lip}((X,d),x_0)$ is weakly compact, then uC_φ is compact.*

Proof. Let the weighted composition operator $uC_\varphi : \mathrm{Lip}((X,d),x_0) \to \mathrm{Lip}((X,d),x_0)$ be weakly compact. Set $A = \mathrm{lip}((X,d),x_0)$ and $B = uC_\varphi(A)$. Since uC_φ is a linear mapping and A is a linear subspace of $\mathrm{Lip}((X,d),x_0)$, we deduce that B is a linear subspace of A. We claim that $uC_\varphi(\mathrm{Lip}((X,d),x_0))$ is contained in B. Define the map $T : A \to B$ by

$$T(f) = uC_\varphi(f) \qquad (f \in A).$$

Then T is a bounded linear operator from A into B. It is easy to see that

$$e_{B,x} \circ T = u(x)E_{X,A}(\varphi(x)) \tag{2.4}$$

for all $x \in X$. Define the map $\Phi_A : A^{**} \to \mathrm{Lip}((X,d),x_0)$ by

$$\Phi_A(\Lambda) = \Lambda \circ E_{X,A} \qquad (\Lambda \in A^{**}). \tag{2.5}$$

By [10, Theorems 3.3.3 and 2.2.2], is A^{**} with the operator norm is isometrically isomorphism to $\mathrm{Lip}((X,d),x_0)$ with the norm $\mathrm{L}_{(X,d)}(\cdot)$ via the map Φ_A since $\mathrm{lip}((X,d),x_0)$ separates uniformly the points of X.

Define the map $\Phi_B : B^{**} \to \mathrm{Lip}((X,d),x_0)$ by

$$\Phi_B(\Lambda) = \Lambda \circ E_{X,B} \qquad (\Lambda \in B^{**}). \tag{2.6}$$

By Lemma 2.3, Φ_B is a bounded linear operator from B^{**} with the operator norm into $\mathrm{Lip}((X,d),x_0)$ with the norm $\mathrm{L}_{(X,d)}(\cdot)$. We show that

$$\Phi_B \circ T^{**} \circ \Phi_A^{-1} = uC_\varphi \tag{2.7}$$

on $\mathrm{Lip}((X,d),x_0)$. Let $f \in \mathrm{Lip}((X,d),x_0)$. The surjectivity of Φ_A implies that there exists $\Lambda \in A^{**}$ such that

$$\Phi_A(\Lambda) = f. \tag{2.8}$$

Since $\Lambda \circ T^* \in B^{**}$, by (2.8) and (2.6) we have

$$(\Phi_B\circ T^{**}\circ\Phi_A^{-1})(f) = (\Phi_B\circ T^{**})(\Lambda) = \Phi_B(\Lambda\circ T^*) = \Lambda\circ T^*\circ E_{X,B}. \quad (2.9)$$

Since (2.4) holds for all $x \in X$, according to (2.8) we get

$$\begin{aligned}(\Lambda \circ T^* \circ E_{X,B})(x) &= (\Lambda \circ T^*)(e_{B,x}) = \Lambda(T^*(e_{B,x}))\\ &= \Lambda(e_{B,x} \circ T) = \Lambda(u(x)E_{X,A}(\varphi(x))\\ &= u(x)\Lambda(E_{X,A}(\varphi(x)) = u(x)(\Lambda \circ E_{X,A})(\varphi(x))\\ &= u(x)\Phi_A(\Lambda)(\varphi(x)) = u(x)f(\varphi(x))\\ &= uC_\varphi(f)(x)\end{aligned}$$

for all $x \in X$. This implies that

$$\Lambda \circ T^{**} \circ E_{X,B} = uC_\varphi(f). \quad (2.10)$$

According to (2.9) and (2.10), we get

$$(\Phi_B \circ T^{**} \circ \Phi_A^{-1})(f) = uC_\varphi(f). \quad (2.11)$$

Since (2.11) holds for all $f \in \mathrm{Lip}((X, d), x_0)$, we deduce that (2.7) holds on $\mathrm{Lip}((X, d), x_0)$. The weak compactness of $uC_\varphi : \mathrm{Lip}((X, d), x_0) \to \mathrm{Lip}((X, d), x_0)$ implies that $T : A \to B$ is weakly compact. This implies that

$$T^{**}(A^{**}) \subseteq \pi_B(B), \quad (2.12)$$

where $\pi_B(B)$ is the natural embedding of B in B^{**}. By (2.12) and $A^{**} = \Phi_A^{-1}(\mathrm{Lip}((X, d), x_0))$, we get

$$T^{**}(\Phi_A^{-1}(\mathrm{Lip}((X, d), x_0)) \subseteq \pi_B(B).$$

It follows that

$$(\Phi_B \circ T^{**} \circ \Phi_A^{-1})(\mathrm{Lip}((X, d), x_0)) \subseteq (\Phi_B \circ \pi_B)(B). \quad (2.13)$$

By (2.13) and (2.7), we get

$$uC_\varphi(\mathrm{Lip}((X, d), x_0)) \subseteq (\Phi_B \circ \pi_B)(B). \quad (2.14)$$

Now, we show that

$$(\Phi_B \circ \pi_B)(B) \subseteq B. \quad (2.15)$$

Let $f \in A$. Then

$$\begin{aligned}(\Phi_B \circ \pi_B)(T(f))(x) &= \Phi_B(\pi_B(T(f)))(x) = (\pi_B(T(f)) \circ E_{X,B})(x)\\ &= \pi_B(T(f))(E_{X,B}(x)) = \pi_B(T(f))(e_{B,x})\\ &= e_{B,x}(T(f)) = T(f)(x)\end{aligned}$$

for all $x \in X$. This implies that

$$(\Phi_B \circ \pi_B)(T(f)) = T(f). \quad (2.16)$$

Since $T(A) = B$ and (2.16) holds for all $f \in A$, we deduce that (2.15) holds. According to (2.14) and (2.15), we get

$$uC_\varphi(\mathrm{Lip}((X,d),x_0)) \subseteq B. \tag{2.17}$$

Therefore, our claim is justified.

Assume towards a contradiction that uC_φ is not compact. By Lemma 2.1, $C_\varphi : \mathrm{Lip}((X,d),x_0) \to \mathrm{Lip}((X,d),x_0)$ is not compact. According to [9, Theorem 1.2], we deduce that φ is not supercontractive. By using [8, Lemma 2.1], there exist $\varepsilon > 0$, two sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in X converging a point z in (X,d) such that $0 < d(x_n, y_n) < \frac{1}{n}$ and $\varepsilon < \frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)}$ for all $n \in \mathbb{N}$, and a function $f \in \mathrm{Lip}((X,d),x_0)$ such that $f(\varphi(x_n)) = d(\varphi(x_n),\varphi(y_n))$ and $f(\varphi(y_n)) = 0$ for all $n \in \mathbb{N}$. According to $f \in \mathrm{Lip}((X,d),x_0)$ and (2.17), we get $uC_\varphi(f) \in B$. This implies that there exists $g \in A$ such that

$$uC_\varphi(f) = T(g) = uC_\varphi(g). \tag{2.18}$$

Since $(C_\varphi f)(x_0) = f(\varphi(x_0)) = 0 = g(\varphi(x_0)) = (C_\varphi f)(x_0)$, $u(x) \neq 0$ for all $x \in X\backslash\{x_0\}$, by (2.18) we get

$$C_\varphi(f) = C_\varphi(g).$$

This implies that $g(\varphi(x_n)) = d(\varphi(x_n),\varphi(y_n))$ and $g(\varphi(y_n)) = 0$. Therefore,

$$\frac{|g(\varphi(x_n)) - g(\varphi(y_n))|}{d(x_n,y_n)} = \frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)} > \varepsilon$$

for all $n \in \mathbb{N}$. This contradict to $g \in \mathrm{lip}((X,d),x_0)$ since $0 < d(x_n,y_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence, the proof is complete. □

Note that Theorem 2.4 is an extension of [8, Corollary 2.4].

As an application of Theorem 2.4, we give a weighted composition operator on a pointed Lipschitz space $\mathrm{Lip}((X,d),x_0)$ which is not weakly compact.

Example 2.5. Let $X = \{z \in \mathbb{C} : |z| \leq 1\}$, ρ be the Euclidean metric on X, $\alpha \in (0,1)$, d be the metric ρ^α on X and 0 be the base point of X. Then $\mathrm{lip}((X,d),0)$ separates uniformly the points of X. Define the function $u : X \to \mathbb{C}$ by $u(z) = 1+|z|$, $z \in X$, and the self-map φ of X by $\varphi(z) = \frac{z}{2}$, $z \in X$. Then $u \in \mathrm{Lip}(X,d)$, $u(z) \neq 0$ for all $z \in X\backslash\{0\}$ and φ is a base point preserving Lipschitz mapping from (X,d) to (X,d). Since

$$|u(z)|\frac{d(\varphi(z),\varphi(w))}{d(z,w)} = (1+|z|)\frac{|\frac{z}{2} - \frac{w}{2}|^\alpha}{|z-w|^\alpha} = 2^{-\alpha}(1+|z|) \leq 2^{1-\alpha}$$

for all $z, w \in X$ with $z \neq w$, by [4, Theorem 2.1] we conclude that the map $T = uC_\varphi$ is a weighted composition operator on $\mathrm{Lip}((X,d),0)$. It is clear that $\operatorname{coz}(u) = X$, where $\operatorname{coz}(u) = \{z \in X : u(z) \neq 0\}$. This implies that $\varphi(\operatorname{coz}(u)) = \varphi(X) = \{z \in X : |z| \leq \frac{1}{2}\}$. Therefore, $\varphi(\operatorname{coz}(u))$ is a totally bounded set in (X, d). It is easy to see that

$$|u(z)|\frac{d(\varphi(z), \varphi(w))}{d(z, w)} \geq 2^{-\alpha}$$

for all $z, w \in X$ with $z \neq w$. This implies that

$$\lim_{d(z,w)\to 0} |u(z)|\frac{d(\varphi(z), \varphi(w))}{d(z, w)} \neq 0.$$

Therefore, $T = uC_\varphi$ is not compact by [4, Theorem 4.3]. According to Theorem 2.4, we deduce that T is not weakly compact.

As an application of Theorem 2.4, we show that certain weakly compact composition operators on $\operatorname{Lip}(X, d)$ are compact. To this aim, we need the following lemma which is a modification of [2, Lemma 2.2] and the paragraph after that.

Lemma 2.6. *Let (X, d) be a bounded metric space, $x_0 \notin X$ and $X_0 = X \cup \{x_0\}$.*

(i) *Define the function $d_0 : X_0 \times X_0 \to \mathbb{R}$ by*

$$d_0(x, y) = \begin{cases} d(x, y) & x, y \in X, \\ \frac{1}{2}\operatorname{diam}(X) & \textit{either} \quad x = x_0, y \in X \quad \textit{or} \quad x \in X, y = x_0, \\ 0 & x = y = x_0. \end{cases} \tag{2.19}$$

Then d_0 is a bounded metric space on (X_0, d_0).

(ii) *(X_0, d_0) is a compact if and only if (X, d) is compact.*

(iii) *If $\varphi : X \to X$ is a Lipschitz mapping from (X, d) to (X, d), then the map $\varphi_0 : X_0 \to X_0$, defined by $\varphi_0 = \varphi$ on X and $\varphi_0(x_0) = x_0$, is a base point preserving Lipschitz mapping from (X_0, d_0) to (X_0, d_0) with* $\mathrm{L}(\varphi_0) \leq \max\{1, \mathrm{L}(\varphi)\}$.

(iv) *If $f \in \operatorname{Lip}(X, d)$, then the function $f_0 : X \to \mathbb{K}$ defined by $f_0 = f$ on X and $f_0(x_0) = 0$, belongs to $\operatorname{Lip}((X_0, d_0), x_0)$ with $\mathrm{L}_{(X_0,d_0)}(f) \leq 2\mathrm{L}_{(X,d)}(f)$.*

(v) *If $g \in \operatorname{Lip}((X_0, d_0), x_0)$ and $f = g|_X$, then $f \in \operatorname{Lip}(X, d)$ with $\mathrm{L}_{(X,d)}(f) \leqslant \mathrm{L}_{(X_0,d_0)}(g)$.*

(vi) *The map $\Psi : \operatorname{Lip}(X, d) \to \operatorname{Lip}((X_0, d_0), x_0)$ defined by*

$$\Psi(f) = f_0 \qquad (f \in \operatorname{Lip}(X, d)), \tag{2.20}$$

is a bijective bounded linear operator from $\operatorname{Lip}(X, d)$ with the sum norm $\|\cdot\|_{\operatorname{Lip}(X,d)}$ to $\operatorname{Lip}((X_0, d_0), x_0)$ with the Lipschitz norm $\mathrm{L}_{(X_0,d_0)}(\cdot)$. In particular, Ψ^{-1} is continuous and bounded linear operator.

(vii) *$\Psi(\operatorname{lip}(X, d))$ is contained in $\operatorname{lip}((X_0, d_0), x_0)$.*

(viii) *If the Lipschitz algebra* $\mathrm{lip}(X, d)$ *separates uniformly the points of* X*, then the Lipschitz space* $\mathrm{lip}((X_0, d_0), x_0)$ *separates uniformly the points of* X_0*.*

Proof. We prove (ii) and (viii).

To prove (ii), we first assume that (X, d) is compact. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X_0. Set

$$\mathrm{K} = \{n \in \mathbb{N} : y_n = x_0\}.$$

Case 1. $\mathrm{K} = \emptyset$. Then $\{y_n\}_{n=1}^{\infty}$ is a sequence in X. The compactness of (X, d) implies that there exist a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ and a point $y \in X$ such that

$$\lim_{k\to\infty} d(y_{n_k}, y) = 0.$$

It follows $y \in X_0$ and

$$\lim_{k\to\infty} d_0(y_{n_k}, y) = 0.$$

Case 2. K is finite and $\mathrm{K} \neq \emptyset$. Assume that $m = \max(\mathrm{K})$. Then $\{y_{m+n}\}_{n=1}^{\infty}$ is a sequence in X. The compactness of (X, d) implies that there exist a subsequence $\{y_{m+n_k}\}_{k=1}^{\infty}$ of $\{y_{m+n}\}_{n=1}^{\infty}$ and a point $y \in X$ such that

$$\lim_{k\to\infty} d(y_{m+n_k}, y) = 0.$$

It follows that $y \in X_0$ and

$$\lim_{k\to\infty} d_0(y_{m+n_k}, y) = 0.$$

Case 3. K is infinite. Then there exists a sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ with $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ such that $y_{n_k} = x_0$. Therefore, $\{y_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{y_n\}_{n=1}^{\infty}$ and

$$\lim_{k\to\infty} d_0(y_{n_k}, x_0) = 0.$$

Hence, (X_0, d_0) is compact.

We now assume that (X_0, d_0) is compact. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X. Then $\{y_n\}_{n=1}^{\infty}$ is a sequence in X_0. The compactness of (X_0, d_0) implies that there exist a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ and a point $y \in X_0$ such that

$$\lim_{k\to\infty} d_0(y_{n_k}, y) = 0. \tag{2.21}$$

We claim that $y \neq x_0$. If $y = x_0$, then $d_0(y_{n_k}, y) = \frac{1}{2}\,\mathrm{diam}(X)$ for all $k \in \mathbb{N}$. Therefore, $\lim_{k\to\infty} d_0(y_{n_k}, y) = \frac{1}{2}\,\mathrm{diam}(X)$ which contradicts to (2.21). Hence, our claim is justified. It follows that $y \in X$ and

$$\lim_{k\to\infty} d(y_{n_k}, y) = 0.$$

Therefore, (X,d) is compact.

To prove (viii), assume that $\mathrm{lip}(X,d)$ separates uniformly the points of X. Then there exists a constant $C>1$ such that, for every $x,y\in X$, there exists a function $f\in \mathrm{lip}(X,d)$ with $\mathrm{L}_{(X,d)}(f)\le C$ such that $|f(x)-f(y)|=d(x,y)$. We show that $\mathrm{lip}((X_0,d_0),x_0)$ separates uniformly the points of X_0. To this aim, take $C_0=2C$. Let $x,y\in X_0$, We show that there exists a function $g\in \mathrm{lip}((X_0,d_0),x_0)$ with $\mathrm{L}_{(X_0,d_0)}(g)\le C_0$ such that $|g(x)-g(y)|=d_0(x,y)$.

Case 1. $x,\ y\in X$. Then there exists a function $f\in \mathrm{lip}(X,d)$ with $\mathrm{L}_{(X,d)}(f)\le C$ such that $|f(x)-f(y)|=d(x,y)$. Take $g=f_0=\Psi(f)$. Then $g\in \mathrm{lip}((X_0,d_0),x_0)$ by (vii). Furthermore,

$$\mathrm{L}_{(X,d)}(g)=\mathrm{L}_{(X,d)}(f_0)\le 2\mathrm{L}_{(X,d)}(f)\le 2C=C_0,$$

$$|g(x)-g(y)|=|f_0(x)-f_0(y)|=|f(x)-f(y)|=d(x,y)=d_0(x,y).$$

Case 2. $x\in X,\ y=x_0$. Take $f=\frac{1}{2}\,\mathrm{diam}(X)1_X$. Then $f\in \mathrm{lip}(X,d)$ and $\mathrm{L}_{(X,d)}(f)=0$. Take $g=f_0$. Then $g\in \mathrm{lip}((X_0,d_0),x_0)$ by (vii). Furthermore,

$$\mathrm{L}_{(X_0,d_0)}(g)=\mathrm{L}_{(X_0,d_0)}(f_0)\le 2\mathrm{L}_{(X,d)}(f)=0\le C_0,$$

$$|g(x)-g(y)|=|f_0(x)-f_0(y)|=|f_0(x)|=\frac{1}{2}\,\mathrm{diam}(X)=d_0(x.y).$$

Case 3. $x=x_0,\ y\in X$. Take $f=\frac{1}{2}\,\mathrm{diam}(X)1_X$ and $g=f_0$. Then $g\in \mathrm{lip}((X_0,d_0),x_0)$ by (vii). Furthermore,

$$\mathrm{L}_{(X_0,d_0)}(g)=\mathrm{L}_{(X_0,d_0)}(f_0)=2\mathrm{L}_{(X,d)}(f)=0\le C_0,$$

$$|g(x)-g(y)|=|f_0(x)-f_0(y)|=|f_0(y)|=\frac{1}{2}\,\mathrm{diam}(X)=d_0(x,y).$$

Case 4. $x=x_0,\ y=x_0$. Take $g=0_{X_0}$. Then $g\in \mathrm{lip}((X_0,d_0),x_0)$, $\mathrm{L}_{(X_0,d_0)}(g)=0\le C_0$ and

$$|g(x)-g(y)|=|0-0|=0=d_0(x_0,x_0)=d_0(x,y).$$

Therefore, $\mathrm{Lip}((X_0,d_0),x_0)$ separates uniformly the points of X. □

Theorem 2.7. *Let (X,d) be a compact metric space, $\varphi:X\to X$ be a Lipschitz mapping from (X,d) to (X,d) and $u\in \mathrm{Lip}(X,d)$ with $u(x)\neq 0$ for all $x\in X$. Let $\mathrm{lip}(X,d)$ separates uniformly the points of X. If $T=uC_\varphi:\mathrm{Lip}(X,d)\to \mathrm{Lip}(X,d)$ is weakly compact, then T is compact.*

Proof. Let $x_0\notin X$, $X_0=X\cup\{x_0\}$ and $d_0:X_0\times X_0\to\mathbb{R}$ be the metric on X which is defined in Lemma 2.6 (i). By Lemma 2.6 (ii), (X_0,d_0) is a compact metric space since (X,d) is compact. Thus, $((X_0,d_0),x_0)$

is a pointed compact metric space with the base point x_0. For each $\mathbb{K}$-valued function f on X, let f_0 be the $\mathbb{K}$-valued function on X_0 defined by $f_0(x) = f(x)$ if $x \in X$ and $f_0(x_0) = 0$. Define the map $\Psi : \mathrm{Lip}(X, d) \to \mathrm{Lip}((X_0, d_0), x_0)$ by $\Psi(f) = f_0$ where $f \in \mathrm{Lip}(X, d)$. According to Lemma 2.6 (vi), we deduce that Ψ is a bounded linear operator from $\mathrm{Lip}(X, d)$ with the sum norm $\|\cdot\|_{\mathrm{Lip}(X,d)}$ onto $\mathrm{Lip}((X_0, d_0), x_0)$ with the Lipschitz norm $\mathrm{L}_{(X_0,d_0)}(\cdot)$. Let φ_0 be the self-map of X_0 which is defined by $\varphi_0(x) = \varphi(x)$ for $x \in X$ and $\varphi_0(x_0) = x_0$. By Lemma 2.6 (iii), φ_0 is a Lipschitz mapping from (X_0, d_0) to (X_0, d_0) with $\varphi_0(x_0) = x_0$. Take $T_0 = u_0 C_{\varphi_0}$. Then T_0 is a weighted composition operator on $\mathrm{Lip}((X_0, d_0), x_0)$. It is easy to see that

$$\Psi \circ T = T_0 \circ \Psi. \tag{2.22}$$

Let $T = uC_\varphi$ be compact. By [5, Theorem VI.4.5], we deduce that $\Psi \circ T$ is a weakly compact operator from $\mathrm{Lip}(X, d)$ with the sum norm $\|\cdot\|_{\mathrm{Lip}(X,d)}$ to $\mathrm{Lip}((X_0, d_0), x_0)$ with the Lipschitz norm $\mathrm{L}_{(X_0,d_0)}(\cdot)$. Again, by [5, Theorem VI.4.5], we deduce that $\Psi \circ T \circ \Psi^{-1}$ is a weakly compact operator on $\mathrm{Lip}((X_0, d_0), x_0)$ with the Lipschitz norm $\mathrm{L}_{(X_0,d_0)}(\cdot)$. By Lemma 2.6(viii), $\mathrm{lip}((X_0, d_0), x_0)$ separates uniformly the points of X_0 since $\mathrm{lip}(X, d)$ separates uniformly the points of X. According to Theorem 2.4, we deduce that the weighted composition operator $T_0 = u_0 C_{\varphi_0}$ is a compact operator on $\mathrm{Lip}((X_0, d_0), x_0)$ with the Lipschitz norm $\mathrm{L}_{(X_0,d_0)}(\cdot)$. By [5, Theorem VI.5.4], $T_0 \circ \Psi$ is a compact operator from $\mathrm{Lip}((X_0, d_0), x_0)$ with the Lipschitz norm $\mathrm{L}_{(X_0,d_0)}(\cdot)$ to $\mathrm{Lip}(X, d)$ with the sum norm $\|\cdot\|_{\mathrm{Lip}(X,d)}$. Again by [5, Theorem VI.5.4], we conclude that $\Psi^{-1} \circ T_0 \circ \Psi$ is a compact operator on $\mathrm{Lip}(X, d)$ with the sum norm $\|\cdot\|_{\mathrm{Lip}(X,d)}$. Therefore, according to (2.22) we conclude that $T = uC_\varphi$ is compact. □

As an application of Theorem 2.7, we give a weighted composition operator on a Lipschitz algebra $\mathrm{Lip}(X, d)$ which is not weakly compact.

Example 2.8. Let $X = \{z \in \mathbb{C} : |z| \le 2\}$, ρ be the Euclidean metric on X, $\alpha \in (0, 1)$ and d be the metric ρ^α on X. Then $\mathrm{lip}(X, d)$ separates uniformly the points of X. Define the function $u : X \to \mathbb{C}$ by $u(z) = e^{|z|}$, $z \in X$, and the self-map φ of X by $\varphi(z) = \frac{z}{3}$, $z \in X$. Then $u \in \mathrm{Lip}(X, d)$, $u(z) \neq 0$ for all $z \in X$ and φ is a Lipschitz mapping from (X, d) to (X, d). Since

$$|u(z)|\frac{d(\varphi(z), \varphi(w))}{d(z, w)} = |e^{|z|}|\frac{|\frac{z}{3} - \frac{w}{3}|^\alpha}{|z - w|^\alpha} = e^{|z|}3^{-\alpha} \le e^2 3^{-\alpha}$$

for all $z, w \in X$ with $z \neq w$, by [1, Theorem 2.4] we deduce that $T = uC_\varphi$ is a weighted composition operator on $\mathrm{Lip}(X, d)$. In addition,

$\varphi(\text{coz}(\text{u})) = \varphi(X) = \{z \in X : |z| \leq \frac{2}{3}\}$ which implies that $\varphi(\text{coz}(\text{u}))$ is a totally bounded set in (X, d). It is easy to see that

$$|u(z)| \frac{d(\varphi(z), \varphi(w))}{d(z, w)} \geq 3^{-\alpha}$$

for all $z, w \in X$ with $z \neq w$. It follows that

$$\lim_{d(z,w) \to 0} u(z) \frac{d(\varphi(z), \varphi(w))}{d(z, w)} \neq 0.$$

Therefore, $T = uC_\varphi$ is not compact by [1, Theorem 4.6]. According to Theorem 2.7, we deduce that T is not weakly compact.

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