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Numerical approximation for generalized fractional Volterra integro-differential equations via parabolic contour

Shiva Eshaghi [1]
Department of Basic Science, Kermanshah University of Technology, Kermanshah, Iran

Abstract. In this article, a numerical scheme is constructed to approximate the generalized fractional Volterra integro-differential equations with the regularized Prabhakar derivative. The solution of the problem is represented in the form of inverse Laplace transform in the complex plane. Then, we select the parabolic contour as an optimal contour and use trapezoidal rule to approximate the inverse Laplace transform. Next, the performance of the numerical scheme is implemented for an example. Further, we obtain the absolute errors for various parameters by using our numerical scheme on parabolic contour and show that the proposed algorithm for the solution of inverse Laplace transform is a very well algorithm with high order accuracy.

Keywords: Laplace Transforms, Parabolic contour, Generalized fractional Volterra integro-differential equations.

2000 Mathematics subject classification: 26A33, 65R10.

1. Introduction

The fractional Volterra integro-differential equations have many applications in physics, engineering, economics, diffusion problems. Since the exact solutions of fractional Volterra integro-differential equations are difficult in many cases, so numerical methods are proposed to obtain

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the solution of these types of equations [3, 2, 23, 24, 25, 27, 28, 39].
From another point of view, some authors [17, 29, 38] generalized the Riemann-Liouville (Caputo) fractional integral and derivative to the Prabhakar fractional integral and derivative with the generalized Mittag-Leffler function [38] in kernel. The Prabhakar fractional derivative has fundamental applications in the applied mathematics [4, 9, 10, 11, 13, 17, 37], the time-evolution of polarization processes [17, 18, 22, 37], the fractional Poisson process [17], the fractional Maxwell model in linear viscoelasticity [20], the generalized model of particle deposition in porous media [46] and the generalized reaction-diffusion equations [1]. The great importance for considering the Prabhakar fractional derivative and integral is related to the description of relaxation and response in the anomalous dielectrics of the Havriliak-Negami models [16, 19, 32, 35].
Our aim in this work is to propose a numerical scheme to approximate the generalized fractional Volterra integro-differential equations (GFVIDEs) with the regularized Prabhakar fractional derivative. To this end, we construct the numerical method based on the Laplace transform. Then, we get the solution of GFVIDEs in the sense of contour integral in the complex plane by applying the Laplace transform. Next, we select the parabolic contour and use the trapezoidal rule with equal step size to approximate this integral. Finally, the performance of the numerical method is tested for an example. The structure of this article is as follows. In Section 2, we state some materials in generalized fractional calculus. In Section 3, numerical method to approximate the GFVIDEs with the regularized Prabhakar fractional derivative is proposed. In Section 4, we give an example to show absolute errors for various parameters by using our numerical scheme on parabolic contour.

2. Preliminaries

In year 1971, Prabhakar introduced the generalized Mittag-Leffler function on his study on singular integral equations as follows [38]

$$E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\rho k + \mu)} \frac{z^k}{k!}, \tag{2.1}$$

where $\gamma, \rho, \mu \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\mu) > 0$. Also, $(\gamma)_k$ is the Pochhammer symbol [7]

$$(\gamma)_0 = 1,$$

$$(\gamma)_k = \gamma(\gamma+1)...(\gamma+k-1), \qquad k = 1, 2, \cdots .$$

For $\gamma = 1$, we get the two-parameter Mittag-Leffler function $E_{\rho,\mu}(z)$ defined by

$$E_{\rho,\mu}(z) := E^1_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \qquad \rho, \mu \in \mathbb{C}, \Re(\rho) > 0, \tag{2.2}$$

in addition, for $\gamma = \mu = 1$, this function coincides with the classical Mittag-Leffler function $E_\rho(z)$ [34]

$$E_\rho(z) := E^1_{\rho,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}, \qquad \rho \in \mathbb{C}, \Re(\rho) > 0. \tag{2.3}$$

Recently, some researchers have established many contributions on the generalized Mittag-Leffler function, especially in the theory of fractional calculus, and have detected some of its applications in the physics. For example, new definitions of generalized fractional derivatives were introduced and solutions of the Cauchy-type initial and boundary value problems were expressed in terms of the generalized Mittag-Leffler function [14, 17, 26, 29, 30, 31, 42, 43]. Also, the Mittag-Leffler function and many different generalizations have been calculated in the whole complex plane [26, 40]. The main conclusions in the classical theory of Mittag-Leffler functions are given by Erdélyi in the handbook [8] and the new conclusions are presented by Dzherbashyan [6]. See more details of the generalized Mittag-Leffler function in [5, 12, 15, 37, 40, 41].

Definition 2.1. For $f \in L^1[0,b]$, the Prabhakar fractional integral operator with the generalized Mittag-Leffler function in its kernel is defined as follows [17]

$$E^\gamma_{\rho,\mu,\omega,0+} f(x) = \int_0^x (x-u)^{\mu-1} E^\gamma_{\rho,\mu}\big(\omega(x-u)^\rho\big) f(u)du, \qquad 0 < x < b \leq \infty. \tag{2.4}$$

Remark 2.2. We note that for $\gamma = 0$, the Prabhakar fractional integral operator (2.4) coincides with the Riemann-Liouville fractional integral of order μ as

$$E^0_{\rho,\mu,\omega,0+} f = I^\mu_{0+} f, \tag{2.5}$$

where the Riemann-Liouville fractional integral of order μ is defined as [30, 36]

$$I^\mu_{0+} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(\xi)}{(x-\xi)^{1-\mu}} d\xi, \qquad \mu > 0,\ f \in L^1[0,b],\ 0 < x < b \leq \infty. \tag{2.6}$$

Definition 2.3. Let $f \in L^1[0,b], 0 < x < b \leq \infty$. The Prabhakar fractional derivative is defined by [17]

$$D^{\gamma}_{\rho,\mu,\omega,0+} f(x) = \frac{d^m}{dx^m} E^{-\gamma}_{\rho,m-\mu,\omega,0+} f(x), \tag{2.7}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}, \Re(\rho), \Re(\mu) > 0$. Also, its regularized Caputo counterpart (the regularized Prabhakar fractional derivative) for $f \in AC^m[o,b]$, $0 < x < b < \infty$, is given by

$$\begin{aligned} {}^C D^{\gamma}_{\rho,\mu,\omega,0+} f(x) &= E^{-\gamma}_{\rho,m-\mu,\omega,0+} \frac{d^m}{dx^m} f(x) \\ &= D^{\gamma}_{\rho,\mu,\omega,0+} f(x) - \sum_{k=0}^{m-1} x^{k-\mu} E^{-\gamma}_{\rho,k-\mu+1}(\omega x^{\rho}) f^{(k)}(0+), \end{aligned} \tag{2.8}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}, \Re(\rho), \Re(\mu) > 0$. The notation $AC^m[0,b]$ is the space of real-valued functions $f(x)$ with continuous derivatives up to order $m-1$ on $[0,b]$ such that $f^{(m-1)}(x)$ belongs to the space of absolutely continuous functions $AC[0,b]$:

$$AC^m[0,b] = \Big\{ f : [0,b] \to \mathbb{R} : \frac{d^{m-1}}{dx^{m-1}} f(x) \in AC[0,b] \Big\}.$$

Remark 2.4. It is obvious that the Prabhakar fractional derivative (2.7) and the regularized Prabhakar fractional derivative (2.8) generalize the Riemann-Liouville and the Caputo fractional derivatives of order μ, respectively, i.e., for $\gamma = 0$ we have

$$D^{0}_{\rho,\mu,\omega,0+} f = D^{\mu}_{0+} f,$$

$${}^C D^{0}_{\rho,\mu,\omega,0+} f = {}^C D^{\mu}_{0+} f,$$

where the Riemann-Liouville and the Caputo fractional derivatives of order μ $(m-1 < \mu < m,\ m \in \mathbb{Z})$ are defined as follows [30, 36]

$$D^{\mu}_{0+} f(x) = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dx^m} \int_0^x (x-\xi)^{m-1-\mu} f(\xi) d\xi, \quad x > 0. \tag{2.9}$$

$${}^C D^{\mu}_{0+} f(x) = I^{m-\mu}_{0+} \frac{d^m}{dx^m} f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-\xi)^{m-1-\mu} \frac{d^m}{d\xi^m} f(\xi) d\xi. \tag{2.10}$$

Lemma 2.5. *The Laplace transform of the generalized Mittag-Leffler function (2.1) has the following form* [38]

$$\mathcal{L}\big[x^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega x^{\rho})\big](s) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}, \qquad |\omega s^{-\rho}| < 1, \tag{2.11}$$

where $\gamma, \rho, \mu, \omega, s \in \mathbb{C}$ *and* $\Re(\mu) > 0,\ \Re(s) > 0$.

3. Numerical approximation of the GFVIDEs

Consider the following GFVIDEs with the regularized Prabhakar fractional derivative

$$ {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}f(t) = g(t) + \int_0^t k(t,\tau)f(\tau)d\tau, \quad m-1 \le \alpha < m,\ m \in \mathbb{N}. \tag{3.1} $$

For the convolution type of equation (3.1), the kernel will be of the form $k(t,\tau) = k(t-\tau)$. Therefore, the equation (3.1) becomes

$$ {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}f(t) = g(t) + \int_0^t k(t-\tau)f(\tau)d\tau. \tag{3.2} $$

By taking the Laplace transform on the both side of (3.2) with respect to t, we have

$$ \mathcal{L}\{{}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}f(t); t\to s\} = \mathcal{L}\{g(t); t\to s\} + \mathcal{L}\{\int_0^t k(t-\tau)f(\tau)d\tau; t\to s\}. \tag{3.3} $$

By using the convolution theorem and the Laplace transform of the generalized Mittag-Leffler function, we obtain the left hand side of the above relation as follows

$$ \begin{aligned} \mathcal{L}\{{}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}f(x); s\} &= \mathcal{L}\{E^{-\gamma}_{\rho,m-\mu,\omega,0+}\frac{d^m}{dx^m}f(x); s\} \\ &= \mathcal{L}\{x^{m-\mu-1}E^{-\gamma}_{\rho,m-\mu}(\omega x^{\rho})\}\mathcal{L}\{\frac{d^m}{dx^m}f(x); s\} \\ &= s^{\mu-m}(1-\omega s^{-\rho})^{\gamma}\Big[s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1}f^{(k)}(0)\Big] \\ &= s^{\mu}(1-\omega s^{-\rho})^{\gamma}F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma}f^{(k)}(0). \end{aligned} $$

So, we rewrite (3.3) as

$$ s^{\mu}(1-\omega s^{-\rho})^{\gamma}F(s) = \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma}f^{(k)}(0) + G(s) + K(s)F(s), \tag{3.4} $$

where

$$ F(s) = \mathcal{L}\{f(t); s\}, \quad G(s) = \mathcal{L}\{g(t); s\} \quad K(s) = \mathcal{L}\{k(t); s\}. $$

Finally, we get

$$F(s) = \frac{G(s)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma} - K(s)} + \frac{\sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^{\gamma} f^{(k)}(0)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma} - K(s)}. \tag{3.5}$$

By taking the inverse Laplace, the problem reduces to compute the following integral in the complex plane

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds. \tag{3.6}$$

The numerical method for the inversion of Laplace transform is based on the approximation of the Bromwich complex contour integral. We select the contour of integration to approximate the path from $c - i\infty$ to $c + i\infty$. To this end we consider the parabolic contour [44, 45]. The parametric equation of parabolic contour is given by [33, 44, 45]

$$s = \beta\Big((1-c)^2 - \zeta^2\Big) + 2i\beta\zeta(1-c), \quad -\infty < \zeta < +\infty, \tag{3.7}$$

where β and c are parameters and need to be optimized for better accuracy. More details about the parabolic contour are given in [33, 45]. The numerical solution can be represented in the following form

$$f(t) = \mathcal{L}^{-1}\{F(s); s \to t\} = \frac{1}{2\pi i}\int_{\Gamma} F(s(\zeta))e^{s(\zeta)t}s'(\zeta)ds. \tag{3.8}$$

If we use equal weight quadrature rule, i.e the trapezoidal rule with the step size h, then the equation (3.8) can be approximated as

$$f_N(t) = \frac{h}{2\pi i}\sum_{j=-N}^{N} F(s(\zeta_j))e^{s(\zeta_j)t}s'(\zeta_j), \quad 1 < \alpha < 2, \quad \zeta_j = jh. \tag{3.9}$$

4. Example

Consider the following GFVIDEs

$${}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}f(t) = t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho}) + \int_0^t (t-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho})f(\tau)d\tau, \tag{4.1}$$

with the initial condition $f(0) = 0$. Where $\rho, \omega, \gamma \in \mathbb{R}$ and $\mu \in (0, 1)$. The exact solution of the GFVIDEs (4.1) is

$$f(t) = \sum_{i=0}^{\infty} t^{2\mu(i+1)-1}E^{2\gamma(i+1)}_{\rho,2\mu(i+1)}(\omega t^{\rho}). \tag{4.2}$$

To give the approximate solution of the GFVIDEs (4.1), we use the presented numerical scheme. By applying the Laplace transform to equation (4.1) with respect to t, we have

$$\begin{aligned} s^{\mu}(1-\omega s^{-\rho})^{\gamma}F(s) &= s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}f(0) \\ &\quad + s^{-\mu}(1-\omega s^{-\rho})^{-\gamma} + s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}F(s). \end{aligned}$$

The initial condition $f(0) = 0$ yields

$$\begin{aligned} F(s) &= \frac{s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}}{s^{\mu}(1-\omega s^{-\rho})^{\gamma} - s^{-\mu}(1-\omega s^{-\rho})^{-\gamma}} \\ &= \frac{s^{-2\mu}(1-\omega s^{-\rho})^{-2\gamma}}{1-s^{-2\mu}(1-\omega s^{-\rho})^{-2\gamma}}, \quad \rho,\omega,\gamma \in \mathbb{R}, \quad 0<\mu<1. \end{aligned} \tag{4.3}$$

From the above equation and the relation (3.9), we get the approximate solution as

$$f_N(t) = \frac{h}{2\pi i}\sum_{j=-N}^{N} \frac{(s(\zeta_j))^{-2\mu}(1-\omega(s(\zeta_j))^{-\rho})^{-2\gamma}}{1-(s(\zeta_j))^{-2\mu}(1-\omega(s(\zeta_j))^{-\rho})^{-2\gamma}} e^{s(\zeta_j)t} s'(\zeta_j),$$

where $\zeta_j = jh$. For our numerical experiments, we choose the parameters such that the absolute error has the least value. So, we take the parameters as follows

$$c = 0.3, h = \frac{3}{N}, \beta = \frac{\pi N}{12t}.$$

Table (1) and Figures (1)-(5) show the absolute errors for approximate solution of the GFVIDEs (4.1). Figure (1) shows the absolute errors for $N = 100$, $\gamma = 2.5$, $\rho = 0.1$, $\omega = -45$ and various μ. Figures (2)-(4) show the absolute errors along the change of γ, μ, ω for $N = 100$ and $t = 0.01$. Figure (5) indicates absolute error for $\gamma = 2.5$, $\rho = 0.1$, $\omega = -45$, $\mu = 0.75$ and different N.

5. Conclusion

In this article, we constructed a numerical scheme to approximate the GFVIDEs with the regularized Prabhakar fractional derivative. The proposed numerical method is based on the Laplace transform and the quadrature rule. Then, we performed the proposed method for an example and showed the absolute value errors for approximating the solution of GFVIDEs.

N	t	γ	ρ	μ	ω	Absolute error
20	0.1	0.2	0.1	0.99	-1	1.9514e-11
30	0.1	0.2	0.1	0.99	-1	4.1930e-17
40	0.1	0.2	0.1	0.99	-1	1.1110e-16
50	0.1	0.2	0.1	0.99	-1	3.8858e-16
100	0.1	0.2	0.1	0.99	-1	3.0341e-14
100	1	0.2	0.1	0.99	-1	4.9028e-14
100	0.01	0.2	0.1	0.99	-1	1.1645e-14
100	0.01	1.5	0.1	0.99	-1	4.2241e-15
100	0.01	2.5	0.1	0.99	-1	2.1133e-15
100	0.01	2.5	0.2	0.99	-1	3.7637e-15
100	0.01	2.5	0.4	0.99	-1	1.1763e-14
100	0.01	2.5	0.1	0.95	-1	3.6462e-15
100	0.01	2.5	0.1	0.75	-1	8.6063e-14
100	0.01	2.5	0.1	0.75	-0.1	3.0590e-13
100	0.01	2.5	0.1	0.75	-10	5.0779e-17
100	0.01	2.5	0.1	0.75	-45	8.1248e-20

TABLE 1. Absolute errors of the GFVIDEs (4.1) using our numerical scheme and the parabolic contour.

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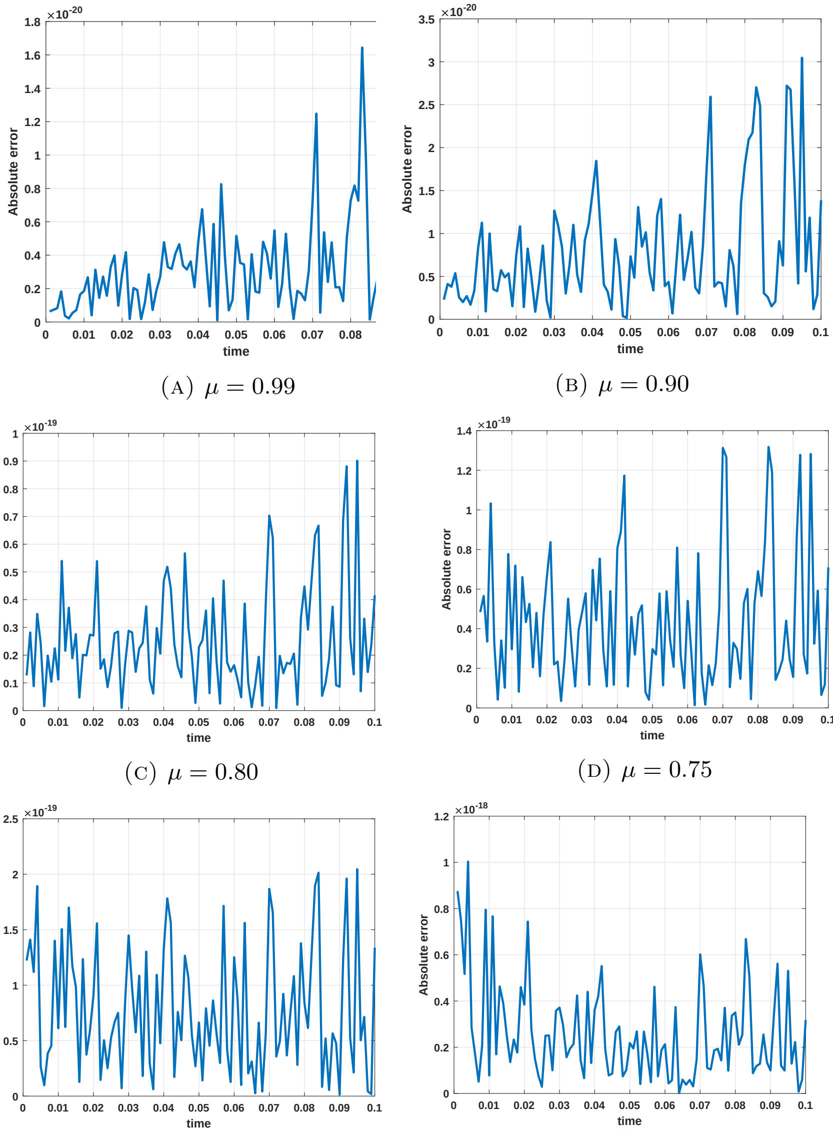

(A) $\mu = 0.99$ (B) $\mu = 0.90$

(C) $\mu = 0.80$ (D) $\mu = 0.75$

(E) $\mu = 0.70$ (F) $\mu = 0.60$

FIGURE 1. Absolute errors for approximate the solution of GFVIDEs (4.1) for $N = 100$, $\rho = 0.1$, $\omega = -45$ and $\gamma - 2.5$.

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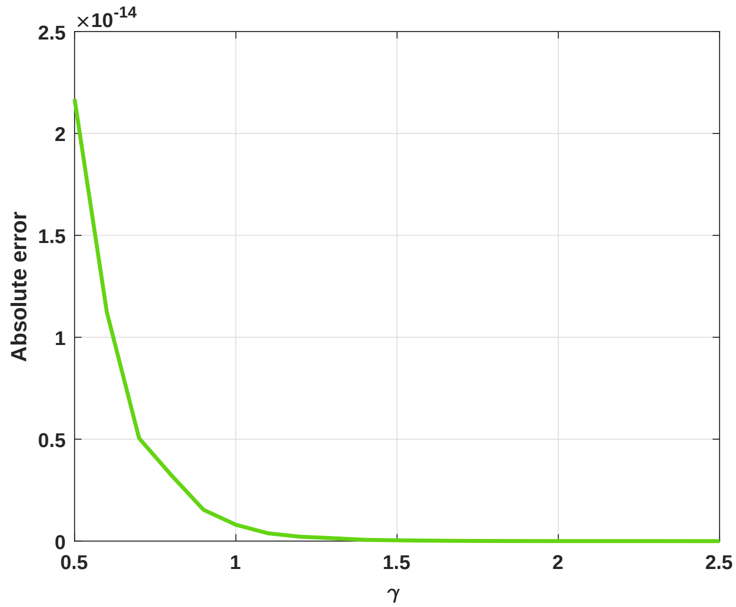

FIGURE 2. Absolute errors for approximate the solution of GFVIDEs (4.1) for $N = 100$, $t = 0.01$, $\rho = 0.1$, $\mu = 0.75$, $\omega = -45$ and different γ.

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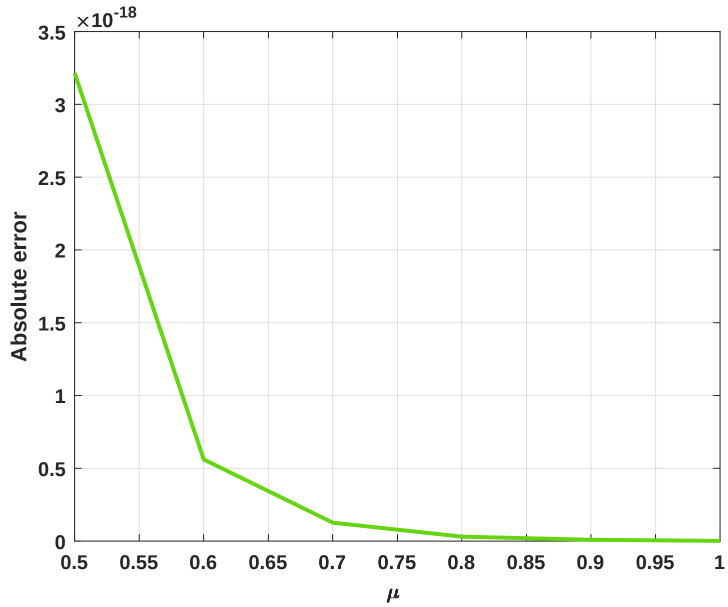

FIGURE 3. Absolute errors for approximate the solution of GFVIDEs (4.1) for $N = 100$, $t = 0.01$, $\gamma = 2.5$, $\rho = 0.1$, $\omega = -45$ and different μ.

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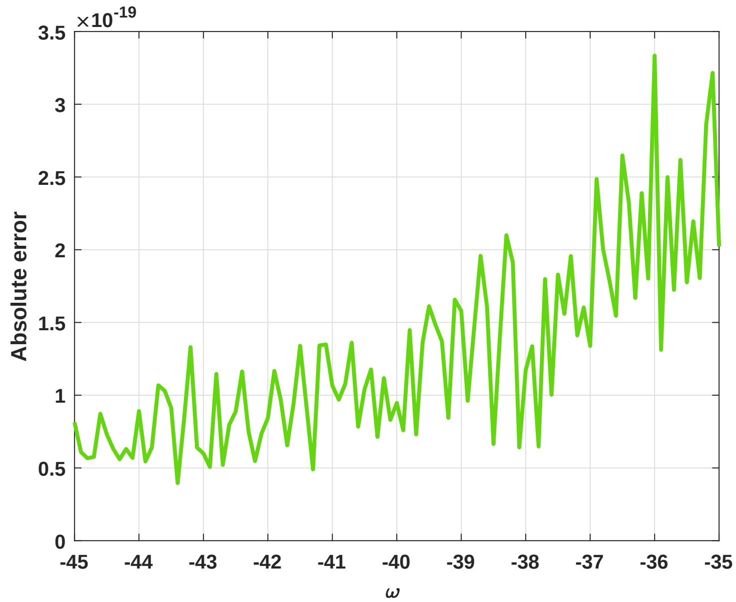

FIGURE 4. Absolute errors for approximate the solution of GFVIDEs (4.1) for $N = 100$, $t = 0.01$, $\gamma = 2.5$, $\rho = 0.1$, $\mu = 0.75$ and different ω.

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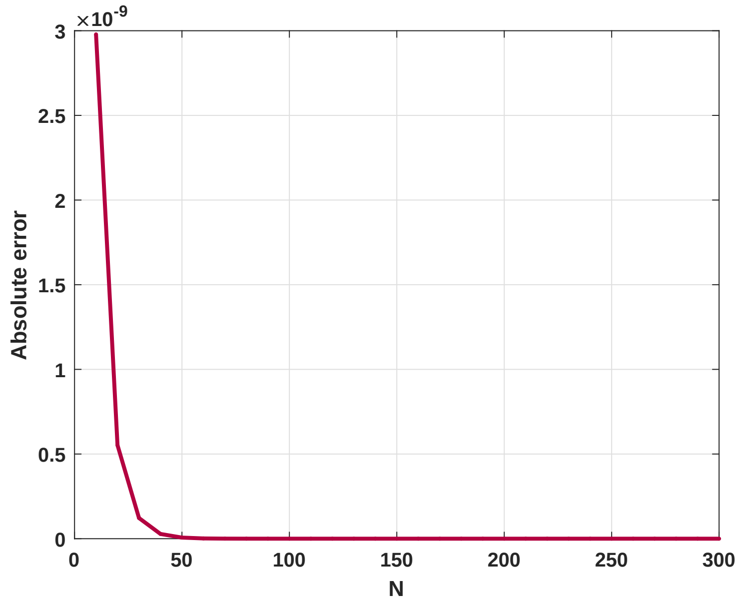

Figure 5. Absolute errors for approximate the solution of GFVIDEs (4.1) for $\gamma = 2.5$, $\rho = 0.1$, $\omega = -45$, $\mu = 0.75$, $t = 0.01$ and different N.

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