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Totally Synchronizing Generated System

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Abstract. We introduce the notion of a minimal generator G for the coded system X; that is a generator for coded system X whenever $u \in G$, then $u \notin W(\overline{< G \setminus \{u\} >})$. Such an X is called *minimally generated system*. We aim to introduce a class of minimally generated subshifts generated by some certain synchronizing blocks. These systems are precisely the tool that will enable us to show that for such subshifts X, each $x \in X$ can be written uniquely as $x = \dots v_{-1}v_0v_1v_2\dots$, where $\{\dots, v_{-1}, v_0, v_1, v_2, \dots\} \in G$. Shows that the converse of that theorem isn't necessarily true. We will show which of the components of the Kreiger graph of such a subshift could be a candidate to be suitable for a Fischer cover.

Keywords: Coded System, Strong Synchronizing, Minimal Generator.

2000 Mathematics subject classification: 37B10, 37B40, 54H20.

1. Introduction

One of the most studied dynamical systems is a subshift of finite type (SFT). SFT X is a system whose set of forbidden blocks is finite [7]; or equivalently, X is SFT iff there is $M \in \mathbb{N}$ such that any block of length greater than M is synchronizing. Recall that a block m is

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synchronizing if whenever v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X as well. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics* where they are factors of SFT's. Synchronized systems, has attracted much attention and extension of them has been of interest since that notion was introduced [3]. One was via *half synchronized systems*; that is, systems having *half synchronizing* blocks. In fact, if for a left transitive point such as rm and mv any block in X one has again $rmv \in X^- = \{x_- := \cdots x_{-1}x_0 : x = \cdots x_{-1}x_0x_1 \cdots \in X\}$, then m is called half synchronizing [3]. Clearly any synchronized system is half synchronized. Dyke (or Dyck!) subshifts and certain β-shifts are non-synchronized but half synchronized systems [8]. Here in Section (3), we will introduce the notion of a *totally synchronizing generated system*, generated by G such that all blocks in G are synchronizing. These systems enable us to show that for such subshifts X, if $x \in X$, then there is unique $\{\ldots, v_{-1}, v_0, v_1, v_2, \ldots\} \in G$ such that $x = \ldots v_{-1}v_0v_1v_2\ldots$. In 1992, Fiebigs in [3], as an extension to the Fischer cover of a synchronized system, introduced a unique component of the Kreiger graph as the Fischer cover of a half synchronized subshift. In Section (4), give another right resolving and follower separated cover for X, denoted by $\mathcal{H}_G$ which is not necessarily Fischer cover of X and gives a sufficient condition on a minimal generator G that the cover $\mathcal{H}_G$ be Fischer coverand gives a sufficient condition on a minimal generator G that the cover $\mathcal{H}_G$ be Fischer cover.

2. Background and definitions

Let $\mathcal{A}$ be a non empty finite set. The full shift $\mathcal{A}$-shift ($\mathcal{A}^{\mathbb{Z}}$), is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. A *block* is a finite sequence of symbols. The *shift map* σ on the $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$ [11]. For a $\mathcal{A}^{\mathbb{Z}}$, set $X_{\mathcal{F}}$ to be the collection of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or *subshift* is a subset X of a full shift such that $X = X_{\mathcal{F}}$ for some subset $\mathcal{F}$.

Let $W_n(X)$ be the set of all admissible n-blocks. A subshift X is *irreducible* if for every blocks u_1, $u_2 \in W(X)$ there is a block $u \in W(X)$ such that $u_1uu_2 \in W(X)$. A shift of *sofic* is the image of an SFT by a factor code [12].

Let $\mathcal{E}(G)$ be the set of vertices and $\mathcal{V}(G)$ be the set of *edge shift* for a graph G. Suppoze that X_G to be

$$\{(\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}$$

where $i(e)$ and $t(e)$ are initiate and terminate vertex of edge e. A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where $\mathcal{E}$ is edge set for graph G and $\mathcal{L}$ is the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G. Set

$$X_\mathcal{G} := \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* of $X = \overline{X_\mathcal{G}}$.

Let X be a subshift and $x \in X$. Set $w_+(x_-) = \{x_+ \in X^+ : x_-x_+ \in X\}$ and for $m \in W(X)$ set $w_+(m) = \{x_+ \in X^+ : mx_+ \in X^+\}$. Analogously, we define predecessor sets $w_-(x_+)$ and $w_-(m)$. Consider the collection of all $w_+(x_-)$ as the set of vertices of a graph. There is an edge labeled a from I_1 to I_2 if and only if there is an x_- such that $x_-a \in X^-$ and $I_1 = w_+(x_-), I_2 = w_+(x_-a)$. This graph is called the *Krieger graph* for X. For synchronized system X with synchronizing m, the irreducible component of the Krieger graph containing $w_+(m)$ is denoted by X_0^+ and is called the *Fischer cover* of X [5].

3. MINIMAL GENERATOR

A coded system is a shift space that can be presented by an irreducible countable labeled graph [7].

Definition 3.1. [10] Let G be a generator for coded system X. Then, G is called *minimal* (resp. *weak minimal*), whenever $u \in G$, then $u \notin W(Z)$, (resp. $X \neq Z$) where $Z = \overline{< G \setminus \{u\} >}$. Such an X is called *minimally* (resp. *weak minimally*) generated system.

Example 3.2. Let $\emptyset \neq S \subseteq \mathbb{N}$. Then, $G := \{10^n1 : n \in S\}$ is a minimal generator for subshift $X := \overline{< G >}$.

Theorem 3.3. *(1) The shift space X has a minimal (resp. minimal weak) generator if and only if X^{-1} has so.*

(2) Let G be a minimal (resp. minimal weak) generator for product shift space $X_1 \times X_2$. Then, X_i has minimal (resp. minimal weak) generator as well for $i = 1, 2$.

Proof. Note that G is a minimal (resp. minimal weak) generator for X, then $G^{-1} := \{v^{-1} = v_i v_{i-1} \dots v_1 : v = v_1 \dots v_{i-1} v_i \in G\}$ is a minimal (resp. minimal weak) generator for X^{-1}. This proves part (i).

(ii) Set $G := \{v_1^1 \times v_1^2, v_2^1 \times v_2^2, \dots\}$ and let $i \in \{1, 2\}$. We claim that

$$G_i := \{v_j^i : j \in \mathbb{N}\}$$

is a minimal (resp. minimal weak) generator for X_i. Since $X_1 \times X_2 = \overline{< G >}$, so $X_i = \overline{< G_i >}$ is trivial. It suffice to show that for any $v_j^i \in G_i$, $v_j^i \notin W(\overline{< G_i \setminus \{v_j^i\} >})$.

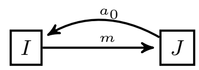

FIGURE 1. The subgraph of X_0^+.

If $v^i_j \in W(\overline{< G_i \setminus \{v^i_j\} >})$, then there is $\{v^i_{j_1}, v^i_{j_2}, \dots v^i_{j_l}\} \subseteq G_i$ such that for $1 \leq k \leq l$, we have $v^i_j \subseteq v^i_{j_1} \dots v^i_{j_l}$ and $v^i_j \neq v^i_{j_k}$. This show that

$$v^1_j \times v^2_j \in W(\overline{< G_1 \times G_2 \setminus \{v^1_j \times v^2_j\} >}).$$

That is absurd. □

Definition 3.4. [9] Let X be a synchronized system. We call a block m an *strong synchronizing* for X if whenever e, e' are finite paths in Fischer cover X_0^+ labeled m, then $e = e'$.

An irreducible shift space with a strong synchronizing block is called *strong synchronized*. Any strong synchronized system is synchronized. we will show that every strong synchronized system is weak minimally system. First, let X be a strong synchronized system and $S_t(X)$ (resp. $S(X)$) denote the set of all strong synchronizing (resp. synchronizing) blocks for X.

Theorem 3.5. *Let X be a strong synchronized system with generator G. Suppoze there is $m \in S_t(X) \cap G$ such that for all $u \in G$, there are not non empty blocks a, b such that $vu = avb$ or $uv = avb$. Then, X has a weak minimal generator.*

Proof. Pick $m \in S_t(X)$ and let π_m be a unique path in Fischer cover X_0^+ such that $\mathcal{L}(\pi_m) = m$. Set $i(\pi_m) := I$, $t(\pi_m) := J$ and

$$G_m := \{ma : \; mam \in W(X) \text{ and } m \nsubseteq a\}. \tag{3.1}$$

We claim that G_m is a weak minimal generator for X. Clearly $X = \overline{< G_m >}$ and so G_m is a generator for X. Thus it suffices to show that for all $ma \in G_m$, $X \neq Z$ where $Z = \overline{< G_m \setminus \{ma\} >}$. Pick $ma_0 \in G_m$. Thus $ma_0m \in W(X)$ and so there is a path π_{a_0} in Fischer cover X_0^+ with initial vertex J and terminal vertex I. Figure 1. Note that if π be a finite path in X_0^+ labeled ma_0m, then $\pi = \pi_m \pi_{a_0} \pi_m$ and so if $ma_0m \subseteq ma_1ma_2 \dots ma_k$, then there is $1 \leq i \leq k$ such that $a_0 = a_i$. Hence $ma_0m \notin W(Z)$ and we are done. □

The next example shows that the converse of theorem 3.5 does not hold.

Example 3.6. (1) Pick $S \subseteq \mathbb{N} \cup \{0\}$ such that $0 \in S$. Set $G := \{10^n : n \in S\}$ and claim that G is a weak minimal generator for S-gap shift $X(S)$. For all $n \in S$, $(10^n 1)^\infty \notin \overline{< G \setminus \{10^n\} >}$ and so $\overline{< G >} \neq \overline{< G \setminus \{10^n\} >}$. Also $\overline{< G >} = X(S)$ is trivial and we are done.

(2) Let D be the Dyke subshift. Add a symbol $*$ to the set of brackets. Let X be the shift space which consists of all sequences of these five symbols such that any finite subblock which doesn't contain a $*$ obeys the rules of standard bracket [8]. Then, X is not a strong synchronized system [9].

It is easy to see that It is easily to see that

$$G_* = \{*u : * \notin u \in W(X)\}$$

is a weak minimal generator for X and we are done.

Note that if $m \in S_t(X)$ and $m^2 \in W(X)$, then G_m as in (3.1) is a minimal generator for X if and only if $G_m = \{m\}$.

Theorem 3.7. *Let X be a strong synchronized system and $m \in S_t(X)$ such that $m^2 \notin W(X)$. Then, G_m as in* (3.1) *is a minimal generator for X if and only if all cycles in the Fischer cover X_0^+ meeting $I := i(\pi_m)$, passes over m.*

Proof. Let G_m be a minimal generator for X and let there is a cycle C passing through $I := i(\pi_m)$ and labeled u such that $m \nsubseteq u$. Pick a finite path π_{u_0} in X_0^+ with initial vertex $J := i(\pi_m)$ and terminal vertex I such that $m \nsubseteq u_0$ as in Figure 5. Then, mu_0, $mu_0u \in G_m$ such that $mu_0 \subseteq mu_0u$ that is absurd.

Conversely, let all cycles in X_0^+ that passing through I, containing m. Pick $ma_0 \in G_m$. If $ma_0 \subseteq ma_1 \dots ma_k$ for some $1 \leq i \leq k$, then there are $1 \leq i \leq k$ and $u_i \in W(X)$ such that $ma_i = ma_0u_i$. Let $u_i \neq \varepsilon$. But $i(\pi_{u_i}) = t(\pi_{a_0}) = I = t(\pi_{a_i}) = t(\pi_{u_i})$, so there is a cycle labeled u_i and passing through I such that $m \nsubseteq u_i$. That is absurd and so u_i is the empty block and so $ma_i = ma_0$. This means that G_m is a minimal generator for X. □

The next example shows that the hypothesis of Theorem 3.7 can not be weakened to synchronized system.

Example 3.8. Let H be the graph as in Figure 2 and $X = X_H$. Then, $m := 101$ is a synchronizing block of X such that $m \notin S_t(X)$ and

$$G_m = \{m0, m010, m012, m01210, m2, m210\}.$$

Pick $a := 210$ and $a_1 := 01210$. Then, $ma \subseteq 10ma = ma_1$ and so $ma \in W(Z)$ where $Z = \overline{< G_m \setminus \{ma\} >}$. Thus G_m is not a minimal

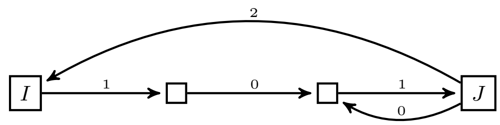

FIGURE 2. The grapg H for the cover of a synchronized system such that G_m is not a minimal generator for X_H.

generator for X. But all cycles in the Fischer cover $X_0^+ = H$ meeting $I := i(\pi_m)$, passes over m.

Note that if $m \in S(X)$ and $m \subseteq u$, then $u \in S(X)$. But it is not true when $m \in S_t(X)$. This fact can be seen by the fact that in Figure 2, $2 \in S_t(X)$ but $012 \notin S_t(X)$.

Let G be a minimal generator for a subshift X. Set G_{ts} denote the set of all $v \in G$ such that for all $u \in G$, there are not non empty blocks a, b such that $vu = avb$ or $uv = avb$.

Theorem 3.9. *Let G be a minimal generator of $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $v := v_1v_2 \dots v_n \in G$. Then, $v \in G_{ts}$ if and only if for each $u \in G$, $v \nsubseteq v_2 \dots v_n u$ and $v \nsubseteq uv_1v_2 \dots v_{n-1}$.*

Proof. Suppose that $v \in G_{ts}$ and let there is a block $u = u_1u_2 \dots u_k \in G$ such that $v \subseteq v_2 \dots v_n u$. Then, $v = v_{n'} \dots v_n u_1 u_2 \dots u_{n'-1}$ such that $n' > 1$. Also

$$vu = v_1v_2 \dots v_{n'-1}v_{n'} \dots v_n u_1 u_2 \dots u_{n'-1} u_{n'} u_{n'+1} \dots u_k.$$

Set $a := v_1v_2 \dots v_{n'-1}$ and $b := u_{n'}u_{n'+1} \dots u_k$. Then, $vu = avb$ that is absurd.

Conversely, suppose that for each $u \in G$, $v \nsubseteq v_2 \dots v_n u$ and $v \nsubseteq uv_1v_2 \dots v_{n-1}$. Also let there is a block $u = u_1u_2 \dots u_k \in G$ such that $vu = avb$ for some non empty blocks $a = a_1a_2 \dots a_i$, $b = b_1b_2 \dots b_j$. Then,

$$v_1v_2 \dots v_i v_{i+1} \dots v_n u_1 u_2 \dots u_k = a_1 a_2 \dots a_i v b.$$

Thus $v_1v_2 \dots v_i = a_1a_2 \dots a_i$ and $v = v_{i+1} \dots v_n u_1 u_2 \dots u_i$ and so $v \subseteq v_{i+1} \dots v_n u$ that is absurd. □

Theorem 3.10. *If $v \in G_{ts}$ and period$(v^\infty) = n$, then $|v| = n$.*

Proof. Set $r := |v|$. If $r > n$, then there is $k > 1$ such that $r = nk$ and so there is $v' \in W_n(X)$ such that $v = (v')^k$. Thus $v^2 = v'v(v')^{k-1}$ that is absurd and so v^∞ has least period $|v|$. □

Theorem 3.11. *Let G be a minimal generator for a subshift X. Then,*

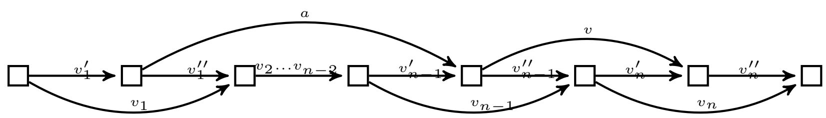

Figure 3. Lemma 3.13.

(1) If $v \in G_{ts}$ and av, $vb \in W(X)$, then a and b are terminal segment and initial segment of a finite concatenation of elements in G respectively.

(2) If v, $v' \in G_{ts}$, then $w_-(v) = w_-(v')$ and $w_+(v) = w_+(v')$.

(3) $G_{ts} \subseteq S(X)$.

Proof. (1) Since $av \in W(X)$, so there is $\{v_1, v_2, \ldots, v_n\} \subseteq G$ such that $av \subseteq v_1 v_2 \ldots v_n$ and so

$$v_1 v_2 \ldots v_n = v_1' v_1'' v_2 \ldots v_j \ldots v_{n-2} v_{n-1}' v_{n-1}'' v_n' v_n''$$

where $v_i = v_i' v_i''$ for $i = 1, n-1, n$, $v = v_{n-1}'' v_n'$ and $a = v_1'' v_2 \ldots v_{n-2} v_{n-1}'$. Figure 3. But $v \subseteq v_{n-1} v_n$, so $v_n = v$ or $v_{n-1} = v$. Suppose that $v_n = v$. Then, $v_{n-1} v = v_{n-1} v_n = v_{n-1}' v v_n''$ and so $v_{n-1}' = \varepsilon$ or $v_n'' = \varepsilon$. If $v_{n-1}' = \varepsilon$, then $a = v_1'' v_2 \ldots v_{n-2}$ and we are done.

Now let $v_{n-1}' \neq \varepsilon$. Then, v_n'' must be an empty block and so $v_n' = v_n = v = v_{n-1}'' v_n'$. Hence $v_{n-1}'' = \varepsilon$. Thus $v_{n-1} = v_{n-1}'$ and so $a = v_1'' v_2 \ldots v_{n-1}$. Similar reasoning works for $v_{n-1} = v$.

If $vb \in W(X)$, then by use same routine as in the before case, to show that there is $\{u_1, u_2, \ldots, u_{n'}\} \subseteq G$ such that $b = u_2 \ldots u_{n'-1} u_{n'}'$ where $u_{n'} = u_{n'}' u_{n'}''$.

(2) Let $a \in w_-(v)$. Then, it follows from (i) that there is

$$\{v_1, v_2, \ldots, v_n\} \subseteq G$$

such that $a = v_1'' v_2 \ldots v_n$ where $v_1 = v_1' v_1''$ and so

$$av' = v_1'' v_2 \ldots v_n v' \subseteq v_1 v_2 \ldots v_n v' \in W(X).$$

Thus $av' \in W(X)$ and so $w_-(v) = w_-(v')$. Similar reasoning works for $b \in w_+(v)$ and so $w_+(v) = w_+(v')$.

(3) Let $v \in G_{ts}$ and av, $vb \in W(X)$. Then, it follows from (i) that $av = v_1'' v_2 \ldots v_n v$ and $vb = v u_2 \ldots u_{n'-1} u_{n'}'$. Thus

$$avb = v_1'' v_2 \ldots v_n v u_2 \ldots u_{n'-1} u_{n'}' \subseteq v_1 v_2 \ldots v_n v u_2 \ldots u_{n'}$$

and so $avb \in W(X)$.

□

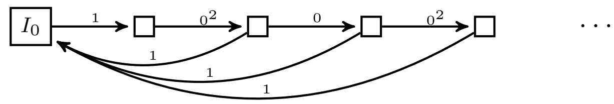

FIGURE 4. The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; $\mathcal{H}_G$ is the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$ with $G := \{10^n1 : n \in P\}$.

Let G be a minimal generator for a subshift X with $G = G_{ts}$. Then, G is called a *totally synchronizing generator*. Such an X is called *totally synchronizing generated system.*

The next example shows that there are non sofic but totally synchronizing generated systems.

Example 3.12. Let P be the set of all prime numbers. Set $G := \{10^n1 : n \in P\}$ and $X := \overline{< G >}$. Then, X is a totally synchronizing generated system. But it is easy to check that for $i = 1, 2, 3, \ldots$ the follower sets $w_+(10^i)$ are all different from each other, so that the shift space X has infinitely many follower sets and so by [7, Theorem 3.2.10], X is not a sofic.

Let $G = \{u_1, u_2, \ldots\}$ be a minimal generator for a subshift X. We give another right resolving and follower separated cover for X, denoted by $\mathcal{H}_G$ which is not necessarily Fischer cover of X. To do so fix $\{a_1, a_2, \ldots\} \subseteq \mathbb{N}$. Let the loop graph $\mathcal{G}$ has one vertex I_0 and infinite self loops e_i labeled a_i at that vertex ($i \geq 1$). We construct a new graph from $\mathcal{G}$ denoted by $\mathcal{G}_{u_i \hookrightarrow a_i}$ by replacing u_i for a_i whenever there is a path in $\mathcal{G}$ labeled a_i for all $i \geq 1$. We can suppose that $\mathcal{G}_{u_i \hookrightarrow a_i}$ is right resolving. Now let $\mathcal{H}_G$ be the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$ [10]. Then, by [7, Lemma 3.3.8] $X = X_{\mathcal{G}_{u_i \hookrightarrow a_i}} = X_{\mathcal{H}_G}$ and $\mathcal{H}_G$ is right resolving and follower separated [4]. For instance see the next example.

Example 3.13. (1) Let $X := \overline{< G >}$ where

$$G := \{(), (()), [()], ((())), [(())], \ldots\} \cup \{[], ([]), [[]], [[[]]], ([[]]), \ldots\}.$$

Then, G is a minimal generator for X. Figure 4 shows $\mathcal{H}_G$ for G.

(2) Let X be a strong synchronized system, $m \in S_t(X)$ and

$$G_m := \{ma_1, ma_2, \ldots\}$$

be a minimal generator for X where $\{a_1, a_2, \ldots\} \subseteq W(X)$. Then, $X_0^+ = X_{\mathcal{H}_G}$ where $\mathcal{H}_G$ is the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$ and $\mathcal{G}_{u_i \hookrightarrow a_i}$ is as the Figure 8.

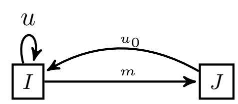

FIGURE 5. The subgrapg of X_0^+.

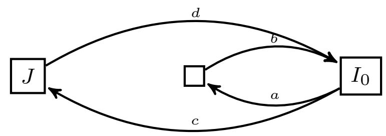

FIGURE 6. A subgraph of $\mathcal{H}_G$ where $ab = v_0 = bc$.

The following gives a sufficient condition on a minimal generator G that the cover $\mathcal{H}_G$ be Fischer cover. For this we first need to define the *magic block* m for a right reasolving cover if there is one and only one vertex I such that $m \in F_-(I)$ where

$$F_-(I) = \{\mathcal{L}\text{-labels of all finite paths terminating at } I\}.$$

Theorem 3.14. *Let G be a minimal generator for the coded system X and assume that $v_0 \in G_{ts}$. Then,*

(1) $\mathcal{H}_G = X_0^+$.
(2) $G_{ts} \subseteq S_t(X)$.

Proof. (i) To show that $\mathcal{H}_G = X_0^+$, it suffices by [3, Theorem 2.16] to show that $\mathcal{H}_G$ has a magic block. The construction of $\mathcal{H}_G$ shows that $v_0 \in F_-(I_0)$. Let $v_0 \in F_-(J)$ and $J \neq I_0$. Then, there are non empty blocks a, b, c, d of X such that $ab = v_0 = bc$ and $v = cd$ as in Figure 6. Then, $v_0 v = a v_0 d$ and so $a = \varepsilon$ or $d = \varepsilon$ that is absurd and so v_0 is a magic block for the $\mathcal{H}_G$ which set over claim.

(ii) Since there is exactly one path labeled v_0 in the Fischer cover $\mathcal{H}_G$, so v_0 is a strong synchronizing block of X. □

The next theorem can be applied in the reference [1].

Theorem 3.15. *Let $G = G_{ts}$ for a subshift X and $x = \dots v_{-1} v_0 v_1 \dots = \dots v'_{-1} v'_0 v'_1 \dots$ where v_j, $v'_j \in G$. Then,*

(1) There are $i, j \in \mathbb{Z}$ such that for all $k \in \mathbb{Z}$, $v_{i+k} = v'_{j+k}$.
(2) If $v_i = x_0 x_1 \dots x_{i_0}$, then there is $j \in \mathbb{Z}$ such that $v'_j = v_i$ and $x = \dots v'_{j-1}.v'_j v'_{j+1} \dots$

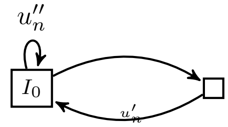

FIGURE 7. A subgraph of $\mathcal{H}_G$.

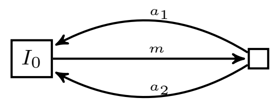

FIGURE 8. The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; $\mathcal{H}_G = (X_{\mathcal{H}_G})_0^+$ is the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$.

Proof. There are i_0, $j_0 \in \mathbb{Z}$ such that $x_0 \in v_{i_0} \cap v'_{j_0}$. Then, $v'_{j_0} \subseteq v_{i_0-1}v_{i_0}$ or $v'_{j_0} \subseteq v_{i_0}v_{i_0+1}$. Without loss of generality, we can assume $v'_{j_0} \subseteq v_{i_0}v_{i_0+1}$. Thus $v'_{j_0} = v_{i_0}$ or $v'_{j_0} = v_{i_0+1}$. Hence there is $l \in \{i_0, i_0+1\}$ such that $v_{l+k} = v'_{j_0+k}$ for all $k \in \mathbb{Z}$.

Part (ii) follows from Part (i). □

The next example shows that the converse of the above theorem is not necessarily true.

Example 3.16. (1) Let G and X to be as in 3.13. Suppose that $x := \cdots v_{-1}v_0v_1 \cdots = \cdots v'_{-1}v'_0v'_1 \cdots$ where v_i, $v'_i \in G$ and $i \in \mathbb{Z}$. Then, there is $n \in \mathbb{X}$ such that

$$\ldots v_{i_0} = v'_{j_0},\ v_{i_0+1} = v'_{j_0+1},\ v_{i_0+2} = v'_{j_0+2},\ \ldots.$$

But G is not minimal generator.

(2) Set $G := \{v_1 := 101,\ v_2 := 010,\ v_3 := 0101\}$ and

$$x := (v_1v_2)^\infty.(v_1v_2)^\infty \in X := \overline{< G >}.$$

Then, $x = (v_3)^\infty$. But there is no $i \in \{1, 2\}$ such that $v_3 = v_i = x_{[0,2]}$. This shows that the hypothesis of Theorem 3.15 can not be weakened to the generator.

4. Conclusion

These systems are precisely the tool that will enable us to create a bridge between dynamical systems and other mathematical branches. By creating such a connection, we will be able to introduce the basic concepts linear independence and dependence from the branch of linear algebra to dynamic systems and enter the topic of applied mathematics.

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