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Semicomplete Lattice of All T-Complex Gradations of Openness on a Complex Fuzzy Topological Space

M. Mostafavi [1] and M. Sadeghi [2]
[1] Faculty of Basic Sciences, University of Qom, Qom, Iran
[2] Department of Mathematics, University of Mazandaran, Babolsar, Iran

Abstract. In this paper, we introduce the complex (anti complex) fuzzy topological space $(X, \mathfrak{T})$ with complex (anti complex) gradation of openness under T-norm (C-conorm), which X is itself a T-complex (C-anti complex) fuzzy subset of a nonempty set M. We show that the set of all T-complex gradations of openness on X is a semicomplete lattice. Some example such as T-complex fuzzy subspace of $\Lambda\mathbb{R}^m$, the exterior algebra on $\mathbb{R}^m$ are given.

Keywords: Spiral T-norm, T-complex fuzzy subset, C-anti complex fuzzy subspace, complex gradation of openess under T-norm, anti complex fuzzy topological space.

2000 Mathematics subject classification: 08A72, 47A30, 54A40, 06B23, 46B20.

1. Introduction

Ramot et al. [24] introduced the notion of a complex fuzzy set (CFS) as an extension of a fuzzy set defined by Zadeh [30] which its range extends from a closed interval [0, 1] to a circle of radius one in a complex plane. The ability of the complex fuzzy set to represent two-dimensional phenomena makes it superior for fuzzy information processing and intuition which is common in time-periodic phenomena.

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Complex fuzzy sets, classes and their logic play an important role in applications such as periodicity event prediction and advanced control systems. To learn more about structures and applications of complex fuzzy sets, see [1, 2, 3, 8, 9, 10, 11, 17, 23, 25, 26, 33]. Complex fuzzy set is used in signals and systems because it behaves like the Fourier transform in certain cases. Zeeshan and khan [18] developed a new algorithm using complex fuzzy sets for applications in signals and systems by which reference signals are identified from a large number of signals detected by a digital receiver. They used the inverse discrete Fourier transform of a complex fuzzy set for the input signals and a reference signal. Therefore, a method is provided to measure the exact values of two signals by which they can identify the reference signal. See also two works [16, 32].

In this paper, we define complex fuzzy subspace of a k-vector space V, under T-norm and anti complex fuzzy subspace of a k-vector space V, under C-conorm. Some example such as T-complex fuzzy subspace of $\Lambda\mathbb{R}^m$, the exterior algebra on $\mathbb{R}^m$ are given. Then we investigate various operations between T-complex fuzzy sets and present a numerical example for each of them. Also we define image and inverse image of a T-complex fuzzy subspace under a function.

Since Chang [5] defined fuzzy topology, various concepts of it were defined such as [6, 7, 12, 19, 20, 21, 27, 28, 29]. In 1985, Shostak [27] introduced a concept of the gradation of openness of fuzzy subsets of a nonempty set. Also many authers investigated graded fuzzy topological spaces such as [6, 7, 12, 20, 21, 29].

The author introduced and discussed properties of a kind of fuzzy topological structure in [22]. Considering the importance and application of the complex fuzzy sets, we study about this topic. In this paper, we define complex (anti complex) fuzzy topological space $(X, \mathfrak{T})$ with complex (anti complex) gradation of openness under T-norm (C-conorm) which X is itself a T-complex (C-anti complex) fuzzy subset of a nonempty set M. We define spiral T-norm of a sequence in $[0, 1]$ and spiral minimum of a sequence in $[0, 2\pi]$ and then by using them, we prove that the set of all T-complex gradations of openness $(\mathfrak{M}_{\mathfrak{T}}(X), \leq)$ on X, is a semicomplete lattice

Definition 1.1. [24] Let M be a nonempty set. A complex fuzzy set A on M is an object having the form $A = \{(x, \mu_A(x))|x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in M$, a complex number $\mu_A(x)$ lies within the unit circle in the complex plane. We shall assume that $\mu_A(x)$ will be represented by $r_A(x)e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, and $r_A : M \to [0, 1]$ and $\omega_A : M \to [0, 2\pi]$.

The term $r_A(x)$ is said to be phase term and $\omega_A(x)$ is said to be amplitude term. Note that by setting $\omega(x) = 0$, we turn back to the traditional fuzzy subset.

Let $\mu_1 = r_1 e^{\omega_1}$ and $\mu_2 = r_2 e^{\omega_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \leq \mu_2$, we mean $r_1 \leq r_2$ and $\omega_1 \leq \omega_2$.

Three constant complex fuzzy sets $\tilde{1}$, $\tilde{0}$ and $\tilde{i}$ are defined by

$$\mu_{\tilde{1}}(x) = r_{\tilde{1}}(x) e^{i\omega_{\tilde{1}}(x)},\ r_{\tilde{1}}(x) = 1,\ \omega_{\tilde{1}}(x) = 2\pi,\ \ \forall x \in M$$

$$\mu_{\tilde{0}}(x) = r_{\tilde{0}}(x) e^{i\omega_{\tilde{0}}(x)},\ r_{\tilde{0}}(x) = 0,\ \omega_{\tilde{0}}(x) = 0,\ \ \forall x \in M$$

$$\mu_{\tilde{i}}(x) = r_{\tilde{i}}(x) e^{i\omega_{\tilde{i}}(x)},\ r_{\tilde{i}}(x) = 0,\ \omega_{\tilde{i}}(x) = \frac{\pi}{2},\ \ \forall x \in M$$

Definition 1.2. [13] A T-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)

for all $x, y, z \in [0,1]$.
We say that T is idempotent if for all $x \in [0,1],\ T(x, x) = x$.

Example 1.3. [13]
(1) Standard intersection T-norm $T_{min}(x, y) = min\{x, y\}$
(2) Bounded sum T-norm $T_b(x, y) = max\{0, x + y - 1\}$
(3) algebraic product T-norm $T_p(x, y) = xy$

Definition 1.4. A C-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(C1) $C(x, 0) = x$ (neutral element),
(C2) $C(x, y) \leq C(x, z)$ if $y \leq z$ (monotonicity),
(C3) $C(x, y) = C(y, x)$ (commutativity),
(C4) $C(x, C(y, z)) = C(C(x, y), z)$ (associativity)

for all $x, y, z \in [0,1]$.
We say that the C-conorm C is idempotent if $\forall x \in [0,1],\ C(x, x) = x$.

Example 1.5. (1) Standard union C-conorm $C_{max}(x, y) = max\{x, y\}$
(2) Bounded sum C-conorm $C_b(x, y) = max\{1, x + y\}$
(3) Algebraic product C-conorm $C_p(x, y) = x + y - xy$

Lemma 1.6. *Consider a T-norm T and a C-conorm C (briefly (T,C)-norm). Then for all $x,y,z,w \in [0,1]$ we have*

$$T(x,y) \leq x \wedge y,$$
$$C(x,y) \geq x \vee y,$$
$$T\big(T(x,y),T(z,w)\big) = T\big(T(x,z),T(y,w)\big),$$
$$C\big(C(x,y),C(z,w)\big) = C\big(C(x,z),C(y,w)\big),$$

2. Main result

In this section after some definitions and theorems, we define (anti) complex gradation of openness under T-norm (C-conorm) and then we introduce T-complex (C-anti complex) fuzzy topological space with (anti) complex gradation of openness. Also we define the concept of the spiral T-norm of a countable subset $\{x_i | i \in \mathbb{N}\}$ of $[0,1]$ and using it we prove that the set of all T-complex gradations of openness on X $(\mathfrak{M}_{\mathfrak{T}}(X), \leq)$ is a semicomplete lattice

Definition 2.1. Let V be a k-vector space. A complex fuzzy subset $B = \{(x, \mu_B(x)) | x \in X\}$ of V is called a complex fuzzy subspace under T-norm if $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, such that

$$r_B(\gamma x + \lambda y) \geq T\big(r_B(x), r_B(y)\big), \quad \omega_B(\gamma x + \lambda y) \geq \min\big(\omega_B(x), \omega_B(y)\big)$$

for any $x,y \in V$ and $\gamma, \lambda \in k$. Then we can write briefly B is a T-complex fuzzy subspace of V or $B \in TCF(V)$.

Example 2.2. Let $E = \Lambda\mathbb{R}^m$ be an exterior algebra on $\mathbb{R}^m$ with anticommutative generators $\{\xi_1, \ldots, \xi_m\}$. Hence $\xi_i^2 = 0$, and $\xi_j \wedge \xi_i = -\xi_i \wedge \xi_j$ for all $1 \leq i,j \leq m$. Then each $\xi \in E$ has the form

$$\xi = \sum_{1 \leq i_1 < \ldots < i_k \leq m} \alpha_{i_1 \ldots i_k} \, \xi_{i_1} \wedge \ldots \wedge \xi_{i_k}, \qquad \alpha_{i_1 \ldots i_k} \in \mathbb{R}.$$

We define complex fuzzy subset B of E by $B = r_B e^{i\omega_B}$,

$$r_B(\xi_i) = r_i, \quad \omega_B(\xi_i) = t_i, \quad r_i \in [0,1],\ t_i \in [0,2\pi]$$

for all $1 \leq i,j \leq m$ and

$$r_B(\xi) = \sup_{1 \leq i_1 < \ldots < i_k \leq m} \big\{T\big(\ldots T(T(r_{i_1}, r_{i_2}), r_{i_3}), \ldots, r_{i_k}\big)\big\}, \tag{2.1}$$

$$\omega_B(\xi) = \sup_{1 \leq i_1 < \ldots < i_k \leq m} \big\{\min\big(\ldots \min(\min(t_{i_1}, t_{i_2}), t_{i_3}), \ldots, t_{i_k}.\big)\big\}. \tag{2.2}$$

We show that B is a T-complex fuzzy subspace of E:
For each $\xi,\ \eta \in E$ and $\gamma, \lambda \in k$, we have

$$\eta = \sum_{1 \leq j_1 < \ldots < j_l \leq m} \beta_{j_1 \ldots j_l} \, \xi_{j_1} \wedge \ldots \wedge \xi_{j_l}, \qquad \beta_{j_1 \ldots j_l} \in \mathbb{R}$$

$$T\big(r_B(\xi), r_B(\eta)\big) = T\Bigg(\sup_{1\le i_1<\ldots<i_k\le m} \big\{T\big(\ldots T(T(r_{i_1}, r_{i_2}), r_{i_3}), \ldots, r_{i_k}\big)\big\},$$

$$\sup_{1\le j_1<\ldots<j_l\le m} \big\{T\big(\ldots T(T(r_{j_1}, r_{j_2}), r_{j_3}), \ldots, r_{j_k}\big)\big\}\Bigg)$$

$$\le \sup_{1\le i_1<\ldots<i_k\le m} \big\{T\big(\ldots T(T(r_{i_1}, r_{i_2}), r_{i_3}), \ldots, r_{i_k}\big)\big\},$$

$$\bigvee \sup_{1\le j_1<\ldots<j_l\le m} \big\{T\big(\ldots T(T(r_{j_1}, r_{j_2}), r_{j_3}), \ldots, r_{j_k}\big)\big\}$$

$$= r_B(\gamma\ \xi + \lambda\ \eta),$$

Similarly we can prove

$$\max\big(\omega_B(\xi), \omega_B(\eta)\big) \le \omega_B(\gamma\ \xi + \lambda\ \eta).$$

Definition 2.3. Let V be a k-vector space. An Anti complex fuzzy subset $B = \{(x, \mu_B(x))|x \in X\}$ of V is called an anti complex fuzzy subspace under C-conorm if $\mu_B(x)) = r_B(x)e^{i\omega_B(x)}$,

$$r_B(\gamma x + \lambda y) \le C\big(r_B(x), r_B(y)\big), \quad \omega_B(\gamma x + \lambda y) \le \max\big(\omega_B(x), \omega_B(y)\big)$$

for any $x, y \in V$ and $\gamma, \lambda \in k$. Then we can write briefly B is a C-anti complex fuzzy subspace of V or $B \in CACF(V)$.

Example 2.4. Let $E = \Lambda\mathbb{R}^2$ be an exterior algebra on $\mathbb{R}^m$ with anti-commutative generators $\{\xi_1,\ \xi_2\}$. Hence $\xi_1^2 = 0,\ \xi_2^2 = 0$, and $\xi_2 \wedge \xi_1 = -\xi_1 \wedge \xi_2$. Then each $\xi \in E$ has the form

$$\xi = \alpha_1\ \xi_1 + \alpha_2\ \xi_2 + \alpha_{12}\ \xi_1 \wedge \xi_2, \qquad 0 \ne \alpha_1, \alpha_2, \alpha_{12} \in \mathbb{R}$$

We define anti complex fuzzy subset G of E by $G = r_G e^{i\omega_G}$,

$$r_G(\xi_i) = r_i, \quad \omega_G(\xi_i) = t_i, \quad r_i \in [0,1], \quad t_i \in [0, 2\pi]$$

for $i = 1, 2$ and $r_G(\xi_1 \wedge \xi_2) = C(r_1, r_2)$, $\omega_G(\xi_1 \wedge \xi_2) = \max(t_1, t_2)$. Also

$$r_G(\xi) = \max\{r_1, r_2, C(r_1, r_2)\} = C(r_1, r_2)$$

$$\omega_G(\xi) = \max\{t_1, t_2, \max(t_1, t_2)\} = \max(t_1, t_2).$$

We show that G is a C-anti complex fuzzy subspace of E:
For each $\xi,\ \eta \in E$ and $\gamma, \lambda \in k$, we have

$$\eta = \beta_1\ \xi_1 + \beta_2\ \xi_2 + \beta_{12}\ \xi_1 \wedge \xi_2, \qquad 0 \ne \beta_1, \beta_2, \beta_{12} \in \mathbb{R}$$

$$\gamma\xi + \lambda\eta = (\gamma\alpha_1 + \lambda\beta_1)\ \xi_1 + (\gamma\alpha_1 + \lambda\beta_1)\ \xi_2 + (\gamma\alpha_{12} + \lambda\beta_{12})\ \xi_1 \wedge \xi_2.$$

$$\begin{aligned} C\big(r_G(\xi), r_G(\eta)\big) &= C\big(C(r_1, r_2), C(r_1, r_2)\big) \\ &\geq C(r_1, r_2) \bigvee C(r_1, r_2) \\ &= C(r_1, r_2) \\ &= r_G(\gamma\ \xi + \lambda\ \eta), \end{aligned}$$

Similarly we can prove

$$\max\big(\omega_G(\xi), \omega_G(\eta)\big) \geq \omega_G(\gamma\ \xi + \lambda\ \eta).$$

Definition 2.5. Let $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$ be two complex fuzzy subsets of a nonempty set M. We define T-complex fuzzy subset $A \cap B$ by $\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)}$,

$$r_{A\cap B}(x) = T\big(r_A(x), r_B(x)\big),\ \ \omega_{A\cap B}(x) = \min\big(\omega_A(x), \omega_B(x).\big)$$

and T-complex fuzzy subset $A \cup B$ by $\mu_{A\cup B}(x) = r_{A\cup B}(x)e^{i\omega_{A\cup B}(x)}$,

$$r_{A\cup B}(x) = C\big(r_A(x), r_B(x)\big), \quad \omega_{A\cup B}(x) = \max\big(\omega_A(x), \omega_B(x)\big).$$

Example 2.6. Let $M = \{x_1, x_2, x_3\}$ and

$$A = \{(x_1, 0.5e^{i0.4\pi}),\ (x_2, 0.6e^{i\pi}),\ (x_3, 0.8e^{i0.7\pi})\}$$

and

$$B = \{(x_1, 0.3e^{i0.6\pi}),\ (x_2, 0.5e^{i2\pi}),\ (x_3, 0.9e^{i0.6\pi})\}$$

be two complex fuzzy subsets of M. Then $A \cap B$ and $A \cup B$ are defined by:

$$A \cap B = \{(x_1, 0.3e^{i0.4\pi}),\ (x_2, 0.5e^{i\pi}),\ (x_3, 0.8e^{i0.6\pi})\}$$

$$A \cup B = \{(x_1, 0.5e^{i0.6\pi}),\ (x_2, 0.6e^{i2\pi}),\ (x_3, 0.9e^{i0.7\pi})\}$$

Definition 2.7. Let $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$ be two complex fuzzy subsets of a k-vector space V. Then T-complex fuzzy subsets $A + B$ and $\gamma.A$ of V for each $\gamma \in k$, are defined by:

$$\mu_{A+B}(x) = r_{A+B}(x)e^{i\omega_{A+B}(x)},$$

$$r_{A+B}(x) = \begin{cases} \sup_{x=a+b}\ \{T\big(r_A(a), r_B(b)\big)\} & if\ x = a + b \\ 0 & elsewhere \end{cases}$$

$$\omega_{A+B}(x) = \begin{cases} \sup_{x=a+b}\ \{\min\big(\omega_A(a), \omega_B(b)\big)\} & if\ x = a + b, \\ 0 & elsewhere \end{cases}$$

for all $x \in X$, and $\mu_{\gamma.A}(x) = r_{\gamma.A}(x)e^{i\omega_{\gamma.A}(x)}$,

$$r_{\gamma.A}(x) = \begin{cases} r_A(\frac{1}{\gamma}x) & if\ \gamma \neq 0 \\ 1 & if\ \gamma = 0,\ x = 0 \\ 0 & if\ \gamma = 0,\ x \neq 0 \end{cases}$$

$$\omega_{\gamma.A}(x) = \begin{cases} \omega_A(\frac{1}{\gamma}x) & if\ \gamma \neq 0 \\ 0 & if\ \gamma = 0. \end{cases}$$

for all $x \in X$. Further if $A \cap B = \tilde{0}$, then $A + B$ is said to be the direct sum and denoted by $A \oplus B$.

Example 2.8. Let $V = \{v | v = c_1e_1 + c_2e_2\}$ where $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $c_1, c_2 \in \mathbb{R}$. Consider T-norm T_{min}. Let

$$A = \{(0,\ 0e^0), (c_1e_1,\ 0.8e^{i\pi}),\ (c_2e_2,\ 0.2e^{i2\pi}),\ (c_1e_1 + c_2e_2,\ 0.2e^{i\pi})\}$$

and

$$B = \{(0,\ 0e^0), (c_1e_1,\ 0.6e^{i0.5\pi}),\ (c_2e_2,\ 0.4e^{i0.7\pi}),\ (c_1e_1 + c_2e_2,\ 0.4e^{i0.5\pi})\}$$

be two complex fuzzy subsets of V. Then T-complex fuzzy subsets $A+B$ and $\gamma.A$ of V for each $\gamma \in k$, are defined by:

$$A + B = \{(0,\ 0e^0), (c_1e_1,\ 0e^{i0}),\ (c_2e_2,\ 0e^{i0}),\ (c_1e_1 + c_2e_2,\ 0.4e^{i0.5\pi})\},$$

and

$$\gamma.A = \{(0,\ 0e^0), (c_1e_1,\ 0.8e^{i\pi}),\ (c_2e_2,\ 0.2e^{i2\pi}),\ (c_1e_1 + c_2e_2,\ 0.2e^{i\pi})\}$$

when $\gamma \neq 0$ and

$$\gamma.A = \{(0,\ 1e^0), (c_1e_1,\ 0e^{i0}),\ (c_2e_2,\ 0e^{i0}),\ (c_1e_1 + c_2e_2,\ 0e^{i0})\}$$

when $\gamma = 0$. We compute $\mu_{A+B}(c_1e_1 + c_2e_2)$ and other cases are obvios: Let $c_1e_1 + c_2e_2 = (a_1e_1 + a_2e_2) + (b_1e_1 + b_2e_2)$. Then

$$T\big(r_A(a_1e_1 + a_2e_2),\ r_B(b_1e_1 + b_2e_2)\big) = T_{min}(0.2,\ 0.4) = 0.2$$

$$T\big(r_A(c_1e_1),\ r_B(c_2e_2)\big) = T_{min}(0.8,\ 0.4) = 0.4$$

Hence $r_{A+B} = \max(0.2,\ 0.4) = 0.4$. Also

$$\min\big(\omega_A(a_1e_1 + a_2e_2),\ \omega_B(b_1e_1 + b_2e_2)\big) = \min(\pi,\ 0.5\pi) = 0.5\pi$$

$$\min\big(\omega_A(c_1e_1),\ \omega_B(c_2e_2)\big) = \min(\pi,\ 0.7\pi) = 0.7\pi$$

Hence $\omega_{A+B} = \max(0.5\pi,\ 0.7\pi) = 0.5\pi$.

Theorem 2.9. *i) Let $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$ be two T-complex fuzzy subspaces of V. Then $A \cap B = \{(x, \mu_{A\cap B}(x))\}$, $A + B = \{(x, \mu_{A+B}(x))\}$ and $\gamma.A = \{(x, \mu_{\gamma.A}(x))\}$ for each $\gamma \in k$, are also T-complex fuzzy subspaces of V.*
ii) Let $A = \{(x, \mu_A(x))\}$ and $B = \{(x, \mu_B(x))\}$ be two C-anti complex

fuzzy subspaces of V. Then $A\cup B=\{(x,\mu_{A\cup B}(x))\}$ *is a C-anti complex fuzzy subspace of V.*

Proof. We prove (ii) and (i) is similar.

$$\begin{aligned} r_{A\cup B}(\gamma x+\lambda y) &= C\big(r_A(\gamma x+\lambda y), r_B(\gamma x+\lambda y)\big) \\ &\le C\Big(C\big(r_A(x), r_A(y)\big), C\big(r_B(x), r_B(y)\big)\Big) \\ &= \mathrm{C}\Big(\ \mathrm{C}\big(\mathrm{r}_A(x), r_B(x)\big), C\big(r_A(y), r_B(y)\big)\Big) \\ &= \mathrm{C}\big(\ \mathrm{r}_{A\cup B}(x), r_{A\cup B}(y)\big) \end{aligned}$$

$$\begin{aligned} \omega_{A\cup B}(\gamma x+\lambda y) &= \max\big(\omega_A(\gamma x+\lambda y), \omega_B(\gamma x+\lambda y)\big) \\ &\le \max\Big(\max\big(\omega_A(x), \omega_A(y)\big), \max\big(\omega_B(x), \omega_B(y)\big)\Big) \\ &= \max\Big(\max\big(\omega_A(x), \omega_B(x)\big), \max\big(\omega_A(y), \omega_B(y)\big)\Big) \\ &= \max\big(\omega_{A\cup B}(x), \omega_{A\cup B}(y)\big) \end{aligned}$$ □

Definition 2.10. Let $B=\{(x,\mu_B(x))\}$ be a complex fuzzy subset of a group G. Then B is called a complex fuzzy subgroup of G, under T-norm (an anti complex fuzzy subgroup of G under C-conorm), if it satisfies two following conditions:

i) $r_B(xy)\ge T\big(r_A(x), r_B(x)\big),\quad \omega_B(xy)\ge \min\big(\omega_A(x), \omega_B(x)\big),$
$\Big(r_B(xy)\le C\big(r_A(x), r_B(x)\big),\quad \omega_B(xy)\le \max\big(\omega_A(x), \omega_B(x)\big)\Big),$

ii) $r_B(x^{-1})\ge r_B(x),\ \ \omega_B(x^{-1})\ge \omega_B(x)$

for any $x,y\in G$.

Example 2.11. Let $G=\{e,a,b,c\}$ be the Klein 4-group. Every element is its own inverse, $a^2=b^2=c^2=e$ and the product is defined by $ab=c,\ ac=b,\ bc=a$. Let

$$A=\{(e,\ 0.5e^{i1.2\pi}),(a,\ 0.5e^{i1.2\pi}),\ (b,\ 0.6e^{i1.2\pi}),\ (c,\ 0.7e^{i1.2\pi})\}$$

$$B=\{(e,\ 0.64e^{i0.7\pi}),(a,\ 0.4e^{i\pi}),\ (b,\ 0.5e^{i1.2\pi}),\ (c,\ 0.6e^{i1.2\pi})\}$$

be two complex fuzzy subsets of G. Consider algebraic product T-norm $T_p(x,y)=xy$ and algebraic product C-conorm $C_p(x,y)=x+y-xy$. Then A is a complex fuzzy subgroup of G under T-norm and B is an anti complex fuzzy subgroup of G under C-conorm.

Definition 2.12. Let f be a mapping from a nonempty set M to a nonempty set M'. Let $A=\{(x,\mu_A(x))\}$ and $B=\{(x,\mu_B(x))\}$ be complex fuzzy subsets of M and M' respectively. Then the inverse image of B under f, is a complex fuzzy subset $f^{-1}[B]$ defined by:

$$\mu_{f^{-1}[B]}(x) = r_{f^{-1}[B]}(x)e^{i\omega_{f^{-1}[B]}(x)}, \qquad r_{f^{-1}[B]}(x) = r_B(f(x)),$$

$$\omega_{f^{-1}[B]}(x) = \omega_B(f(x))$$

for all $x \in V$ and the image of A under f is a complex fuzzy subset $f[A] = \{(x, \mu_{f[A]}(x))\}$ defined by: $\mu_{f[A]}(y) = r_{f[A]}(y)e^{i\omega_{f[A]}(y)}$,

$$r_{f[A]}(y) = \begin{cases} \sup\{r_A(x) |\ x \in f^{-1}(y)\} & if\ \ y \in f(M) \\ 0 & if\ \ y \notin f(M) \end{cases}$$

$$\omega_{f[A]}(y) = \begin{cases} \sup\{\omega_A(x) |\ x \in f^{-1}(y)\} & if\ \ y \in f(M) \\ 0 & if\ \ y \notin f(M) \end{cases}$$

for all $y \in M'$

Theorem 2.13. *Let f be a linear mapping from the k-vector space V to the k-vector space V'.*
$i)$ If $B = \{(x, \mu_B(x))\}$ is a T-complex (C-anti complex) fuzzy subspace of V, then $f^{-1}[B]$ is a T-complex (C-anti complex) fuzzy subspace of V.
$ii)$ If $A = \{(x, \mu_A(x))\}$ is a T-complex fuzzy subspace of V, then $f[A]$ is a T-complex fuzzy subspace of V'.

Proof. $i)$ Let B be a T-complex fuzzy subspace of V. For each $x, z \in V$ and $\gamma, \delta \in k$, we have

$$\begin{aligned} T\big(r_{f^{-1}[B]}(x), r_{f^{-1}[B]}(z)\big) &= T\big(r_B(f(x)),\ r_B(f(z))\big) \\ &\leq r_B\big(\gamma f(x) + \delta f(z)\big), \\ &= r_B\big(f(\gamma x + \delta z)\big), \\ &= r_{f^{-1}[B]}(\gamma x + \delta z) \end{aligned}$$

Similarly we can prove

$$\min\big(\omega_{f^{-1}[B]}(x), \omega_{f^{-1}[B]}(z)\big) \geq \omega_{f^{-1}[B]}(\gamma x + \delta z).$$

Hence $f^{-1}[B]$ is a complex fuzzy subspace of V.
Now if B be a C-anti complex fuzzy subspace of V, then for each $x, z \in V$

and $\gamma, \delta \in k$, we have

$$\begin{aligned} C\big(r_{f^{-1}[B]}(x), r_{f^{-1}[B]}(z)\big) &= C\big(r_B(f(x)),\ r_B(f(z))\big) \\ &\geq r_B\big(\gamma f(x) + \delta f(z)\big), \\ &= r_B\big(f(\gamma x + \delta z)\big), \\ &= r_{f^{-1}[B]}(\gamma x + \delta z) \end{aligned}$$

Similarly we can prove

$$\max\big(\omega_{f^{-1}[B]}(x), \omega_{f^{-1}[B]}(z)\big) \geq \omega_{f^{-1}[B]}(\gamma x + \delta z).$$

Hence $f^{-1}[B]$ is a C-anti complex fuzzy subspace of V.
ii) Let A be a T-complex fuzzy subspace of V. To prove that $f[A]$ is a T-complex fuzzy subspace of V', we show that for each $y, w \in V'$ and $\gamma, \delta \in k$, we have

$$T\big(r_{f[A]}(y), r_{f[A]}(w)\big) \geq r_{f[A]}(\gamma y + \delta w).$$

1) If $y, w \in f(V)$, then we have

$$\begin{aligned} T\big(r_{f[A]}(y), r_{f[A]}(w)\big) &= T\big(\sup\ \{r_A(x) |\ x \in f^{-1}(y)\},\ \sup\ \{r_A(z) |\ z \in f^{-1}(w)\}\big) \\ &\geq \sup\ \{T\big(r_A(x), r_A(z)\big) |\ x \in f^{-1}(y), z \in f^{-1}(w)\} \\ &\geq \sup\ \{r_A\big(\gamma x + \delta z\big) |\ x \in f^{-1}(y), z \in f^{-1}(w)\} \\ &\geq r_{f[A]}(\gamma y + \delta w). \end{aligned}$$

2) If $y \in f(V)$ and $w \notin f(V)$, we have

$$\begin{aligned} T\big(r_{f[A]}(y), r_{f[A]}(w)\big) &= T\big(\sup\ \{r_A(x) |\ x \in f^{-1}(y)\},\ 0\big) \\ &\geq \sup\{T(r_A(x), 0) |\ x \in f^{-1}(y)\} = 0 \\ &\geq 0 = r_{f[A]}(\gamma y + \delta w). \end{aligned}$$

3) If $y, w \notin f(V)$, we have

$$T\big(r_{f[A]}(y), r_{f[A]}(w)\big) = T\big(0,\ 0\big) = 0 \geq 0 = r_{f[A]}(\gamma y + \delta w).$$

Similarly we can show that

$$\min\big(\omega_{f[A]}(y), \omega_{f[A]}(w)\big) \geq \omega_{f[A]}(\gamma y + \delta w).$$

□

Theorem 2.14. *Let $f : V \to V'$ be a linear mapping between the k-vector spaces. Then for any T-complex fuzzy subspace $A = \{(x, \mu_A(x))\}$ and $D = \{(x, \mu_D(x))\}$ of V and all $\lambda \in k$, we have*

1) $f[A + D] = f[A] + f[D]$,
2) $f[\lambda A] = \lambda f[A]$.

Proof. 1) Let $w \in V'$. We want to show that $a = b$ where $a = r_{f[A+D]}(w)$ and $b = r_{f[A]+f[D]}(w)$. Suppose first that $w \notin Imf$. Then $a = 0$. Also if $x, y \in V'$ with $x + y = w$, then at least one of the x, y is not in Imf and thus $r_{f[A]}(x) \wedge r_{f[D]}(y) = 0$. So we have
$T(r_{f[A]}(x), r_{f[D]}(y)) = 0$. Hence $b = 0 = a$.
Assume next that $w \in Imf$. Given $\varepsilon > 0$, there exists $z \in V$ with $f(z) = w$ such that $r_{A+D}(z) > a - \varepsilon$. Then there exist $x, y \in V$ with $x + y = z$, such that $T\big(r_A(x), r_D(y)\big) > a - \varepsilon$. Since $f(x) + f(y) = w$, we have

$$
\begin{aligned}
b &= \sup_{w=u+v} \{T\big(r_{f[A]}(u), r_{f[D]}(v)\big)\} \\
&\geq T\big(r_{f[A]}(f(x)), r_{f[D]}(f(y))\big) \\
&\geq T\big(r_A(x), r_D(y)\big) \\
&> a - \varepsilon
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we get $b \geq a$. On the other hand given $\varepsilon > 0$, there exist u_1, u_2 with $u_1 + u_2 = w$ such that

$$b - \varepsilon < T\big(r_{f[A]}(u_1), r_{f[D]}(u_2)\big)$$

Taking $\varepsilon < b$ (if $b = 0$ then $a = 0$ and we have nothing to prove), we have that $u_1, u_2 \in Imf$. Therefore, there exist x_1, x_2 in V with $u_1 = f(x_1)$, $u_2 = f(x_2)$ such that

$$b - \varepsilon < T\big(r_A(x_1), r_D(x_2)\big).$$

Since $f(x_1 + x_2) = w$, we get $a > b - \varepsilon$ and hence $a \geq b$, because $\varepsilon > 0$ was arbitrary. So $a = b$. Similarly we can prove that $\omega_{f[A+D]} = \omega_{f[A]+f[D]}$.

2) Let $w \in V'$, $c = r_{f[\lambda A]}(w)$ and $d = r_{\lambda f[A]}(w)$. If $w \notin Imf$. Then $c = d = 0$. Assume that $w \in Imf$. If $\lambda \neq 0$,

$$\begin{aligned} c &= \sup \{r_{\lambda A}(x)|\ f(x) = w\} \\ &= \sup \{r_A(\tfrac{1}{\lambda}x)|\ f(x) = w\} \\ &= \sup \{r_A(y)|\ f(\lambda y) = w\} \\ &= \sup \{r_A(y)|\ \lambda f(y) = w\} \\ &= r_{\lambda f[A]}(w) = d. \end{aligned}$$

Next suppose that $\lambda = 0$. If $w \neq 0$, then $c = 0$ and $d = r_{0f[A]}(w) = 0$. If $w = 0$, we have

$$\begin{aligned} c &= \sup \{r_{0A}(x)|\ f(x) = 0\} \\ &= \sup \{1|\ f(x) = 0\} \\ &= \sup \{r_A(y)|\ y \in V\} \\ &= r_{0f[A]}(0) = d. \end{aligned}$$

In a similar manner, we can show that $\omega_{f[\lambda A]} = \omega_{\lambda f[A]}$ and this completes the proof. □

Definition 2.15. Let $X = \{(x, \mu_A(x))|x \in X\}$ be a complex fuzzy subset of M. We denote the set of all complex fuzzy subsets of M which are less or equal to X (called complex fuzzy subsets of X) by $S^1{}^M_X$. If τ is a collection of complex fuzzy subsets of X, that satisfies the following conditions:

1) $X,\ \phi \in \tau$,
2) $\{A_i\}_{i \in I} \subseteq \tau \ \Rightarrow\ \bigcup_{i \in I} A_i \in \tau$,
3) $A, B \in \tau \ \Rightarrow\ A \cap B \in \tau$,

then (X, τ) is called a complex fuzzy topological space (Cfts).

Example 2.16. Let $M = \mathbb{R}^n$ and $X = \tilde{1}$. Let $B(a, r, b, c) = \{(x, \mu_{B(a,r,b,c)})|x \in X\}$ be a complex fuzzy subset of M that $\mu_{B(a,r,b,c)}$ equals to $\tilde{0}$ outside or on the sphere $B(a, r)$ and equals to the function be^{ic} on M which $b : B(a, r) \to [0, 1],\ c : B(a, r) \to [0, 2\pi]$ are two arbitrary functions. We call the fuzzy topology induced by

$\beta_{Cn} = \{B(a, r, b, c),\ a \in \mathbb{R}^n,\ r \in \mathbb{R}^+,\ b : B(a, r) \to [0, 1], c : B(a, r) \to [0, 2\pi]\}$

the complex fuzzy Euclidean topology of dimension n (denoted by τ_{Cn}).

Definition 2.17. Let $\mathfrak{T} : S^{1}{}_{X}^{M} \to S^1$, be a mapping, lies within the unit circle in the complex plane and be represented by
$\mathfrak{T}(A) = r_{\mathfrak{T}}(A)e^{i\omega_{\mathfrak{T}}(A)}\ \forall A = \{(x, \mu_{A}(x))|x \in X\} \in S^{1}{}_{X}^{M}$, where
$r_{\mathfrak{T}} : S^{1}{}_{X}^{M} \to [0, 1]$ and $\omega_{\mathfrak{T}} : S^{1}{}_{X}^{M} \to [0, 2\pi]$ satisfy:

(i) $r_{\mathfrak{T}}(X) = r_{\mathfrak{T}}(\tilde{0}) = 1,\ \ \omega_{\mathfrak{T}}(X) = \omega_{\mathfrak{T}}(\tilde{0}) = 2\pi$,

(ii) $\forall A = \{(x, \mu_A(x))|x \in X\},\ B = \{(x, \mu_B(x))|x \in X\} \in S^{1}{}_{X}^{M}$

$r_{\mathfrak{T}}(A \cap B) \geq T\big(r_{\mathfrak{T}}(A), r_{\mathfrak{T}}(B)\big)$, $\omega_{\mathfrak{T}}(A \cap B) \geq \big(\omega_{\mathfrak{T}}(A) \wedge \omega_{\mathfrak{T}}(B)\big)$,

(iii) $\forall\{A_j = \{(x, \mu_{A_j}(x))|x \in X\},\ j \in J\} \subseteq CI_X^M$

$r_{\mathfrak{T}}(\bigcup_{j\in J} A_j) \geq \bigwedge_{i,j\in J} T\big(r_{\mathfrak{T}}(A_i), r_{\mathfrak{T}}(A_j)\big)$,

$\omega_{\mathfrak{T}}(\bigcup_{j\in J} A_j) \geq \inf\{\ \omega_{\mathfrak{T}}(A_j), j \in J\}$,

Then $\mathfrak{T}$ is called complex gradation of openness under T-norm and $(X, \mathfrak{T})$ is called a T-complex fuzzy topological space with T-complex gradation of openness (TCG-fts).

Example 2.18. Let $M = \mathbb{R}^n$ and $X = \tilde{1}$. As two useful examples, we define $\mathfrak{T}_{Cn} : S^{1}{}_{X}^{M} \to I$ by:

$$\mathfrak{T}_{Cn}(B) = \begin{cases} \tilde{1}(B) & B \in \tau_{Cn}, \\ \tilde{0}(B) & elsewhere. \end{cases}$$

and $\mathfrak{T}_{Cinf} : S^{1}{}_{X}^{M} \to I$ by: $\mathfrak{T}_{Cinf}(B) = r_{\mathfrak{T}_{Cinf}}(B)e^{i\omega_{\mathfrak{T}_{Cinf}}(B)}$,

$$r_{\mathfrak{T}_{Cinf}}(B) = \begin{cases} 1 & B = \tilde{0}, \\ inf\{r_{B}(x) : x \in M\} & \tilde{0} \neq B \in \tau_{Cn}, \\ 0 & elsewhere, \end{cases}$$

$$\omega_{\mathfrak{T}_{Cinf}}(B) = \begin{cases} 2\pi & B = \tilde{0}, \\ inf\{\omega_{B}(x) : x \in M\} & \tilde{0} \neq B \in \tau_{Cn}, \\ 0 & elsewhere, \end{cases}$$

Obviously both are complex gradation of openness under T-norm T_{min}. In general if $\mathfrak{T}$ be any complex gradation of openness under T-norm T on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T} = \tau_{Cn}$, then we call $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Cn})$ the T-complex fuzzy Euclidean topological space with complex gradation of openness.

Definition 2.19. Let $\mathfrak{T} : S^{1M}_{X} \to S^1$, be a mapping, lies within the unit circle in the complex plane and be represented by $\mathfrak{T}(A) = r_{\mathfrak{T}}(A)e^{i\omega_{\mathfrak{T}}(A)}$ $\forall A = \{(x, \mu_A(x))|x \in X\} \in S^{1M}_{X}$, where $r_{\mathfrak{T}} : S^{1M}_{X} \to [0,1]$ and $\omega_{\mathfrak{T}} : S^{1M}_{X} \to [0, 2\pi]$ satisfy:

(i) $, r_{\mathfrak{T}}(X) = r_{\mathfrak{T}}(\tilde{0}) = 0,\ \ \omega_{\mathfrak{T}}(X) = \omega_{\mathfrak{T}}(\tilde{0}) = \dfrac{\pi}{2}$,

(ii) $\forall A = \{(x, \mu_A(x))|x \in X\},\ B = \{(x, \mu_B(x))|x \in X\} \in S^{1M}_{X}$,

$$r_{\mathfrak{T}}(A \cap B) \le C\big(r_{\mathfrak{T}}(A), r_{\mathfrak{T}}(B)\big), \qquad \omega_{\mathfrak{T}}(A \cap B) \le max\big(\omega_{\mathfrak{T}}(A), \omega_{\mathfrak{T}}(B)\big),$$

(iii) $\forall \{A_j = \{(x, \mu_{A_j}(x))|x \in X\},\ j \in J\} \subseteq CI^M_X$,

$$r_{\mathfrak{T}}(\textstyle\bigcup_{j\in J} A_j) \le \bigvee_{i,j\in J} C\big(r_{\mathfrak{T}}(A_i), r_{\mathfrak{T}}(A_j)\big),$$

$$\omega_{\mathfrak{T}}(\textstyle\bigcup_{j\in J} A_j) \le \sup\{\ \omega_{\mathfrak{T}}(A_j), j \in J\}.$$

Then $\mathfrak{T}$ is called anti complex gradation of openness under C-conorm, and $(X, \mathfrak{T})$ is called a C-anti complex fuzzy topological space with anti complex gradation of openness ($CACG$-fts).

Example 2.20. Let $M = \mathbb{R}^n$ and $X = \tilde{1}$. We define $\mathfrak{T}_{ACn} : S^{1M}_{X} \to I$ by:

$$\mathfrak{T}_{ACn}(B) = \begin{cases} \tilde{i}(B) & B \in \tau_{Cn}, \\ \tilde{0}(B) & elsewhere. \end{cases}$$

and $\mathfrak{T}_{ACinf} : S^{1M}_{X} \to I$, by: $\mathfrak{T}_{ACsup}(B) = r_{\mathfrak{T}_{ACsup}}(B)e^{i\omega_{\mathfrak{T}_{ACsup}}(B)}$,

$$r_{\mathfrak{T}_{ACsup}}(B) = \begin{cases} sup\{r_B(x) : x \in M\} & \tilde{0} \ne B \in \tau_{Cn}, \\ 0 & elsewhere, \end{cases}$$

$$\omega_{\mathfrak{T}_{ACsup}}(B) = \begin{cases} \frac{\pi}{2} & B = \tilde{0}, \\ sup\{\omega_B(x) : x \in M\} & \tilde{0} \ne B \in \tau_{Cn}, \\ 0 & elsewhere, \end{cases}$$

Obviously both are anti complex gradation of openness under C-conorm C_{max}. In general if $\mathfrak{T}$ be any anti complex gradation of openness under C-conorm C on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T} = \tau_{Cn}$, then we call $(1_{\mathbb{R}^n}, \mathfrak{T}_{ACn})$ the C-anti complex fuzzy Euclidean topological space with complex gradation of openness.

Theorem 2.21. *Let T be idempotent and $(X, \mathfrak{T})$ be a T-complex fuzzy topological space. For any $r, s \in [0,1]$, we define*
$\mathfrak{T}_{r,s} = \{A \in L^M_X : r_{\mathfrak{T}}(A) \ge r,\ \omega_{\mathfrak{T}}(A) \ge s\}$. *Then $(X, \mathfrak{T}_{r,s})$ is a complex fuzzy topologiacal space.*

Proof. Since Dom$\mathfrak{T} = S^{1M}_{X}$ for all $A \in supp\mathfrak{T}$, we have A is less than or equal to X. Hence $suppA \subseteq suppX$. Also we have

i) $r_{\mathfrak{T}}(\tilde{0}) = r_{\mathfrak{T}}(X) = 1 \geq r,\ \omega_{\mathfrak{T}}(\tilde{0}) = \omega_{\mathfrak{T}}(X) = 2\pi \geq s$. Hence $\phi,\ X \in \mathfrak{T}_{r,s}$.

ii) For any $A, B \in \mathfrak{T}_{r,s}$, using the condition (ii) of Definition 2.8 and $(T2)$ we have

$$r_{\mathfrak{T}}(A \cap B) \geq T\big(r_{\mathfrak{T}}(A), r_{\mathfrak{T}}(B)\big) \geq T(r, r) = r,$$

$$\omega_{\mathfrak{T}}(A \cap B) \geq \min\big(\omega_{\mathfrak{T}}(A), \omega_{\mathfrak{T}}(B)\big) \geq \min(s, s) = s.$$

Thus $A \cap B \in \mathfrak{T}_{r,s}$.

iii) For all family $\{A_j = (\mu_{A_j}, \nu_{A_j}),\ j \in J\} \subseteq CI^M_X$, we have

$$r_{\mathfrak{T}}(\bigcup_{j\in J} A_j) \geq \bigwedge_{i,j\in J} T\big(r_{\mathfrak{T}}(A_i), r_{\mathfrak{T}}(A_j)\big) \geq \bigwedge_{i,j\in J} T(r, r)) = r$$

$$\omega_{\mathfrak{T}}(\bigcup_{j\in J} A_j) \geq \inf\{\ \omega_{\mathfrak{T}}(A_j), j \in J\} \geq s.$$

Hence $\bigcup_{j\in J} A_j \in \mathfrak{T}_{r,s}$.

Therefore, $(X, \mathfrak{T}_{r,s})$ is a complex fuzzy topological space. □

Definition 2.22. Let T be a T-norm and $\{x_i | i \in \mathbb{N}\}$ be a countable subset of $[0, 1]$. Define

$$T^1_{\circledS}(\{x_i\}) = x_1, \quad T^2_{\circledS}(\{x_i\}) = T(x_1, x_2)$$

$$T^3_{\circledS}(\{x_i\}) = T(T(x_1, x_2), x_3), \quad T^4_{\circledS}(\{x_i\}) = T\big(T(T(x_1, x_2), x_3), x_4\big)$$

$$T^k_{\circledS}(\{x_i\}) = T\big(\ldots T(T(x_1, x_2), x_3), \ldots, x_k\big)$$

Then we define

$$T^{\infty}_{\circledS}(\{x_i\}) = \lim_{k\to\infty} T^k_{\circledS}(\{x_i\})$$

called spiral T-norm of $\{x_i\}$.

Lemma 2.23. *Let T be a T-norm. Then the definition of spiral T-norm of a countable subset $\{x_i | i \in \mathbb{N}\}$ of $[0, 1]$, is well defined. Also we have*

$$T^{\infty}_{\circledS}(\{x_i\}) \leq x_i\ \forall i \in \mathbb{N}.$$

Proof. Using $(T1)$ and $(T2)$, we have

$$T^2_{\circledS}(\{x_i\}) \leq T(x_1, 1) = x_1 = T^1_{\circledS}(\{x_i\})$$

$$T^3_{\circledS}(\{x_i\}) = T(T(x_1, x_2), x_3) \leq T(T(x_1, x_2), 1) = T(x_1, x_2) = T^2_{\circledS}(\{x_i\})$$

By contradiction on k, we can prove that $\{T^k_{\circledS}(\{x_i\})\}$ is a decreasing sequence in $[0, 1]$. Since we assumed that the lattis $[0, 1]$ is complete, $\lim_{k\to\infty} T^k_{\circledS}(\{x_i\})$ exists. Becouse of $(T4)$, the associativity of T, this definition is independent of the ordering of the elements of this subset. Hence definition of spiral T-norm of $\{x_i\}$ is well defined. □

Definition 2.24. Let $\{x_i | i \in \mathbb{N}\}$ be a countable subset of $[0, 2\pi]$. Define

$$\min_{\circledS}^{1}(\{x_i\}) = x_1, \quad \min_{\circledS}^{2}(\{x_i\}) = \min(x_1, x_2)$$

$$\min_{\circledS}^{3}(\{x_i\}) = \min(\min(x_1, x_2), x_3),$$

$$\min_{\circledS}^{4}(\{x_i\}) = \min\big(\min(\min(x_1, x_2), x_3), x_4\big)$$

$$\min_{\circledS}^{k}(\{x_i\}) = \min\big(\dots\min(\min(x_1, x_2), x_3), \dots, x_k\big)$$

Then we define

$$\min_{\circledS}^{\infty}(\{x_i\}) = \lim_{k\to\infty} \min_{\circledS}^{k}(\{x_i\})$$

called spiral minimum of $\{x_i\}$.

Lemma 2.25. *The definition of spiral minimum of a countable subset $\{x_i | i \in \mathbb{N}\}$ of $[0, 2\pi]$, is well defined. Also we have*

$$\min_{\circledS}^{\infty}(\{x_i\}) \le x_i \ \forall i \in \mathbb{N}.$$

Proof. Since $[0, 2\pi]$ is a complete lattis, setting $T = T_{\min} = \min$, we can prove this lemma similar to the proof of Lemma 2.22. □

Definition 2.26. Let L be a lattice. If any countable subset $\{x_i \mid i \in J \subseteq \mathbb{N}\}$ of L, has an infimum in L, then L is called a semicomplete lattice.

Theorem 2.27. *Assume that X is a complex fuzzy subset of M and T, C are T-norm and C-conorm on $[0, 1]$ respectively. Let $\mathfrak{M}_{\mathfrak{T}}(X)$ be the set of all T-complex gradations of openness on X. We write $\mathfrak{T}_1 \le \mathfrak{T}_2$ if $r_{\mathfrak{T}_1}(A) \le r_{\mathfrak{T}_2}(A)$, $\omega_{\mathfrak{T}_1}(A) \le \omega_{\mathfrak{T}_2}(A)$ for all $A \in S^1{}^M_X$. Then $(\mathfrak{M}_{\mathfrak{T}}(X), \le)$, is a semicomplete lattice.*

Proof. It is clear that $\le$ between functions from $S^1{}^M_X$ to $[0, 1]$, is an equivalence relation. Hence $(\mathfrak{M}_{\mathfrak{T}}(X), \le)$ is a partialy orderd set. Define

$$r_{\mathfrak{T}_0}(\tilde{0}) = r_{\mathfrak{T}_0}(X) = 1, \quad \omega_{\mathfrak{T}_0}(\tilde{0}) = \omega_{\mathfrak{T}_0}(X) = 2\pi,$$

$$r_{\mathfrak{T}_0}(A) = 0, \ \ , \omega_{\mathfrak{T}_0}(A) = 0 \quad \forall A \in CI^M_X - \{\tilde{0}, X\},$$

$$r_{\mathfrak{T}_1}(A) = 1, \quad \omega_{\mathfrak{T}_1}(A) = 2\pi \quad \forall A \in L^M_X.$$

Then $\mathfrak{T}_0$ and $\mathfrak{T}_1$ are two T-complex gradation of openness on X. Since

$$r_{\mathfrak{T}_0}(A) \le r_{\mathfrak{T}}(A) \le r_{\mathfrak{T}_1}(A), \quad \omega_{\mathfrak{T}_0}(A) \le \omega_{\mathfrak{T}}(A) \le \omega_{\mathfrak{T}_1}(A) \quad \forall A \in S^1{}^M_X,$$

we have $\mathfrak{T}_0,\ \mathfrak{T}_1$ are respectively $0, 1$ in the lattice set $\mathfrak{M}_{\mathfrak{T}}(X)$.

We show that every countble subset $\{\mathfrak{T}_j \mid j \in \mathbb{N}\}$ of $\mathfrak{M}_{\mathfrak{T},\mathfrak{T}^*}(X)$ has an infimum in it.

Define $\mathfrak{T}$ by $\mathfrak{T}(A) = r_{\mathfrak{T}}(A)e^{i\omega_{\mathfrak{T}}(A)}$, $\quad r_{\mathfrak{T}}(A) = T^{\infty}_{Ⓢ}(\{r_{\mathfrak{T}_i}(A)\})$ and $\omega_{\mathfrak{T}}(A) = \min^{\infty}_{Ⓢ}(\{\omega_{\mathfrak{T}_i}(A)\})$. Since for each $i \in \mathbb{N}$

$$r_{\mathfrak{T}_i}(X) = r_{\mathfrak{T}_i}(\tilde{0}) = 1, \quad \omega_{\mathfrak{T}_i}(X) = \omega_{\mathfrak{T}_i}(\tilde{0}) = 2\pi,$$

we have

$$T\big(r_{\mathfrak{T}_1}(X), r_{\mathfrak{T}_2}(X)\big) = T(1,1) = 1$$

$$T\big(T(r_{\mathfrak{T}_1}(X), r_{\mathfrak{T}_2}(X)), r_{\mathfrak{T}_3}(X)\big) = T(1,1) = 1,$$

$$\min\big(\omega_{\mathfrak{T}_1}(X), \omega_{\mathfrak{T}_2}(X)\big) = \min(2\pi, 2\pi) = 2\pi$$

$$\min\big(\min(\omega_{\mathfrak{T}_1}(X), \omega_{\mathfrak{T}_2}(X)), \omega_{\mathfrak{T}_3}(X)\big) = \min(2\pi, 2\pi) = 2\pi,$$

By contradiction on k, we can show $T^{k}_{Ⓢ}(\{r_{\mathfrak{T}_i}(X)\}) = 1$ and $\min^{k}_{Ⓢ}(\{\omega_{\mathfrak{T}_i}(X)\}) = 0$ for each $k \in \mathbb{N}$. Therefore, $r_{\mathfrak{T}}(X) = 1$ and $\omega_{\mathfrak{T}}(X) = 2\pi$. Similarly we can show $r_{\mathfrak{T}}(\tilde{0}) = 1$ and $\omega_{\mathfrak{T}}(\tilde{0}) = 0$.
Also for each $A, B \in IL^{M}_{X}$, we have

$$\begin{aligned}
T^{3}_{Ⓢ}(\{r_{\mathfrak{T}_i}(A\cap B)\}) &= T\Big(T\big(r_{\mathfrak{T}_1}(A\cap B)), r_{\mathfrak{T}_2}(A\cap B))\big), r_{\mathfrak{T}_3}(A\cap B)\Big)\\
&\geq T\Big(T\Big(T\big(r_{\mathfrak{T}_1}(A), r_{\mathfrak{T}_1}(B)\big), T\big(r_{\mathfrak{T}_2}(A), r_{\mathfrak{T}_2}(B)\big)\Big), T\big(r_{\mathfrak{T}_3}(A), r_{\mathfrak{T}_3}(B)\big)\Big)\\
&= T\Big(T\Big(T\big(r_{\mathfrak{T}_1}(A), r_{\mathfrak{T}_2}(A)\big), T\big(r_{\mathfrak{T}_1}(B), r_{\mathfrak{T}_2}(B)\big)\Big), T\big(r_{\mathfrak{T}_3}(A), r_{\mathfrak{T}_3}(B)\big)\Big)\\
&= T\Big(T\Big(T\big(r_{\mathfrak{T}_1}(A), r_{\mathfrak{T}_2}(A)\big), r_{\mathfrak{T}_3}(A)\Big),\ T\Big(T\big(r_{\mathfrak{T}_1}(B), r_{\mathfrak{T}_2}(B)\big), r_{\mathfrak{T}_3}(B)\Big)\Big)\\
&= T\Big(T^{3}_{Ⓢ}(\{r_{\mathfrak{T}_i}(A)\}), T^{3}_{Ⓢ}(\{r_{\mathfrak{T}_i}(B)\})\Big).
\end{aligned}$$

By contradiction on k, we can show for each $k \in \mathbb{N}$ we have

$$T^{k}_{Ⓢ}(\{r_{\mathfrak{T}_i}(A\cap B)\}) \geq T\Big(T^{k}_{Ⓢ}(\{r_{\mathfrak{T}_i}(A)\}), T^{k}_{Ⓢ}(\{r_{\mathfrak{T}_i}(B)\})\Big).$$

Therefore,

$$\begin{aligned}\mathfrak{T}(A\cap B) &= T^{\infty}_{\circledS}(\{r_{\mathfrak{T}_i}(A\cap B)\})\\ &= \lim_{k\to\infty} T^{k}_{\circledS}(\{r_{\mathfrak{T}_i}(A\cap B)\})\\ &\geq \lim_{k\to\infty} T\Big(T^{k}_{\circledS}(\{r_{\mathfrak{T}_i}(A)\}), T^{k}_{\circledS}(\{r_{\mathfrak{T}_i}(B)\}),\Big) \quad \textit{by } (3.1)\\ &\geq T\Big(T^{\infty}_{\circledS}(\{r_{\mathfrak{T}_i}(A)\}), T^{\infty}_{\circledS}(\{r_{\mathfrak{T}_i}(B)\})\Big)\\ &= T\big(r_{\mathfrak{T}}(A), r_{\mathfrak{T}}(B)\big),\end{aligned}$$

Similarly we can prove that $\omega_{\mathfrak{T}}(A\cap B)\geq \min\big(\omega_{\mathfrak{T}}(A), \omega_{\mathfrak{T}}(B)\big)$.
For any arbitrary family $\{A_k,\ k\in K\}\subseteq IL^M_X$, we have

$$r_{\mathfrak{T}_j}(\bigcup_{k\in K} A_k)\geq \bigwedge_{k,l\in K} T\big(r_{\mathfrak{T}_j}(A_k), r_{\mathfrak{T}_j}(A_l)\big)$$

for each $j\in\mathbb{N}$. Hence

$$\begin{aligned}T^{3}_{\circledS}(\{r_{\mathfrak{T}_i}(\textstyle\bigcup_{k\in K} A_k)\}) &= T\Big(T\big(r_{\mathfrak{T}_1}(\textstyle\bigcup_{k\in K} A_k)), r_{\mathfrak{T}_2}(\textstyle\bigcup_{k\in K} A_k))\big), r_{\mathfrak{T}_3}(\textstyle\bigcup_{k\in K} A_k)\Big)\\ &\geq T\Big(T\Big(\textstyle\bigwedge_{k,l\in K} T\big(r_{\mathfrak{T}_1}(A_k), r_{\mathfrak{T}_1}(A_l)\big),\ \textstyle\bigwedge_{k,l\in K} T\big(r_{\mathfrak{T}_2}(A_k), r_{\mathfrak{T}_2}(A_l)\big)\Big),\\ &\quad \textstyle\bigwedge_{k,l\in K} T\big(r_{\mathfrak{T}_3}(A_k), r_{\mathfrak{T}_3}(A_l)\big)\Big)\\ &\geq \textstyle\bigwedge_{k,l\in K} T\Big(T\Big(T\big(r_{\mathfrak{T}_1}(A_k), r_{\mathfrak{T}_1}(A_l)\big),\ T\big(r_{\mathfrak{T}_2}(A_k), r_{\mathfrak{T}_2}(A_l)\big)\Big),\\ &\quad T\big(r_{\mathfrak{T}_3}(A_k), r_{\mathfrak{T}_3}(A_l)\big)\Big)\\ &= \textstyle\bigwedge_{k,l\in K} T\Big(T\Big(T\big(r_{\mathfrak{T}_1}(A_k), r_{\mathfrak{T}_2}(A_k)\big),\ T\big(r_{\mathfrak{T}_1}(A_l), r_{\mathfrak{T}_2}(A_l)\big)\Big),\\ &\quad T\big(r_{\mathfrak{T}_3}(A_k), r_{\mathfrak{T}_3}(A_l)\big)\Big)\\ &= \textstyle\bigwedge_{k,l\in K} T\Big(T\Big(T\big(r_{\mathfrak{T}_1}(A_k), r_{\mathfrak{T}_2}(A_k)\big), r_{\mathfrak{T}_3}(A_k)\Big),\end{aligned}$$

$$T\Big(T\big(r_{\mathfrak{T}_1}(A_l), r_{\mathfrak{T}_2}(A_l)\big), r_{\mathfrak{T}_3}(A_l)\Big)\Big)$$

$$= \bigwedge_{k,l\in K} T\Big(T^3_{\circledS}(\{r_{\mathfrak{T}_i}(A_k)\}), T^3_{\circledS}(\{r_{\mathfrak{T}_i}(A_l)\})\Big)$$

By contradiction on k, we can show for each $k \in \mathbb{N}$ we have

$$T^k_{\circledS}(\{r_{\mathfrak{T}_i}(\bigcup_j A_j)\}) \geq \bigwedge_{k,l\in K} T\Big(T^k_{\circledS}(\{r_{\mathfrak{T}_i}(A_k)\}), T^k_{\circledS}(\{r_{\mathfrak{T}_i}(A_l)\})\Big)$$

Therefore,

$$\begin{aligned}
\mathfrak{T}(\textstyle\bigcup_j A_j) &= T^\infty_{\circledS}(\{r_{\mathfrak{T}_i}(\textstyle\bigcup_j A_j)\}) \\
&= \lim_{k\to\infty} T^k_{\circledS}(\{r_{\mathfrak{T}_i}(\textstyle\bigcup_j A_j)\}) \\
&\geq \lim_{k\to\infty} \textstyle\bigwedge_{k,l\in K} T\Big(T^k_{\circledS}(\{r_{\mathfrak{T}_i}(A_k)\}), T^k_{\circledS}(\{r_{\mathfrak{T}_i}(A_l)\})\Big) \\
&= \textstyle\bigwedge_{k,l\in K} \lim_{k\to\infty} T\Big(T^k_{\circledS}(\{r_{\mathfrak{T}_i}(A_k)\}), T^k_{\circledS}(\{r_{\mathfrak{T}_i}(A_l)\})\Big) \\
&= \textstyle\bigwedge_{k,l\in K} T\Big(T^\infty_{\circledS}(\{r_{\mathfrak{T}_i}(A_k)\}), T^\infty_{\circledS}(\{r_{\mathfrak{T}_i}(A_l)\})\Big) \\
&= \textstyle\bigwedge_{k,l\in K} T\big(r_{\mathfrak{T}}(A_k), r_{\mathfrak{T}}(A_l)\big).
\end{aligned}$$

Similarly we can prove that

$$\omega_{\mathfrak{T}}(\bigcup_{j\in J} A_j) \geq \bigwedge_{i,j\in J} \min\big(\omega_{\mathfrak{T}}(A_i) \vee \omega_{\mathfrak{T}}(A_j)\big).$$

Hence $\mathfrak{T} \in \mathfrak{M}_{\mathfrak{T}}(X)$. Therefore, this lattis is semicomplete. □

3. Conclusion

In this paper, we define (anti) complex fuzzy subspaces of a k-vector space V, under T-norm (C-conorm). Then we discuss various operations between T-complex fuzzy sets and also the image and inverse image of a T-complex fuzzy subspace under a function. We introduce complex (anti complex) fuzzy topological space $(X, \mathfrak{T})$ with complex (anti complex) gradation of openness under T-norm (C-conorm) which X is itself a T-complex (C-anti complex) fuzzy subset of a nonempty set M. Finally we define spiral T-norm of a sequence in $[0, 1]$ and spiral minimum of a sequence in $[0, 2\pi]$ and then using them, we prove that the set of all T-complex gradations of openness $(\mathfrak{M}_{\mathfrak{T}}(X), \leq)$ on X, is a semicomplete lattice

In the continuation of our research, the question arises, how can this model be extended to complex fuzzy topological manifolds or fuzzy vector bundles?

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