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On the Topological Indices on Double Graphs

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Abstract. In this paper first, we study the elementary properties of double graphs and then we present explicit formulas for a few topological indices on double graphs, i.e., of graphs that are the direct product of a simple graph G with the graph obtained by the complete graph K_2 adding a loop to each vertex. Finally, we compute the eccentric connectivity index of the double of lexicographic product and complete sum of two graphs.

Keywords: Double graph, Topological index, Direct product of graphs, Lexicographic product of graphs, Complete sum, Fibonacci cubes.

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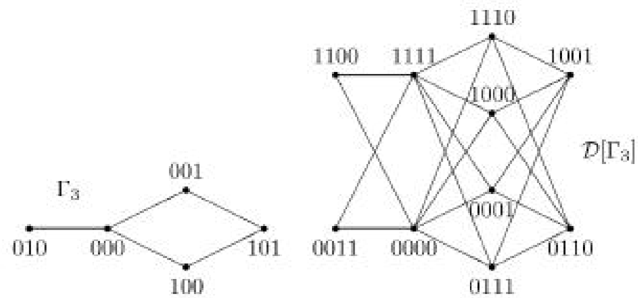

FIGURE 1. A Fibonacci cube and its double

1. Introduction

In [30] it was observed that the binary strings of length $n+1$ without zigzags, i.e. without 010 and 101 as factors, can be reduced to the Fibonacci strings, i.e. binary strings without two consecutive 1's, of length n. The set of Fibonacci strings can be endowed with a graph structure saying that two strings are adjacent when they differ exactly in one position. The graphs obtained in this way are called Fibonacci cubes [21] and have been studied in several recent papers. One interesting such graph structure is the one induced by the graph structure of Fibonacci strings, that is the one obtained defining the adjacency saying that two binary strings without zigzags are adjacent if and only if the corresponding Fibonacci strings are adjacent as vertices of the Fibonacci cube. The resulting graph can be built up by taking two distinct copies of the Fibonacci cube Γ_n and joining every vertex v in one component to every vertex w' in the other component corresponding to a vertex w adjacent to v in the first component. See Figure 1 for an illustration. This is a general construction that can be performed on every simple graph. E. Munarini and et al. in [31] called *double graphs* all the graphs that can be obtained in such a way and studied their extensive properties in detail.

In this paper, we present explicit formulas for some topological indices on double graphs. In the next section, we give necessary definitions and propositions for the reader's convenience, but we refer the reader to [31] for a more thorough exposition. In section 3 first, we recall a few topological indices and then in five subsections, we prove new formulas for these indices.

2. Definitions and preliminaries

In this paper, we will consider only finite simple graphs (i.e. without loops and multiple edges). As usual, $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and adj denotes the adjacency relation of G. For a graph G, the degree of a vertex u is the number of edges incident to v denoted by $d_G(u)$. If $|E(G)| = m$, then [19]:

$$\sum_{u \in V(G)} d_G(u) = 2m. \tag{2.1}$$

For all definitions not given here see [6, 19, 22].

The *direct product* of two graphs G and H is the graph $G \times H$ with $V(G \times H) = V(G) \times V(H)$ and with adjacency defined by (v_1, w_1) adj (v_2, w_2) if and only if v_1 adj v_2 in G and w_1 adj w_2 in H.

The *total graph* T_n on n vertices is the graph associated with the total relation (where every vertex is adjacent to every vertex). It can be obtained from the complete graph K_n by adding a loop to every vertex.

The *double* of a simple graph G is defined as the graph $\mathcal{D}[G] = G \times T_2$. Since the direct product of a simple graph with any graph is always a simple graph, it follows that the double of a simple graph is still a simple graph. See Figure 2 for some examples.

In $\mathcal{D}[G]$ we have (v, h) adj (w, k) if and only if v adj w in G. Then, if $V(T_2) = \{0, 1\}$, we have that $G_0 = \{(v, 0) \mid v \in V(G)\}$ and $G_1 = \{(v, 1) \mid v \in V(G)\}$ are two subgraphs of $\mathcal{D}[G]$ both isomorphic to G such that $G_0 \cap G_1 = \emptyset$ and $G_0 \cup G_1$ is a spanning subgraph of $\mathcal{D}[G]$. Moreover, we have an edge between $(v, 0)$ and $(w, 1)$, and similarly we have an edge between $(v, 1)$ and $(w, 0)$ whenever v adj w in G. We will call $\{G_0, G_1\}$ the canonical decomposition of $\mathcal{D}[G]$.

From the above observations, it follows that if G has n vertices and m edges then $\mathcal{D}[G]$ has $2n$ vertices and $4m$ edges. In particular,

$$d_{\mathcal{D}[G]}(v, k) = 2d_G(v). \tag{2.2}$$

The *lexicographic product* (or *composition*) of two graphs G and H is the graph $G \circ H$ with $V(G) \times V(H)$ as vertex set and with adjacency defined by (v_1, w_1) adj (v_2, w_2) if and only if $v_1 = v_2$ and w_1 adj w_2 in H or v_1 adj v_2 in G. In other words, the graph $G \circ H$ can be obtained from G substituting to each vertex v of G a copy H_v of H and joining every vertex of H_v with every vertex of H_w whenever v and w are adjacent in G [22].

Example 2.1. A composition of C_n and K_2 is called a closed fence. We have:

$$\mathcal{D}[C_n \circ K_2] = C_n \circ \mathcal{D}[K_2] = C_n \circ K_{2,2}$$

We state the following three results which will be used in the sequel.

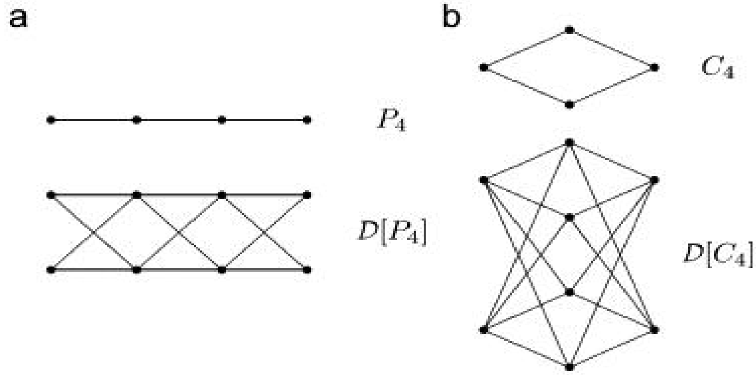

FIGURE 2. (a) a path P_4 and its double (b) a cycle C_4 and its double

Lemma 2.2. [31, Lemma 1] *For any graph G we have $G \times T_n \cong G \circ N_n$, where N_n is the graph on n vertices without edges.*

Proposition 2.3. [31, Proposition 2] *For any graph G on n vertices, $\mathcal{D}[G] \cong G \circ N_2$ and $\mathcal{D}[G]$ is n-partite.*

Proposition 2.4. [31, Proposition 5] *For any graph $G \neq K_1$, G is connected if and only if $\mathcal{D}[G]$ is connected.*

Proposition 2.5. *For any graph G_1 and G_2 the following properties hold:*
1. $\mathcal{D}[G_1 \times G_2] = G_1 \times \mathcal{D}[G_2] = \mathcal{D}[G_1] \times G_2$,
2. $\mathcal{D}[G_1 \circ G_2] = G_1 \circ \mathcal{D}[G_2]$.

Let G_1 and G_2 be two graphs. The sum $G_1 + G_2$ of G_1 and G_2 is the disjoint union of the two graphs. *The complete sum* (or *join*) $G_1 \boxplus G_2$ of G_1 and G_2 is the graph obtained from $G_1 + G_2$ by joining every vertex of G_1 to every vertex of G_2. We will use the following proposition in the last section.

Proposition 2.6. [31, Proposition 10] *For any graph G_1 and G_2 the following properties hold:*
1. $\mathcal{D}[G_1 + G_2] = \mathcal{D}[G_1] + \mathcal{D}[G_2]$,
2. $\mathcal{D}[G_1 \boxplus G_2] = \mathcal{D}[G_1] \boxplus \mathcal{D}[G_2]$.

3. Main Results

In this section, we will compute some topological indices for double graphs. A topological index is a numerical quantity related to a graph that is invariant under graph automorphisms. As usual, the distance between the vertices u and v of G is denoted by $d_G(u, v)$ and it is defined as the length of a shortest path connecting them. Thus distance (shortest-path) between two vertices (u,x) and (v,y) of $\mathcal{D}[G]$, where $u,v \in V(G)$ and $x,y \in V(T_2)=\{0,1\}$, is given by

$$d_{\mathcal{D}[G]}((u,x),(v,y)) = \begin{cases} 2, & \text{if} \quad u=v \\ d_G(u,v), & \text{if} \quad u \neq v \end{cases} \tag{3.1}$$

Now, we recall some topological indices and then obtain the explicit formulas for these indices on double graphs.

3.1. **Topological index with respect to the distance between vertices.** The *Wiener index* $W(G)$ is the first distance-based topological index defined as the sum of all distances between vertices of G, [36] i.e.

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v). \tag{3.2}$$

The name Wiener index or Wiener number for the quantity defined in Equation (3.2) is usual in chemical literature, since Harold Wiener [36], in 1947, seemed to be the first to consider it. In chemical language, the Wiener index is equal to the sum of all shortest carbon-carbon bond paths in a molecule. Wiener himself used the name path number, but denoted his quantity by w. Wiener's original definition was slightly different - yet equivalent - to (3.2). The definition of the Wiener index in terms of distances between vertices of a graph, such as in Equation (3.2), was first given by Hosoya [20]. For more information on the Wiener index and its applications, we encourage the reader to consult papers by Dobrynin and co-authors [9, 10] and references therein.

Next, we start with the following result.

Proposition 3.1. *For any connected graph G on n vertices we have:*

$$W(\mathcal{D}[G]) = 4W(G) + 2n$$

Proof. The claim is obtained by formulas (3.1) and a direct computation as

$$\begin{aligned} W(\mathcal{D}[G]) &= \sum_{(u,x)\neq(v,y)} d_{\mathcal{D}[G]}((u,x),(v,y)) \\ &= \sum_{\substack{u\neq v \\ x=y}} d_{\mathcal{D}[G]}((u,x),(v,y)) + \sum_{\substack{u\neq v \\ x\neq y}} d_{\mathcal{D}[G]}((u,x),(v,y)) \\ &+ \sum_{u=v} d_{\mathcal{D}[G]}((u,x),(v,y)) \\ &= 2\sum_{u\neq v} d_G(u,v) + 2\sum_{u\neq v} d_G(u,v) + 2n \\ &= 4W(G) + 2n. \end{aligned}$$

□

The *Harary index* of a graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [32] and by Ivanciuc et al. [23] in 1993. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. This index $H(G)$, a parallel to the *Wiener index*, is reasonably well-correlated with many physical and chemical properties of organic compounds, and chemists are hence interested in computing it for a variety of classes of graphs. The *Harary index* is defined by:

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)} \tag{3.3}$$

where the summation goes over all unordered pairs of vertices of G and $d_G(u,v)$ denotes the distance between the two vertices u and v in the graph G [37].

Proposition 3.2. *If $H(G)$ is the Harary index of a connected graph G with n vertices, then*

$$H(\mathcal{D}[G]) = 4H(G) + \frac{n}{2}$$

Proof. According to the definition of Harary index of the graph G and from formula (3.1), we have:

$$\begin{aligned} H(\mathcal{D}[G]) &= \sum_{(u,x)\neq(v,y)} \frac{1}{d_{\mathcal{D}[G]}((u,x),(v,y))} \\ &= \sum_{u\neq v} \frac{1}{d_{\mathcal{D}[G]}((u,x),(v,y))} + \sum_{u=v} \frac{1}{d_{\mathcal{D}[G]}((u,x),(v,y))} \\ &= 4\sum_{u\neq v} \frac{1}{d_G(u,v)} + \sum_{u=v\in V(G)} \frac{1}{2} \\ &= 4H(G) + \tfrac{n}{2} \end{aligned}$$

□

3.2. **Topological index with respect to degrees of vertices.** The first and second Zagreb indices of graph G were originally defined as

$$M_1(G) = \sum_{u\in V(G)} d_G^2(u) \tag{3.4}$$

and

$$M_2(G) = \sum_{uv=e\in E(G)} d_G(u)d_G(v), \tag{3.5}$$

respectively. The first Zagreb index of G can be also expressed as a sum over edges of G, i.e., $M_1(G) = \sum_{uv=e\in E(G)} (d_G(u) + d_G(v))$. We encourage the reader to consult [18, 26] and references therein for historical background, computational techniques, and mathematical properties of Zagreb indices.

Proposition 3.3. *Let $\mathcal{D}[G]$ be the double graph of a graph G. Then*

$$M_1(\mathcal{D}[G]) = 8M_1(G)$$

Proof. First of all, we follow by $M_1(G) = \sum_{u\in V(G)} d_G^2(u)$:

$$\begin{aligned} M_1(\mathcal{D}[G]) &= \sum_{(u,x)\in V(\mathcal{D}[G])} (d_{\mathcal{D}[G]}(u,x))^2 \\ &= \sum_{(u,x)\in V(\mathcal{D}[G])} (2d_G(u))^2 \\ &= 4 \sum_{(u,x)\in V(\mathcal{D}[G])} d_G^2(u) \\ &= 8 \sum_{u\in V(G)} d_G^2(u) = 8M_1(G). \end{aligned}$$

□

For the second Zagreb index of the double graph G, we have:

Proposition 3.4. *Let $\mathcal{D}[G]$ be the double graph of a graph G. Then*

$$M_2(\mathcal{D}[G]) = 16M_2(G)$$

Proof. Similar to the first Zagreb index we have

$$\begin{aligned} M_2(\mathcal{D}[G]) &= \sum_{(u,x)(v,y)=e'\in E(\mathcal{D}[G])} d_{\mathcal{D}[G]}(u,x)d_{\mathcal{D}[G]}(v,y) \\ &= \sum_{e'\in E(\mathcal{D}[G])} (2d_G(u).2d_G(v)) \\ &= 4 \sum_{e'\in E(\mathcal{D}[G])} d_G(u).d_G(v) \\ &= 4.4 \sum_{e=uv\in E(G)} d_G(u).d_G(v) \\ &= 16M_2(G). \end{aligned}$$

□

The sum of weights over all edges of G, which is called the *Randić index* or *molecular connectivity index* or simply *connectivity index* of G and is denoted by $R(G)$, has been closely correlated with many chemical properties [8] and found to parallel the boiling point, Kovats constants, and a calculated surface [33]. In addition, the Randić index appears to predict the boiling points of alkanes more closely, and only it takes into account the bonding or adjacency degree among carbons in alkanes.

Note that according to different applications, different weights may be assigned to the edges. Hence, we indicate that the weight of each edge $e = uv$ is $\frac{1}{\sqrt{d_G(u)d_G(v)}}$. Accordingly, this index is defined as:

$$R(G) = \sum_{e=uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}} \tag{3.6}$$

Proposition 3.5. *For a graph G, we have:*

$$R(\mathcal{D}[G]) = 2R(G)$$

Proof.

$$\begin{aligned} R(\mathcal{D}[G]) &= \sum_{e'=(u,x)(v,y)\in E(\mathcal{D}[G])} \frac{1}{\sqrt{d_{\mathcal{D}[G]}(u,x)d_{\mathcal{D}[G]}(v,y)}} \\ &= \sum_{e'\in E(\mathcal{D}[G])} \frac{1}{\sqrt{2d_G(u).2d_G(v)}} \\ &= \tfrac{1}{2} \sum_{e'\in E(\mathcal{D}[G])} \frac{1}{\sqrt{d_G(u)d_G(v)}} \\ &= \tfrac{1}{2}.4 \sum_{e=uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}} \\ &= 2R(G). \end{aligned}$$

□

The atom-bond connectivity index is a valuable predictive index in the study of the heat of formation in alkanes [12, 13]. The mathematical properties of this index and its new version were reported in [7, 8, 16, 15]. It is defined as:

$$ABC(G) = \sum_{e=uv\in E(G)} \sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}} \tag{3.7}$$

where, $d_G(u)$ is the degree of u, etc.
If no ambiguity is possible, the subscripts G and $\mathcal{D}[G]$ may be omitted.

Proposition 3.6. *Let $\mathcal{D}[G]$ be the double of graph G. Then:*

$$ABC(\mathcal{D}[G]) = 2\sqrt{2} \sum_{e=uv\in E(G)} \sqrt{\frac{d(u)+d(v)-1}{d(u)d(v)}}$$

Proof. By formula (2.2) we have

$$\begin{aligned}
ABC(\mathcal{D}[G]) &= \sum_{e'=(u,x)(v,y)\in E(\mathcal{D}[G])} \sqrt{\frac{d(u,x)+d(v,y)-2}{d(u,x)d(v,y)}} \\
&= \sum_{e'\in E(\mathcal{D}[G])} \sqrt{\frac{2d(u)+2d(v)-2}{2d(u).2d(v)}} \\
&= \tfrac{\sqrt{2}}{2} \sum_{e'\in E(\mathcal{D}[G])} \sqrt{\frac{d(u)+d(v)-1}{d(u)d(v)}} \\
&= 4.\tfrac{\sqrt{2}}{2} \sum_{uv\in E(G)} \sqrt{\frac{d(u)+d(v)-1}{d(u)d(v)}} \\
&= 2\sqrt{2} \sum_{e=uv\in E(G)} \sqrt{\frac{d(u)+d(v)-1}{d(u)d(v)}}.
\end{aligned}$$

□

Let G be a graph and $e = uv$ be an edge of G. The *geometric-arithmetic (GA) index* is defined as [35]:

$$GA = GA(G) = \sum_{e=uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \tag{3.8}$$

Proposition 3.7. *Let $\mathcal{D}[G]$ be the double of graph G. For the GA index of $\mathcal{D}[G]$, we have:*

$$GA(\mathcal{D}[G]) = 4GA(G)$$

Proof. Similar to the ABC index one can write

$$\begin{aligned}
GA(\mathcal{D}[G]) &= \sum_{(u,x)(v,y)e'\in E(\mathcal{D}[G])} \frac{2\sqrt{d(u,x)d(v,y)}}{d(u,x)+d(v,y)} \\
&= \sum_{e'\in E(\mathcal{D}[G])} \frac{2\sqrt{2d(u).2d(v)}}{2d(u)+2d(v)} \\
&= \sum_{e'\in E(\mathcal{D}[G])} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \\
&= 4 \sum_{e=uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \\
&= 4GA(G).
\end{aligned}$$

□

3.3. **Topological index with respect to degrees and distances.** Schultz in [34] introduced a graph-theoretical descriptor for characterizing alkanes by an integer, namely the *Schultz index*, defined as [5]

$$S(G) = \sum_{\{u,v\} \subseteq V(G)} (d_G(u) + d_G(v)) d_G(u, v).$$

Also, Klavžar and Gutman defined the *modified Schultz index* of graph G as follows [27]

$$S^*(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u) d_G(v) d_G(u, v).$$

Regarding to Shultz index we have:

Proposition 3.8. *If G is a connected graph with $|E(G)| = m$, then*

$$S(\mathcal{D}[G]) = 8S(G) + 16m$$

Proof. According to the definition of the Schultz index of G and from formulas (3.1) and (2.1), we have:

$$\begin{aligned} S(\mathcal{D}[G]) &= \sum_{(u,x),(v,y) \in V(\mathcal{D}[G])} [d_{\mathcal{D}[G]}(u,x) + d_{\mathcal{D}[G]}(v,y)] d_{\mathcal{D}[G]}((u,x),(v,y)) \\ &= \sum_{u \neq v} (2d_G(u) + 2d_G(v)) d_G(u,v) + \sum_{u=v} [4d_G(u)] \cdot 2 \\ &= 2\sum_{u \neq v} [d_G(u) + d_G(v)] d_G(u,v) + 8 \sum_{u \in V(G)} d_G(u) \\ &= 4.2 \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v) + 8.2m \\ &= 8S(G) + 16m. \end{aligned}$$

□

Proposition 3.9. *Let G be a connected graph and let $S^*(G)$ and $M_1(G)$ be the modified Schultz index and the first Zagreb index of G, respectively. Then:*

$$S^*(\mathcal{D}[G]) = 16S^*(G) + 8M_1(G)$$

Proof.

$$\begin{aligned} S^*(\mathcal{D}[G]) &= \sum_{(u,x),(v,y)\in V(\mathcal{D}[G])} d_{\mathcal{D}[G]}(u,x).d_{\mathcal{D}[G]}(v,y).d_{\mathcal{D}[G]}((u,x),(v,y)) \\ &= \sum_{u\neq v} 2d_G(u).2d_G(v).d_G(u,v) + \sum_{u=v} 2d_G(u).2d_G(u).2 \\ &= 4\sum_{u\neq v} d_G(u).d_G(v).d_G(u,v) + 8\sum_{u\in V(G)} d_G^2(u) \\ &= 4.4\sum_{u,v\in V(G)} d_G(u).d_G(v).d_G(u,v) + 8M_1(G) \\ &= 16S^*(G) + 8M_1(G). \end{aligned}$$

□

3.4. **Szeged, PI, and GA_2 topological indices on double graphs.** Suppose $e = uv$ is an edge of a connected graph G. Let $N_n(e)$ be the vertices of G that are closer to u than to v and let $N_v(e)$ be those vertices which are closer to v than to u. More formally,

$$N_u(e) = \{w \in V(G) : d_G(w,u) < d_G(w,v)\} \tag{3.9}$$

and

$$N_v(e) = \{w \in V(G) : d_G(w,v) < d_G(w,u)\}. \tag{3.10}$$

Let $n_u(e) = |N_u(e)|$ and $n_v(e) = |N_v(e)|$. Then the *Szeged index* of a graph G, denoted by $Sz(G)$, is defined as

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e)n_v(e). \tag{3.11}$$

Notice that vertices equidistant from both ends of the edge $e = uv$ are not counted. This topological index is a mathematically elegant topological index defined by Ivan Gutman [17].

Suppose now that $e' = (u,h)(v,k)$ is an arbitrary edge in $E(\mathcal{D}[G])$, and $e = uv$ is the corresponding edge in G. Then it is straightforward to see that

$$n_{(u,h)}(e') = 2n_u(e). \tag{3.12}$$

With this introduction, we have the following result:

Proposition 3.10. *If $\mathcal{D}[G]$ is the double graph of G, then*

$$Sz(\mathcal{D}[G]) = 16Sz(G).$$

Proof. According to the definition of the Szeged index and from formula (3.12), we have:

$$\begin{aligned} Sz(\mathcal{D}[G]) &= \sum_{(u,h)(v,k)=e'\in E(\mathcal{D}[G])} n_{(u,h)}(e')n_{(v,k)}(e') \\ &= \sum_{e'\in E(\mathcal{D}[G])} 2n_u(e).2n_v(e) \\ &= 4\sum_{e'\in E(\mathcal{D}[G])} n_u(e)n_v(e) \\ &= 4.4\sum_{e\in E(G)} n_u(e)n_v(e) \\ &= 16Sz(G). \end{aligned}$$

Here, note that in the second equality above, the subscript $e = uv$ is the corresponding edge in G with respect to the edge $e' = (u, h)(v, k)$ in $\mathcal{D}[G]$. □

The Padmakar-Ivan *(PI) index* of a graph G is defined as

$$PI(G) = \sum_{uv=e\in E(G)} (n_u(e) + n_v(e)). \tag{3.13}$$

In this definition, similar to the Szeged index $Sz(G)$ of G, edges equidistant from the two ends of the edge $e = uv$ are not counted. The PI index is very simple to calculate and has disseminating power similar to that of the Wiener and Szeged indices. Khadikar and Karmarkar [24, 25] investigated the chemical applications of the PI index. They showed that the proposed PI index correlates highly with W and Sz as well as with the physicochemical properties and biological activities of a large number of diverse and complex compounds. We encourage the readers to consult papers [1, 2, 3, 4] for further studies on the mathematical properties of the PI index and its applications in chemistry and nanoscience.

Proposition 3.11. *If G is a connected graph and $\mathcal{D}[G]$ is the double graph of G, then*

$$PI(\mathcal{D}[G]) = 8PI(G).$$

Proof. Applying the formula (3.12) and using a similar method as in proposition (3.10), we have:

$$\begin{aligned} PI(\mathcal{D}[G]) &= \sum_{(u,h)(v,k)=e'\in E(\mathcal{D}[G])} (n_{(u,h)}(e') + n_{(v,k)}(e')) \\ &= \sum_{e'\in E(\mathcal{D}[G])} (2n_u(e) + 2n_v(e)) \\ &= 2.4 \sum_{e\in E(G)} (n_u(e) + n_v(e)) \\ &= 8PI(G). \end{aligned}$$

□

Following [14] the second *geometric–arithmetic index* of a connected graph G is defined as follows:

$$GA_2 = GA_2(G) = \sum_{e=uv\in E(G)} \frac{2\sqrt{n_u(e).n_v(e)}}{n_u(e) + n_v(e)} \tag{3.14}$$

where, as we mentioned in the introduction of the Szeged index, $n_u(e)$ is the number of vertices of G lying closer to u than v, and $n_v(e)$ is defined analogously. Using a similar proof as we did for the Szeged index in Proposition (3.10) and applying formula (3.12) we get the following formula for the GA_2 index of the double graph:

$$GA_2(\mathcal{D}[G]) = 4GA_2(G).$$

3.5. **Eccentric connectivity index on double graph.** Let G be a connected graph. The eccentricity, $\epsilon(u)$ of a vertex $u \in V(G)$ is the maximum distance between u and any other vertex in G.
The *eccentric connectivity index* of G is defined as [29]:

$$\xi^c(G) = \sum_{u\in V(G)} \epsilon(u)d(u)$$

A vertex $v \in V(G)$ is well–connected in G if v is adjacent to every other vertex of G. Obviously, the eccentricity of a well–connected vertex is equal to 1. The number of well–connected vertices of G is denoted by $w(G)$ [11].

It is clear from the definition that if $u \in V(G)$ and $a \in V(T_2) = \{0, 1\}$ then, the eccentricity of the vertex (u, a) of the double graph of G is given by:

$$\epsilon(u,a) = \begin{cases} 2, & \text{if} \quad \epsilon(u) = 1 \\ \epsilon(u), & \text{if} \quad \epsilon(u) > 1. \end{cases} \tag{3.15}$$

Proposition 3.12. *Let G be a connected graph with $|V(G)| = n$ and $w(G)$ the number of well-connected vertices of G. Then*

$$\xi^c(\mathcal{D}[G]) = 4\xi^c(G) + 4w(G)(n-1)$$

Proof. The claim follows by Eq. (3.15)

$$\begin{aligned}
\xi^c(\mathcal{D}[G]) &= \sum_{(u,x)\in V(\mathcal{D}[G])} \epsilon(u,x)d(u,x) \\
&= \sum_{\substack{(u,x)\in V(\mathcal{D}[G]),\\ \epsilon(u)\neq 1}} \epsilon(u,x)d(u,x) + \sum_{\substack{(u,x)\in V(\mathcal{D}[G]),\\ \epsilon(u)=1}} \epsilon(u,x)d(u,x) \\
&= \sum_{\substack{(u,x)\in V(\mathcal{D}[G]),\\ \epsilon(u)\neq 1}} \epsilon(u).2d(u) + \sum_{\substack{(u,x)\in V(\mathcal{D}[G]),\\ \epsilon(u)=1}} 2.2d(u) \\
&= 4\sum_{\substack{u\in V(G)\\ \epsilon(u)\neq 1}} \epsilon(u)d(u) + 8\sum_{\substack{u\in V(G)\\ \epsilon(u)=1}} d(u) \\
&= 4\xi^c(G) + 4w(G)(n-1).
\end{aligned}$$

□

The total eccentricity of a given graph G is the sum of eccentricities of all vertices of G and is denoted by $\zeta(G)$. For a k-regular graph G we have $\xi^c(G) = k\zeta(G)$ [11].

For two connected graphs G_1 and G_2, by the definition, it is obvious that $G_1 \times G_2$ and $G_1 + G_2$ are unconnected graphs. Thus we can just compute the eccentric connectivity indices of $\mathcal{D}[G_1 \circ G_2]$ and $\mathcal{D}[G_1 \boxplus G_2]$.

Proposition 3.13. *Let G_1 and G_2 be two graphs. Then*

$$\epsilon_{\mathcal{D}[G_1\circ G_2]}(u_1,u_2) = \begin{cases} 2, & \epsilon_{G_1}(u_1) = 1, \\ \epsilon_{G_1}(u_1), & \epsilon_{G_1}(u_1) \geq 2. \end{cases}$$

Next, to state our last two results we need to recall the following (see Theorem 3.10 and Corollary 3.14 of [11]).

Proposition 3.14.

$$\xi^c(G_1[G_2]) = w(G_1)w(G_2)(1-n_1n_2)+w(G_1)(n_2^2(n_1-1)+2m_2)+n_2^2\xi^c(G_1)+2m_2\zeta(G_1).$$

Proposition 3.15. *If there are no well-connected vertices in G_1 and G_2, then*

$$\xi^c(G_1 + G_2) = 4(m_1 + m_2 + n_1n_2) = 4|E(G_1 + G_2)|.$$

Now we have

Proposition 3.16. *For two connected graphs G_1 and G_2, let $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, for $i = 1, 2$, $w(G_1)$ and $\zeta(G_1)$ are the number of well-connected vertices and total eccentricity of the graph G_1, respectively. Then*

$$\xi^c(\mathcal{D}[G_1 \circ G_2]) = w(G_1)(4n_2^2(n_1 - 1) + 8m_2) + 4n_2\xi^c(G_1) + 8m_2\zeta(G_1).$$

Proof. Suppose $n_i^{'} = |V(\mathcal{D}[G_i])|$ and $m_i^{'} = |E(\mathcal{D}[G_i])|$, we have $n_i' = 2n_i$, $m_i' = 4m_i$ for $i = 1, 2$. Since in the double graph, we have no vertex of degree $n_i' - 1$, hence $w(\mathcal{D}[G_2]) = 0$. Therefore by Proposition (2.5) and Proposition (3.14) we have

$$\begin{aligned}
\xi^c(\mathcal{D}[G_1 \circ G_2]) &= \xi^c(G_1 \circ \mathcal{D}[G_2]) \\
&= w(G_1)w(\mathcal{D}[G_2])(1 - n_1 n_2^{'}) + w(G_1)({n_2^{'}}^2(n_1 - 1) + 2m_2^{'}) \\
&+ {n_2^{'}}^2\xi^c(G_1) + 2m_2^{'}\zeta(G_1) \\
&= w(G_1)(4n_2^2(n_1 - 1) + 8m_2) + 4n_2^2\xi^c(G_1) + 8m_2\zeta(G_1)
\end{aligned}$$

□

Finally, we close the article with the following result.

Proposition 3.17. *Let G_1 and G_2 be two graphs. Then*

$$\xi^c(\mathcal{D}[G_1 \boxplus G_2]) = 16|E(G_1 \boxplus G_2)|.$$

Proof. We apply the notation in proof of the previous Proposition. According to the Proposition (2.6) we have

$$\xi^c(\mathcal{D}[G_1 \boxplus G_2]) = \xi^c(\mathcal{D}[G_1] \boxplus \mathcal{D}[G_2]).$$

As double graphs have no well-connected vertices, by Proposition (3.15), one gets

$$\begin{aligned}
\xi^c(\mathcal{D}[G_1] \boxplus \mathcal{D}[G_2]) &= 4(m_1' + m_2' + n_1' \cdot n_2') \\
&= 4(4m_1 + 4m_2 + 2n_1.2n_2) \\
&= 16|E(G_1 \boxplus G_2)|.
\end{aligned}$$

The proof now is complete. □

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