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Numerical approximation based on Bernouli polynomials for solving second-order hyperbolic telegraph equations

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Abstract. In this paper, a practical matrix method is presented for solving a particular type of telegraph equations. This procedure is based on Bernouli Polynomials. This matrix method with collocation suited nodes, decreases the supposed equations into system of algebric equations with unknown Bernouli coefficients. The obtained system is solved and approximate solutions are achieved. The well-conditioning of problems is also considered. The indicated method creates the well-conditioned problems. Some numerical problems are comprised to confirm the efficacy and fitting of the suggested method. The presented technique is easy to implement and produces accurate results. The precision of the method is demonstrated by measuring the errors between exact solutions and approximate solutions for each problem.

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1. Introduction

The telegraph equations are partial differential equations that outline the voltage and current on an electrical transmission line with distance x and time t. This paper deals with a particular type of telegraph equations which is called hyperbolic telegraph equations. Two classes of this type are examined, the first class has the linear form [29]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial t} + \beta g = F(x,t), \quad (x,t) \in [0,1] \times [0,1] \tag{1.1}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= f_1(x), \qquad \frac{\partial u}{\partial t}(x,0) = f_2(x) \\ u(0,t) &= \phi_1(t), \qquad u(1,t) = \phi_2(t), \quad t \geq 0 \end{aligned} \tag{1.2}$$

while the second class has the nonlinear form [15]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial t} + \beta \Phi(g) = F(x,t), \quad (x,t) \in [0,1] \times [0,1] \tag{1.3}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= f_1(x), \qquad \frac{\partial u}{\partial t}(x,0) = f_2(x) \\ u(0,t) &= \phi_1(t), \qquad u(1,t) = \phi_2(t), \quad t \geq 0 \end{aligned} \tag{1.4}$$

where in both two classes, u, f_1, f_2, ϕ_1 and ϕ_2 are known functions. For the exhaustive research of (1.1), we suggest [24], [35], [28] and references therein.

In recent years, different numerical procedures have been expanded and applied in the studies of solving one dimensional and two dimensional hyperbolic telegraph equations. Authors in [1] and [10] have been examined the proposed equations with different boundary conditions. Lately, different numerical methods based on the finite difference procedures have been employed to obtain numerical solutions of two dimensional hyperbolic telegraph equation [13, 26]). Dehghan et al. have examined this type of equations by using a meshless technique [8], a collocation method by applying radial basis functions [11] and the high order implicit collocation method [9]. B*ü*lb*ü*l and Sezer [5] have offered a matrix method based on Taylor polynomials to estimate the numerical solution of the two-space dimensional linear hyperbolic equation. Jiwari et al. [14] have extended a numerical method based on polynomial differential quadrature method to obtain approximate solution of the proposed equation with Dirichlet and Neumann boundary conditions.

Polynomial series and orthogonal functions have gotten significant notice in diverse problems. Polynomials are extremely applicable mathematical implements as they are clearly clarified, can be computed rapidly on computer systems and appeared for a enormous diversity of functions. Also, they can be differentiated and integrated simply and can be composed together to organize spline curves that can approximate any function to any exactness desired. The principal property of using polynomials for solving differential equations numerically, is transforming the difficult problems to a system of algebraic equations, therefore considerably clarifies the problems.

Bernoulli polynomials are involved in diverse extensions and approximation formulas which are applicable in analytic and numerical analysis. The Bernoulli polynomials and numbers have been spreaded by Norlund [27] and Vandiver [34] to the Bernoulli polynomials and numbers of higher order. Similar polynomials and numbers have been expounded in different periods, like the Euler polynomials and numbers and the reputed Bernoulli polynomials of second kind. These polynomials can be explained by diverse methods relying on the implementations [18], [19], [7], [23], [21], [3].

The current paper is organized as follows: in Section 2, we express the main characteristics of the Bernoulli polynomials requisites for our presented expansion and operational matrix. Section 3 is assigned to discretization of telegraph equation. In Section 4, the method is explained. In Section 5, the numerical solutions are provided and accurate solutions are comparing to evaluate the precision of the determined method.

2. Bernouli Polynomials

The Bernouli polynomials of order m, are explained in [16] by

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} \beta_i x^{n-i}, \tag{2.1}$$

where $\beta_i, i = 0, 1, ..., n$ are Bernouli numbers. These numbers are a sequence of signed rational numbers, which are obtained from the series extension of trigonometric functions [4] and can be described by

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \beta_j \frac{t^j}{j!}.$$

The first four Bernouli numbers are

$$\beta_0 = 1,\ \beta_1 = \frac{-1}{2},\ \beta_2 = \frac{1}{6},\ \beta_4 = -\frac{1}{30},$$

with $\beta_{2i+1} = 0, \quad i = 1, 2, 3, \cdots$. The first four Bernouli Polynomials are

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6},\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

The subsequent features apply to Bernouli polynomials [23], [32]

$$B_n(0) = \beta_n, \quad n \geq 0, \tag{2.2}$$

$$\int_a^x B_n(t)dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1}, \tag{2.3}$$

$$\int_0^1 B_m(t)B_n(t)dt = (-1)^{m-1}\frac{n!m!}{(n+m)!}, \quad n, m \geq 1, \tag{2.4}$$

and

$$\sum_{i=0}^{m} B_i(t) = (m+1)t^m. \tag{2.5}$$

It can be simply illustrated that any supposed polynomials of degree n can be extended with regard to linear combination of Bernouli polynomials as

$$p(x) = \sum_{k=0}^{m} c_k B_k(x) = C^T B(x),$$

where C and $B(x)$ are defined by

$$C = [c_0, c_1, ..., c_n]^T \tag{2.6}$$

and

$$B(x) = [B_0(x), B_1(x), ..., B_n(x)]^T, \tag{2.7}$$

where

$$B_k(x) = \binom{k}{k}B_k + \binom{k}{k-1}B_{k-1}x + ... + \binom{k}{1}B_1x^{k-1} + \binom{k}{0}B_0x^k,$$

for $k = 0, 1, ..., m$, therefore

$$B(x) = MT(x), \tag{2.8}$$

where

$$T = [1 \quad x \quad x^2 \quad ...x^n], \tag{2.9}$$

and M is a lower triangular $(n+1)\times(n+1)$ matrix has the form

$$M = \begin{bmatrix} B_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{1}B_1 & \binom{1}{0}B_0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{2}{2}B_2 & \binom{2}{1}B_1 & \binom{2}{0}B_0 & 0 & 0 & \cdots & 0 \\ \binom{3}{3}B_3 & \binom{3}{2}B_2 & \binom{3}{1}B_1 & \binom{3}{0}B_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ \binom{}{n}B_n & \binom{n}{n-1}B_{n-1} & \binom{n}{n-2}B_{n-2} & \binom{n}{n-3}B_{n-3} & \cdots & \binom{n}{0}B_0 & \end{bmatrix}, \tag{2.10}$$

and $det(M)=1$, then M is an invertible matrix. Using (2.8), we have

$$T(x) = M^{-1}B(x). \tag{2.11}$$

2.1. **Approximation of functions.** Suppose that $H = L^2[0,1]$ and $\{B_0(x), B_1(x), \cdots B_N(x)\} \subset H$, where $B_i(x)$' s are Bernoulli polynomials and

$$V = span\{B_0(x), B_1(x), \cdots B_N(x)\},$$

and f be an arbitrary member in H. Since V is a finite dimensional vector space, f has the unique best approximation $\hat{f} \in V$, that is

$$\forall v \in V, \quad \|f-\hat{f}\| \leq \|f-v\|.$$

Since $\hat{f} \in V$, then there exists the unique coefficients $f_0, f_1, \cdots f_N$ such that

$$f \approx \hat{f} = \sum_{n=0}^{N} f_n B_n(x) = F^T B(x), \quad F = [f_0, f_1, \cdots f_N]. \tag{2.12}$$

2.2. **Opeartioanal matrix of integration.**

Theorem 2.1. [31] *For vector $B(t)$ defined in* (2.7)*, the following formula is defined*

$$\int_0^x B(t)dt \simeq PB(x), \tag{2.13}$$

where P is the $(N+1)\times(N+1)$ operational matrix of integration, which is obtained from $P = UM^{-1}$, where

$$U = [U_1, U_2, ..., U_N, \Xi^T M]^T,$$

and

$$U_i = [0 \quad \frac{1}{i}\binom{i}{i-1}B_{i-1} \quad \frac{1}{i}\binom{i}{i-2}B_{i-2} \quad ... \quad \frac{1}{i}\binom{i}{1}B_1 \quad \frac{1}{i}B_0 \quad \underbrace{...}_{N-i}],$$

and $\Xi = [c_1, c_2, ..., c_N]^T$ *which* $\frac{B_{N+1}(x) - B_{N+1}(0)}{N+1} \simeq \Xi^T B(x)$, Ξ *can be computed by* (2.12).

The operational matrix of integration P is a sparse matrix, for example, for $N = 3$, we have

$$P = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 \\ \frac{-1}{12} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{129} & 0 & \frac{-1}{14} & 0 \end{bmatrix}.$$

It is not difficult to see that the operation matrix P as N increases, becomes more sparse. This is one of the advantages of using Bernouli polynomials for solving equations.

3. Semi-discretization of telegraph equation

The Crank–Nicolson procedure is a finite difference technique based on Euler's method. In fact, this method is a combination of forward Euler's method at ith level with the backward Euler method at $(i+1)$th level. By using forward Euler method on (1.1), we obtain

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} + \alpha(\frac{u_{i+1} - u_{i-1}}{2\Delta t}) = (u_{xx})_i - \beta(u)_i + F(x, t_i), \quad 0 \leq i \leq N-1. \tag{3.1}$$

Correspondingly, by using backward Euler method at $(i+1)$th level in time direction on (1.1), one gets

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} + \alpha(\frac{u_{i+1} - u_{i-1}}{2\Delta t}) = (u_{xx})_{i+1} - \beta(u)_{i+1} + F(x, t_{i+1}), \quad 0 \leq i \leq N-1. \tag{3.2}$$

Adding two Eqs. (3.1) and (3.2) gives

$$\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta t^2}+\alpha\frac{u_{i+1}-u_{i-1}}{2\Delta t}=\frac{(u_{xx})_i+(u_{xx})_{i+1}}{2}-\beta\frac{(u)_i+(u)_{i+1}}{2}+ \tag{3.3}$$

$$\frac{F(x,t_i)+F(x,t_{i+1})}{2}, \quad 0\le i\le N-1,$$

as well as the conditions

$$u_0=f_1(x), \quad (u_0)_t=f_2(x), \tag{3.4}$$

$$u_{i+1}(0)=\phi_1(t_{i+1}), \quad (u_{i+1})(1)=\phi_2(t_{i+1}), \quad i=0,1,...,N-1, \tag{3.5}$$

By simplifying Eq. (3.3), we obtain

$$2(u_{i+1}-2u_i+u_{i-1})+\alpha\Delta t(u_{i+1}-u_{i-1})=\Delta t^2((u_{xx})_i+(u_{xx})_{i+1}) \tag{3.6}$$

$$-\Delta t^2\beta((u)_i+(u)_{i+1})+\Delta t^2(F(x,t_i)+F(x,t_{i+1})), \quad 0\le i\le N-1,$$

where u_{i+1} is the solution of the differential equation (3.6) at $(i+1)$th time step. To solve (3.6), we rewrite it as follows:

$$(2+\alpha\Delta t+\Delta t^2\beta)(u)_{i+1}-\Delta t^2(u_{xx})_{i+1}=(4-\Delta t^2\beta)(u)_i+(-2+\alpha\Delta t)u_{i-1} \tag{3.7}$$

$$+\Delta t^2(u_{xx})_i+\Delta t^2F(x,t_i)+\Delta t^2F(x,t_{i+1}), \quad 0\le i\le N-1,$$

with boundary conditions

$$u_{i+1}(0)=\phi_1(t_{i+1}), \quad u_{i+1}(1)=\phi_2(t_{i+1}), \quad i=0,1,\cdots,N-1. \tag{3.8}$$

4. Description of the method

To solve the system of equations (3.7), the Bernouli polynomials approximation are applied in the following manner

$$(u_{i+1})_{xx}(x)=A^TB(x)=\sum_{j=1}^{j=N}a_jB_j(x), \tag{4.1}$$

where

$$A^T=[a_0,a_1,\cdots,a_N],$$

and

$$B^T(x)=[B_0(x),B_1(x),\cdots,B_N(x)],$$

where $(u_{i+1})_{xx}$ is the second order derivative of u_{i+1} with respect to x. By integrating (4.1) from 0 to x, we get

$$(u_{i+1})_x(x)=(u_{i+1})_x(0)+A^TPB(x). \tag{4.2}$$

In Eq.(4.2), the value of $(u_{i+1})_x(0)$ is unknown. This value can be computed by integrating (4.1) from 0 to 1 and using the conditions (1.2). Then we get

$$(u_{i+1})_x(0) = \phi_2(t_{i+1}) - \phi_1(t_{i+1}) - A^T PD, \tag{4.3}$$

where $D = \int_0^1 B(x)dx$. Using two Eqs. (4.2) and (4.3) leads to

$$(u_{i+1})_x(x) = \phi_2(t_{i+1}) - \phi_1(t_{i+1}) - A^T PD + A^T PB. \tag{4.4}$$

Ultimately, integrating (4.4) from 0 to x, we obtain the approximate solution as

$$(u_{i+1})(x) = \phi_1(t_{i+1}) + x(\phi_2(t_{i+1} - \phi_1(t_{i+1})) + A^T P^2 B(x) - xA^T PD. \tag{4.5}$$

By substituting $(u_{i+1})_{xx}(x)$ and $(u_{i+1})(x)$ from (4.1) and (4.5), respectively, into Eq.(3.7), we obtain the system of algebric equations

$$\begin{aligned}(2 + \alpha\Delta t + \Delta t^2\beta)(\phi_1(t_{i+1}) + x(\phi_2(t_{i+1} - \phi_1(t_{i+1})) + A^T P^2 B(x) - xA^T PD \\ -\Delta t^2 A^T B(x) = (4 - \Delta t^2\beta)(u)_i + (-2 + 2\alpha\Delta t)u_{i-1} + \Delta t^2(u_{xx})_i \\ +\Delta t^2 F(x, t_i) + \Delta t^2 F(x, t_{i+1}), \quad 0 \le i \le N-1\end{aligned} \tag{4.6}$$

Now, by using discretization procedure on (4.6) at collocation points $x_j = (j - 0.5)/(N + 1), j = 1, 2, \cdots, N + 1$, we obtain the following linear system

$$\begin{aligned}(2 + \alpha\Delta t + \Delta t^2\beta)(\phi_1(t_{i+1}) + x_j(\phi_2(t_{i+1}) - \phi_1(t_{i+1})) + A^T P^2 B(x_j) - x_j A^T PD \\ -\Delta t^2 A^T B(x_j) = (4 - \Delta t^2\beta)(u)_i + (-2 + \alpha\Delta t)u_{i-1} + \Delta t^2(u_{xx})_i \\ +\Delta t^2 F(x_j, t_i) + \Delta t^2 F(x_j, t_{i+1}), \quad 0 \le i \le N-1\end{aligned} \tag{4.7}$$

These equations can be written as in the matrix form $CA^T = b$, where C and b are known vectors. By using suitable solver, the unknown vector A^T can be computed. For nonlinear (1.3), we obtain nonlinear system of equations which can be solved by using Newton's method and find A^T. Thus, in any case, substituting by the calculated vector coefficients A^T into (4.5), we can compute the approximate solutions.

5. Convergence analysis

To explain the convergence analysis of the method, we have

Proposition 5.1. [17] *suppose f is approximated on $[0, 1]$, by Bernouli polynomials. Then the coefficients f_i in (2.12) satisfy*

$$f_i \le \frac{F_n}{n!}, \tag{5.1}$$

where F_n is the maximum value of f_i in the $[0,1]$. Then Bernouli coefficients are quickly reduced.

Theorem 5.2. [2] *Suppose that $f(x)$ is an enough smooth function in the $[0,1]$ and $P_N[f](x)$ is the approximation of $f(x)$ based on the Bernouli polynomials, if $R_N[f](x)$ be the residual term, then*

$$\begin{aligned} f(x) &= P_N[f](x) + R_N[f](x), \quad (5.2)\\ P_N[f](x) &= \int_0^1 f(x)dx + \sum_{j=1}^{N} \frac{B_j(x)}{j!}(f^{(j-1)}(1) - f^{(j-1)}(0)),\\ R_N[f](x) &= \frac{-1}{N!}\int_0^1 B_N^*(x-t)f_N(t)dt, \end{aligned}$$

where $B_N^*(x) = B_N(x - [x])$ and $[x]$ is the largest integer smaller than or equal to x.

Theorem 5.3. [2] *Suppose that $u(x) \in c^{\infty}[0,1]$ is the exact solution of the (1.1) and $P_N[u](x)$ is the approximation of $u(x)$ based on the Bernouli polynomials, then*

$$||error(u(x))|| \leq \frac{1}{N!}B_N U_N,$$

where B_N and U_N are the maximum values of $B_N(x)$ and $U_N(x)$ in $[0,1]$, respectively.

6. Illustrative Examples

Example 6.1. Consider the Telegraph equation (1.1) with following initial and boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = e^{-t}sin(1), t \geq 0. \quad (6.1)$$

The exact solution is $u(x,t) = e^{-t}sin(x)$, where $F(x,t) = -2e^{-t}sinx$, $\alpha = 8$ and $\beta = 4$. Two Tables 1 and 2 show the RMS error using the proposed technique with $N = 4$ and different values of Δt for different values of x at different times t. Fig 1 represents the absolute error and two Figs 2 and 3 represent the approximate and exact solution. It is found that the obtained errors by using the proposed method are better than the corresponding errors obtained by [15].

Table 1. RMS error for Example 6.1 with $t = 1$, $N = 4$,

x	RMS,	RMS	order of Convergence
	$\Delta t = 0.001$	$\Delta t = 0.01$	
0.2	$4.6365e-005$	$8.3355e-005$	0.3600
$\frac{0.2}{2}$	$1.1743e-005$	$1.7265e-005$	0.3600
$\frac{0.2}{4}$	$5.1287e-005$	$9.1790e-005$	0.4930
$\frac{0.2}{8}$	$7.5457e-005$	$4.9036e-005$	0.4930
$\frac{0.2}{16}$	$2.1261e-005$	$1.3632e-004$	1.5480
$\frac{0.2}{32}$	$6.1333e-005$	$1.2281e-004$	1.5480
$\frac{0.2}{64}$	$1.4538e-005$	$1.6824e-004$	2.8711

Table 2. RMS error for Example 6.1 with $t = 5$, $N = 4$,

x	RMS	RMS	order of Convergence
	$\Delta t = 0.001$	$\Delta t = 0.01$	
0.2	$1.0597e-006$	$1.5510e-006$	3.9259
$\frac{0.2}{2}$	$4.9675e-007$	$3.5226e-007$	
$\frac{0.2}{4}$	$1.1193e-006$	$2.0669e-006$	3.5776
$\frac{0.2}{8}$	$1.5829e-006$	$4.4846e-006$	
$\frac{0.2}{16}$	$1.2617e-006$	$2.9199e-006$	1.5480
$\frac{0.2}{32}$	$4.3167e-007$	$3.6818e-006$	
$\frac{0.2}{64}$	$1.7261e-006$	$3.1403e-006$	0.2584

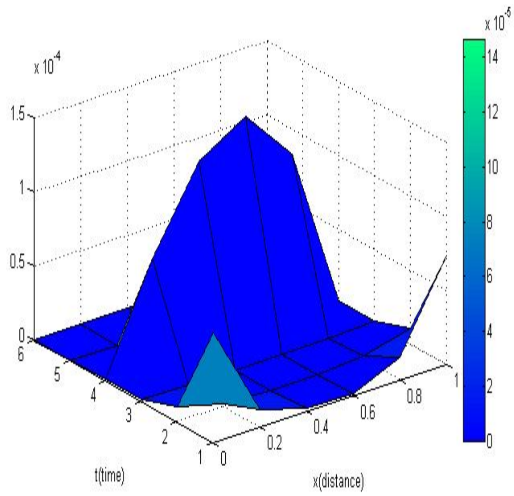

Figure 1. The absolute error function of Example 6.1

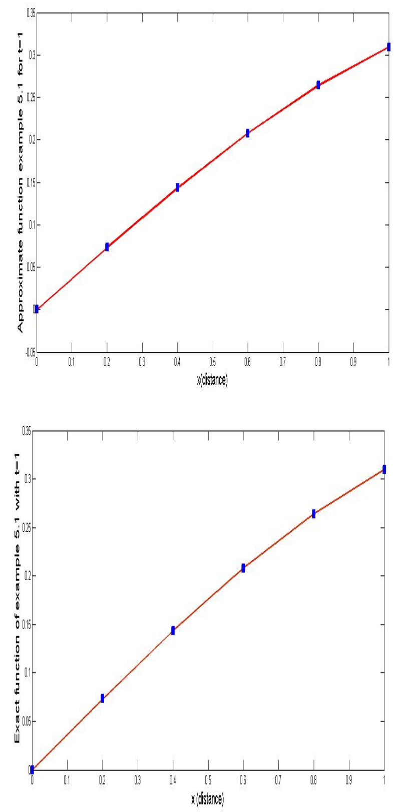

FIGURE 2. The exact and approximate solutions of Example 6.1 for t=1

Example 6.2. Now, we consider the Telegraph equation (1.1) with following initial and boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = e^{-2t} sinh(1), t \geq 0 \tag{6.2}$$

The exact solution is $u(x,t) = e^{-2t} sinh(x)$ and $F(x,t) = -12e^{-2t} sinhx$, with $\alpha = 2$ and $\beta = 1$. Two Tables 3 and 4 show the RMS error with $N = 4$, $\alpha = 4$, $\beta = 2$ for different values of t and x. Fig 4 represents the absolute error and two Figs 5 and 6 represent the approximate and exact solutions.

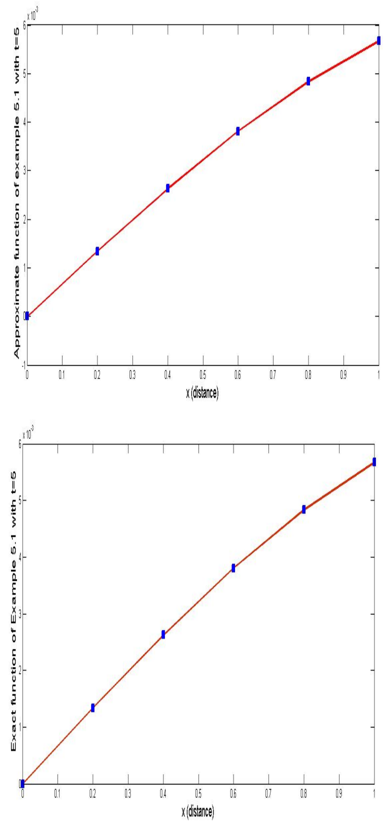

FIGURE 3. The exact and approximate solutions of Example 6.1 for t=5

TABLE 3. RMS error for Example 6.2 with $t = 5$, $N = 4$,

x	RMS	RMS	order of Convergence
	$\Delta t = 0.001$	$\Delta t = 0.01$	
0.2	$1.1716e - 008$	$1.1420e - 008$	3.9700
$\frac{0.2}{2}$	$2.5797e - 009$	$5.0033e - 001$	
$\frac{0.2}{4}$	$1.3003e - 008$	$1.2504e - 008$	2.6486
$\frac{0.2}{8}$	$6.4580e - 009$	$6.8411e - 009$	
$\frac{0.2}{16}$	$1.9370e - 008$	$1.8581e - 008$	1.1798
$\frac{0.2}{32}$	$1.7879e - 008$	$1.6903e - 008$	
$\frac{0.2}{64}$	$2.3777e - 008$	$2.2881e - 008$	0.2545

Table 4. RMS error for Example 6.2 with $t = 1$, $N = 4$,

x	RMS	RMS	order of Convergence
	$\Delta t = 0.001$	$\Delta t = 0.01$	
0.2	$1.9045e-005$	$3.4042e-005$	6.2761
$\frac{0.2}{2}$	$4.9397e-005$	$1.5007e-005$	
$\frac{0.2}{4}$	$2.1039e-005$	$3.7275e-005$	2.5117
$\frac{0.2}{8}$	$3.1267e-005$	$2.0393e-005$	
$\frac{0.2}{16}$	$8.7089e-006$	$5.5390e-005$	1.2067
$\frac{0.2}{32}$	$2.5227e-005$	$5.0388e-005$	
$\frac{0.2}{64}$	$5.5206e-006$	$6.8207e-005$	0.2580

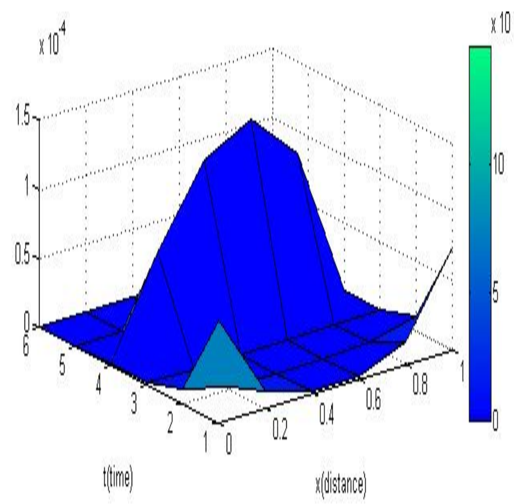

Figure 4. The absolute error function of Example 5.2

Example 6.3. Now, we consider the nonlinear Telegraph equation (1.3) with following initial and boundary conditions:

$$u(0, t) = e^{-t}, \quad u(1, t) = e^{-t}cosh(1), \quad t \geq 0. \tag{6.3}$$

The exact solution is given by $u(x, t) = e^{-t}cosh(x)$ with $F(x, t) = e^{-2t}cos^2hx - 2e^{-t}coshx$, and $\alpha = 2$ and $\beta = 1$. Two Tables 5 and 6 show the RMS error with $N = 4$ using different values of Δt and different values of x at different times t. Fig 3 represents the absolute error and two Figs 8 and 9 represent the approximate and exact solutions. It is found that the errors in proposed method get reduced in comparison with the errors in [29]

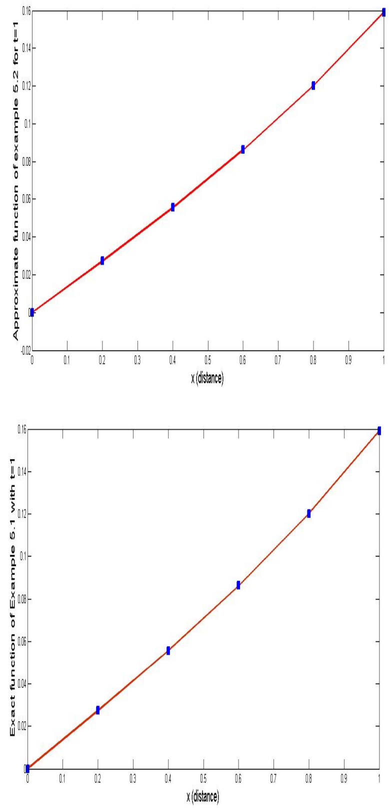

FIGURE 5. The exact and approximate solutions of Example 6.2 for t=1

TABLE 5. RMS error for Example 6.3 with $t = 1$, $N = 4$,

x	RMS	RMS	order of Convergence
	$\Delta t = 0.001$	$\Delta t = 0.01$	
0.2	$1.5451e - 004$	$2.0474e - 004$	3.3569
$\frac{0.2}{2}$	$4.8190e - 005$	$6.9105e - 005$	
$\frac{0.2}{4}$	$1.5733e - 004$	$2.2392e - 004$	2.6975
$\frac{0.2}{8}$	$2.1804 - 004$	$1.0915 - 004$	
$\frac{0.2}{16}$	$1.3837e - 004$	$3.3153e - 004$	1.3014
$\frac{0.2}{32}$	$2.8513e - 004$	$2.8728e - 004$	
$\frac{0.2}{64}$	$3.9030e - 004$	$4.1297e - 004$	0.2784

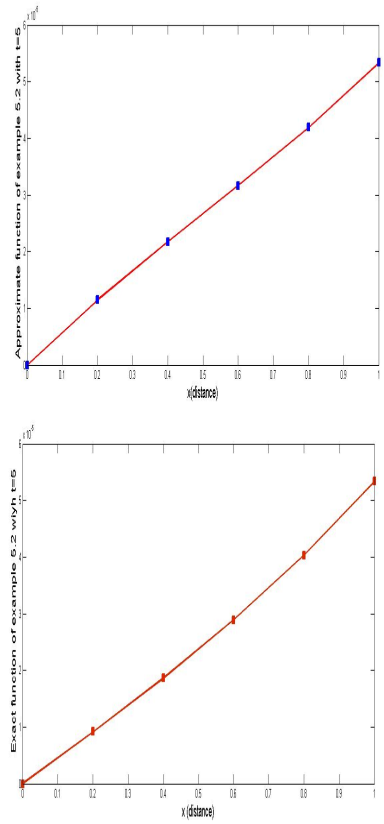

Figure 6. The exact and approximate solutions of Example 6.2 for t=5

Table 6. RMS error for Example 6.3 with $t = 5$, $N = 4$,

x	RMS	RMS	order of Convergence
	$\Delta t = 0.001$	$\Delta t = 0.01$	
0.2	$2.0826e - 006$	$3.7498e - 006$	3.3571
$\frac{0.2}{2}$	$3.9163e - 006$	$1.2655e - 006$	
$\frac{0.2}{4}$	$2.3114e - 006$	$4.1012e - 006$	2.6975
$\frac{0.2}{8}$	$3.5614 - 006$	$1.9990 - 006$	
$\frac{0.2}{16}$	$1.3762e - 006$	$6.0721e - 006$	1.3014
$\frac{0.2}{32}$	$1.4984e - 006$	$5.2616e - 006$	
$\frac{0.2}{64}$	$2.0314e - 006$	$7.5638e - 006$	0.2784

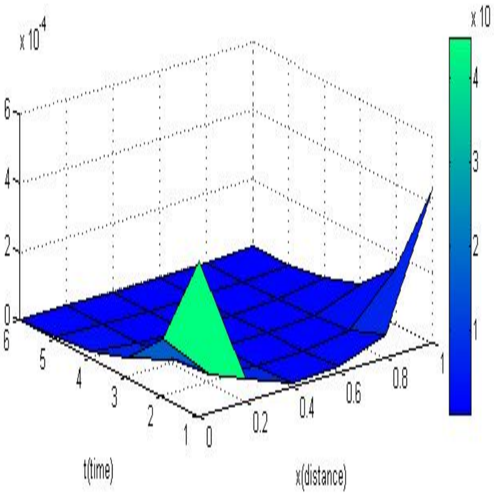

FIGURE 7. The absolute error function of Example 6.3

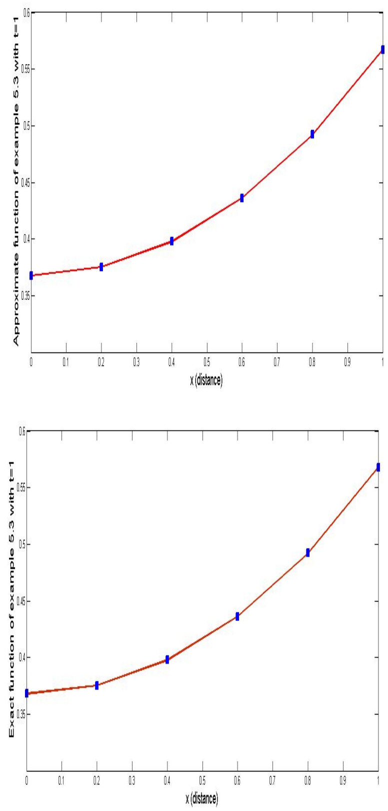

FIGURE 8. The exact and approximate solution of Example 6.3 for t=1

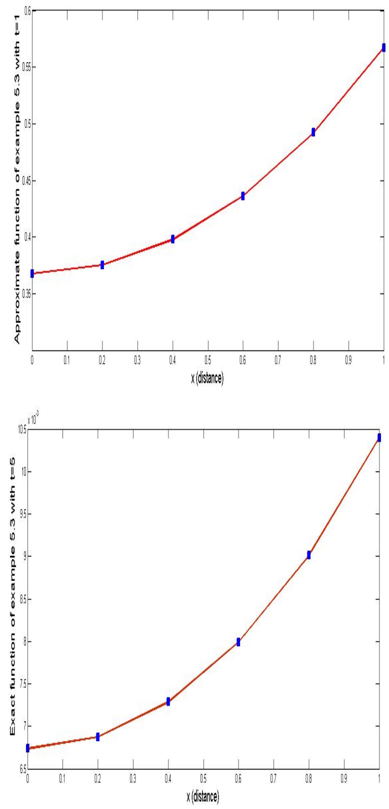

FIGURE 9. The exact and approximate solutions of Example 6.3 for t=5

7. Conclusion

The efficacy and relevancy of the suggested method are explained in the given examples. It is found that the errors in Bernouli matrix method is reduced compared to the errors in [20]. Most of the elements of two matrices D and P in (5.1) are zeros, that is, they are sparse and therefore the suggested method is very interesting and lessens the CPU time and the computer memory, as well as the problems are well-conditioned. Finally, the proposed method can be extended to solve systems of telegraph equations numerically.

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