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Analysis of a fractional SIQR model with Caputo-Fabrizio derivative

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Abstract. The main purpose of this paper is to develop and analyse a fractional SIQR epidemic model with Caputo-Fabrizio derivative. It is shown that the model to have a disease-free and an endemic equilibrium point. Some conditions are derived for the existence and stability of these equilibrium points. Finally, three-step fractional Adams-Bashforth method applied to the model and some numerical simulations are illustrate the results.

Keywords: Fractional derivative, SIQR model, Adams-Bashforth method, Forward bifurcation.

2010 Mathematics subject classification: 34A08, 37N25; Secondary 34K20.

1. Introduction

Since the outbreak of the Covid-19 pandemic, many scientists and authors have focused on studying mathematical epidemic models, as seen in [3, 17, 21]. These models are essential for better understanding disease transmission patterns and evaluating control strategies. The classical SIR model introduced by Kermack and McKendrick [11]. Afterwards, it has been extended by many authors to include extra equations for vaccination, recovery, quarantine, etc. Quarantine is an effective method for preventing the spread of disease in the community, especially for infectious people. Therefore, some researchers have introduced

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a new compartment called Quarantine in the SIR model, resulting in an SIQR model, as described in [17, 24]. It is a mathematical model that divides a population into four compartments: susceptible (S), infected (I), quarantined (Q), and recovered (R). The model assumes that a susceptible person can become infected through contact with an infected person, and that an infected person can be quarantined before showing symptoms of the disease. This model is particularly suited for diseases where individuals who are infected can be quarantined before eventually recovering, preventing them from spreading the disease further, such as tuberculosis, flu, Covid [8, 16]. The SIQR model has gained significant attention from researchers and public health officials in recent years, especially during the COVID-19 pandemic, as it helps to better understand the dynamics of disease transmission and evaluate control strategies [1, 13].

In recent years, many researchers have found that fractional models with fractional calculus, such as Riemann-Liouville, Caputo, Caputo-Fabrizio, Jumarie, Atangana-Baleanu, etc., describe natural phenomena better than the classic integer-order counterparts with ordinary time derivatives [5, 7, 10, 23]. For instance, authors in [20] investigated a fractional order model with vaccine efficacy and waning immunity to understand the dynamics of coronavirus infection. DarAssi et al. [6] studied the competition between two banking systems, rural and commercial, in Indonesia based on Caputo's parametric fractional derivative. Asma et al. [2] investigated a simple SVIR type of model to investigate the coronavirus's dynamics in Saudi Arabia with the recent cases of the coronavirus. The Caputo-Fabrizio fractional derivative is a modification of the Caputo fractional derivative, which is commonly used in fractional calculus. It has been shown to have some advantages over the Caputo derivative in certain applications, particularly in solving fractional differential equations numerically. One of the main benefits of the Caputo-Fabrizio fractional derivative is that it can better approximate the initial conditions of a fractional differential equation, particularly when the initial conditions are non-smooth or discontinuous [4]. This is because the Caputo-Fabrizio derivative is defined in terms of an exponential function, which provides a smoothing effect that helps to regularize the initial conditions.

In [12, 15, 21], Caputo-Fabrizio fractional derivative have been used to study the epidemic diseases. Here, we present our SIQR epidemic model as a system of fractional differential equations with Caputo-Fabrizio derivative. In this article, we assume that a person can be quarantined as soon as they are diagnosed with the disease, before the symptoms appear. To reflect this, we add a new parameter, q_0, representing the rate of

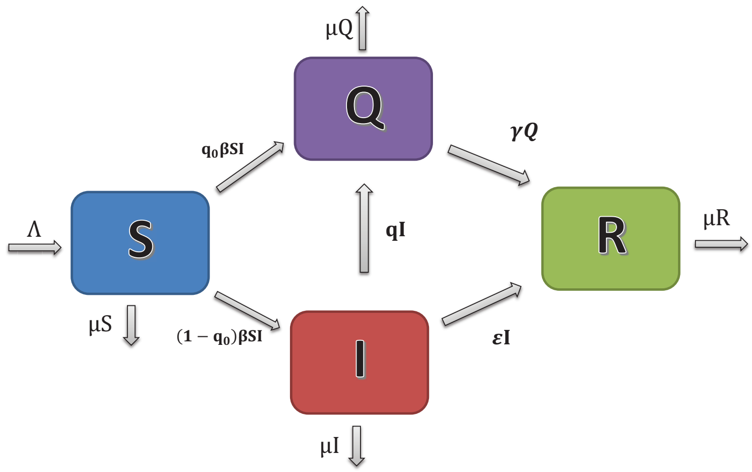


FIGURE 1. Diagram of the SIQR model (3.1). Variables and parameters will be presented in Section 3.

quarantine for new patients immediately after infection. The compartmental relations of the model are shown in Figure 1. We demonstrate that the value of q_0 plays a crucial role in combating the spread of the disease.

The paper is structured as follows: Section 2, presents some definitions and basic concepts. Section 3 introduces the SIQR model and presents some dynamical analysis. Indeed, we study the stability of equilibria and we prove that the forward bifurcation occurs in the system. In section 4, the existence of solution for our model is obtained. In 5, we apply the three-steps Adam-Bashforth numerical technique to obtain numerical solutions of our model. The paper concludes with some results and graphical representations. Finally, the paper ends with the conclusions section.

2. Preliminaries

Caputo and Fabrizio in [4] defined the new definition of fractional derivative without any singularity in its kernel as follows.

Definition 2.1. [4] Let $0 < \alpha < 1$. If $a \in (-\infty, x)$, the Caputo-Fabrizio fractional derivative of a function $f(x) \in H^1(a,b)$, $b > a$ is defined as

$$^{CF}D_x^{\alpha} f(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x f'(s) \exp(-\frac{\alpha(x-s)}{1-\alpha}) ds, \qquad (2.1)$$

where $M(a)$ is the normalization function such that $M(0) = M(1) = 1$. Its fractional integral related to derivative (2.1) is defined as

$$ {}^{CF}I_x^{\alpha} f(x) = \frac{1-\alpha}{M(\alpha)} f(x) + \frac{\alpha}{M(\alpha)} \int_a^x f(t)dt. \tag{2.2} $$

Consider the following fractional-order linear system

$$ {}^{CF}D_x^{\alpha} x(t) = Ax(t), \tag{2.3} $$

where, $0 < \alpha \leq 1$, $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.2. [12] The characteristic equation of system (2.3) is

$$ P(\lambda) = \det\{\lambda(I - (1-\alpha)A) - \alpha A\}. \tag{2.4} $$

Theorem 2.3. [12] *If matrix $(I - (1-\alpha)A)$ is invertible, then system* (2.3) *is asymptotically stable if and only if the real parts of the roots to the characteristic equation* (2.4) *are negative.*

Theorem 2.4. [12] *The system* (2.3) *is asymptotically stable if eigenvalues λ of matrix A satisfy one of the following conditions:*

(a) $\|\lambda\| \geq \frac{1}{1-\alpha}, \quad \lambda \neq \frac{1}{1-\alpha}$;
(b) $Re(\lambda) > \frac{1}{1-\alpha}$;
(c) $Re(\lambda) < 0$;
(d) $|Im(\lambda)| > \frac{1}{2(1-\alpha)}$.

Lemma 2.5. *The Laplace transform of the Caputo-Fabrizio derivative of order $0 \leq \alpha < 1$ is given by*

$$ \mathcal{L}\{{}^{CF}D_x^{\alpha} f(x)\} = \frac{s\mathcal{L}\{f(x)\} - f(0)}{s + \alpha(1-s)}. \tag{2.5} $$

Definition 2.6. [9] A continuous function $g : [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $g(0) = 0$.

Theorem 2.7. [9] *Consider the system described by the equation*

$$ {}^{CF}D_x^{\alpha} y(x) = f(y(x)), \tag{2.6} $$

where $y = 0$ is an equilibrium point. Suppose there exists a class $\mathcal{K}$ function $V(y)$ that is continuously differentiable and defined in a neighborhood $\mathfrak{U} \subset \mathbb{R}^n$ of the origin, satisfying the following conditions:

(1) $V(0) = 0$ *and* $V(y) > 0$ *for all* $y \in \mathfrak{U}\backslash\{0\}$;
(2) ${}^{CF}D_x^{\alpha} y(x) \leq 0$ *for all* $y \in \mathfrak{U}\backslash\{0\}$.

Then the equilibrium point $y = 0$ is locally stable. Moreover, if these two conditions hold globally over $\mathbb{R}^n$, then $y = 0$ is globally stable.

Lemma 2.8. [9] *Let $u(t) \in [0, \infty)$ be a continuously differentiable function and $u^* > 0$. Then, for any time $t \geq t_0$, we have*

$$ {}^{CF}D_t^{\alpha}\big(u(t) - u^* - u^* \ln(\frac{u(t)}{u*}\big) \leq \big(1 - \frac{u^*}{u(t)}\big)\, {}^{CF}D_t^{\alpha} u(t). $$

3. The SIQR model and its dynamics

We introduce an epidemic model which in this model the total population is divided into four classes at any time $t \geq 0$: the susceptible individuals (S), the infected individuals (I), the quarantine individuals (Q) and the recovered individuals (R). Moreover, we introduce our model as a system of fractional differential equations with Caputo-Fabrizio derivative as:

$$\begin{aligned} {}^{CF}D_t^{\alpha} S &= \Lambda - \beta S I - \mu S, \\ {}^{CF}D_t^{\alpha} I &= (1-q_0)\beta S I - (q+\varepsilon+\mu) I, \\ {}^{CF}D_t^{\alpha} Q &= qI + q_0 \beta S I - (\gamma+\mu) Q, \\ {}^{CF}D_t^{\alpha} R &= \varepsilon I + \gamma Q - \mu R. \end{aligned} \tag{3.1}$$

In model (3.1), all parameters are positive and describe in Table 1.

Table 1. Parameters of the SIQR model (3.1).

Parameters	Discription
Λ	The recruitment rate of susceptibles
β	Transmission coefficient
q_0	Per capita rate of quarantine of the new patients
μ	Natural mortality rate
q	Quarantine rate
ε	Recovered rate after infected
γ	Recovered rate of quarantined patients

Here, we study the dynamics of the model. For this reason, first we calculate the equilibrium points of the model. By some calculation, we obtain two equilibrium points, namely, a disease-free equilibrium (DFE) $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ and an endemic equilibrium point, $E^* = (S^*, I^*, Q^*, R^*)$, as

$$\begin{aligned} S^* &= \frac{\Lambda}{\mu \mathcal{R}_0}, \qquad I^* = \frac{\mu}{\beta}(\mathcal{R}_0 - 1), \\ Q^* &= \frac{(\mathcal{R}_0 - 1)(q_0\varepsilon + q_0\mu + q)}{\mu\beta(\mu+\gamma)(1-q_0)}, \\ R^* &= \frac{(\mathcal{R}_0 - 1)[\gamma(\varepsilon + q_0\mu + q) + \varepsilon\mu(1-q_0)]}{\beta(\mu+\gamma)(1-q_0)}, \end{aligned}$$

where, $\mathcal{R}_0 = \frac{\beta\Lambda(1-q_0)}{\mu(q+\mu+\varepsilon)}$ is the basic reproduction number.

Theorem 3.1. *The disease free equilibrium E_0 always exists and it is locally asymptotically stable if $\mathcal{R}_0 < 1$ and it is unstable if $\mathcal{R}_0 > 1$. The endemic equilibrium E_1 exists and locally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix of the system (3.1) is given by:

$$J(S,I,Q,R) = \begin{bmatrix} -\beta\, I - \mu & -\beta\, S & 0 & 0 \\ (1-q_0)\,\beta\, I & (1-q_0)\,\beta\, S - q - \mu - \varepsilon & 0 & 0 \\ \beta\, I\, q_0 & \beta\, q_0\, S + q & -\gamma - \mu & 0 \\ 0 & \varepsilon & \gamma & -\mu \end{bmatrix}. \tag{3.2}$$

By simple calculation we can find the eigenvalues at E_0 are $-\mu, -\mu, -(\mu+\gamma)$ and $(q+\mu+\varepsilon)(\mathcal{R}_0 - 1)$. Therefore, E_0 is asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

At E^*, the eigenvalues are $\lambda_1 = -\mu, \lambda_2 = -(\mu+\gamma)$ and the two others are roots of the following equation

$$\lambda^2 + (\mu\mathcal{R}_0)\lambda + (q+\mu+\varepsilon)(\mathcal{R}_0 - 1) = 0. \tag{3.3}$$

Hence, $\lambda_3 + \lambda_4 = -\mu\mathcal{R}_0 < 0$ and $\lambda_3\lambda_4 = (q+\mu+\varepsilon)(\mathcal{R}_0 - 1) > 0$, and E^* is stable for $\mathcal{R}_0 > 1$. □

In the following two theorems, we will explore the global dynamics of equilibria. Since the first three equations of system (3.1) are independent of the last one, therefore, we can disregard the fourth equation and focus on the equivalent 3-dimensional system, which will make it easier to analyze the system's properties.

Theorem 3.2. *When $\mathcal{R}_0 < 1$, the DFE is globally asymptotically stable.*

Proof. To demonstrate the global stability of the model at the DFE when $\mathcal{R}_0 < 1$, we consider the Lyapunov function $V(I) = I$. Therefore,

$$\begin{aligned} {}^{CF}D_t^{\alpha} V =& (1-q_0)\beta S I - (q+\varepsilon+\mu) I \\ =& (1-q_0)\beta S_0 I - (q+\varepsilon+\mu) I - (1-q_0)\beta(S_0 - S) I \\ =& (q+\varepsilon+\mu)(\mathcal{R}_0 - 1) I - (1-q_0)\beta(\frac{\Lambda}{\mu} - S) I. \end{aligned}$$

Clearly, we observe that ${}^{CF}D_t^{\alpha} V \leq 0$, leading to the conclusion. □

Theorem 3.3. *When $\mathcal{R}_0 > 1$, the endemic equilibrium E^* is globally asymptotically stable.*

Proof. Suppose $\mathcal{R}_0 > 1$, and take following Lyapunov function

$$V(S,I,Q) = S^*\Theta(\frac{S}{S^*}) + \frac{1}{1-q_0} I^*\Theta(\frac{I}{I^*}), \tag{3.4}$$

where, $\Theta(y) = y - 1 - \ln(y)$, for $y > 0$. It is clear that Θ attains its global minimum at $y = 1$ and $\Theta(1) = 0$. This implies $\Theta(y) \geq 0$ for all $y \geq 0$. Hence, $V(S, I, Q) \geq 0$ and $V(S^*, I^*, Q^*) = 0$.

By applying Lemma 2.8 , we get

$$ {}^{CF}D_t^\alpha V(S,I,Q) \leq \big(1 - \frac{S^*}{S}\big)\, {}^{CF}D_t^\alpha S + \frac{1}{1-q_0}\big(1 - \frac{I^*}{I}\big)\, {}^{CF}D_t^\alpha I. \quad (3.5) $$

Using endemic condition $\Lambda = (\beta I^* + \mu))S^*$, $q + \varepsilon + \mu = (1 - q_0)\beta S^*$, we obtain

$$ \begin{aligned} \big(1 - \frac{S^*}{S}\big)\, {}^{CF}D_t^\alpha S =& \big(1 - \frac{S^*}{S}\big)\big(\Lambda - \beta SI - \mu S\big) \\ =& \big(1 - \frac{S^*}{S}\big)\big((\beta I^* + \mu))S^* - \beta SI - \mu S\big) \\ =& \beta S^* I^* \big(1 - \frac{S^*}{S}\big)\big(1 - \frac{SI}{S^* I^*}\big) + \mu S^* \big(1 - \frac{S^*}{S}\big)\big(1 - \frac{S}{S^*}\big) \\ =& \beta S^* I^* \big(1 - \frac{S^*}{S} + \frac{I}{I^*} - \frac{SI}{S^* I^*}\big) + \mu S^* \big(2 - \frac{S^*}{S} - \frac{S}{S^*}\big), \end{aligned} $$

and

$$ \begin{aligned} \frac{1}{1-q_0}\big(1 - \frac{I^*}{I}\big)\, {}^{CF}D_t^\alpha I =& \big(1 - \frac{I^*}{I}\big)\big(\beta SI - \frac{(q + \varepsilon + \mu)}{(1 - q_0)} I\big) \\ =& \big(1 - \frac{I^*}{I}\big)\big(\beta SI - \beta S^* I\big) \\ =& \beta S^* I^* \big(1 - \frac{S}{S^*} - \frac{I}{I^*} + \frac{SI}{S^* I^*}\big). \end{aligned} $$

Hence,

$$ \begin{aligned} {}^{CF}D_t^\alpha V(S,I,Q) \leq& \big(\beta S^* I^* + \mu S^*\big)\big(2 - \frac{S^*}{S} - \frac{S}{S^*}\big) \\ =& -\frac{\beta I^* + \mu}{S}(S - S^*)^2. \end{aligned} $$

This means ${}^{CF}D_t^\alpha V(S, I, Q) \leq 0$, for $\mathcal{R}_0 > 1$, and based on Theorem 2.7, the endemic equilibrium is globally asymptotically stable. □

In the last part of this section, we investigate the forward bifurcation in the our model. A forward bifurcation in an epidemic model occurs when a small change in a parameter of the model causes a qualitative change in the behavior of the system. In the context of an epidemic model, this means that a small change in a parameter can cause the epidemic to either die out or become an ongoing, endemic infection. Before the bifurcation point, the only stable equilibrium is the disease-free equilibrium (DFE), meaning that the infection cannot establish itself in the population. After the bifurcation point, there are two equilibria: the unstable disease-free equilibrium and an stable endemic equilibrium,

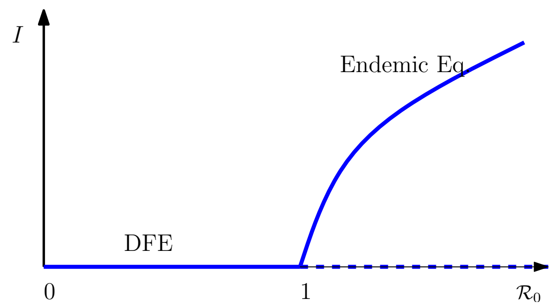


FIGURE 2. Qualitative diagrams for the forward bifurcation. Solid line shows the stability and dashed line shows the instability.

where the infection persists indefinitely in the population (see Fig. 2). The forward bifurcation is an important concept in epidemiology because it helps us understand how changes in the transmission rate, such as interventions or behavior changes, can affect the long-term outcome of an epidemic. If the transmission rate is below the bifurcation threshold, then the epidemic will die out on its own. If it is above the bifurcation threshold, then the epidemic will become an ongoing, endemic infection [14, 19]. Therefore, straightforward result is obtained from Theorem 3.1.

Theorem 3.4. *The fractional system* (3.1) *has a forward bifurcation at* $\mathcal{R}_0 = 1$.

4. Existence of solutions

In this section, we establish the existence of the SIQR model (3.1) by utilizing the fixed point hypothesis. First, consider system (3.1) with non-negative initial values

$$S(0) = S_0 \geq 0,\ I(0) = I_0 \geq 0,\ Q(0) = Q_0 \geq 0,\ R(0) = R_0 \geq 0. \quad (4.1)$$

Let $(\Omega, \|.\|)$ be a Banach space and $\mathfrak{T}$ a self-map on Ω. Let also $u_{n+1} = F(\mathfrak{T}, u_n)$ be some recurrent procedure, such as Picard's iterations sequence. Assume $Fix(\mathfrak{T})$, be the set of fixed points of $\mathfrak{T}$ and $Fix(\mathfrak{T}) \neq \emptyset$. Moreover, $\lim_{n\to\infty} u_n = u^* \in Fix(\mathfrak{T})$. Furthermore, let $\{v_n\} \subset \Omega$ and define $\varepsilon_n = \|v_{n+1} - F(\mathfrak{T}, u_n)\|$. If $\lim_{n\to\infty} \varepsilon_n = 0$, implies $\lim_{n\to\infty} y_n = u^*$, then the recurrent procedure $u_{n+1} = F(\mathfrak{T}, u_n)$ is called to be T-stable [18, 22].

Applying the Laplace transform on both sides of (3.1) and using initial values (4.1), we obtain

$$\begin{aligned}
\frac{s\mathcal{L}\{S(t)\}-S(0)}{s+\alpha(1-s)} &= \mathcal{L}\{\Lambda-\beta SI-\mu S\},\\
\frac{s\mathcal{L}\{I(t)\}-I(0)}{s+\alpha(1-s)} &= \mathcal{L}\{(1-q_0)\beta SI-(q+\varepsilon+\mu)I\},\\
\frac{s\mathcal{L}\{Q(t)\}-Q(0)}{s+\alpha(1-s)} &= \mathcal{L}\{qI+q_0\beta SI-(\gamma+\mu)Q\},\\
\frac{s\mathcal{L}\{R(t)\}-R(0)}{s+\alpha(1-s)} &= \mathcal{L}\{\varepsilon I+\gamma Q-\mu R\}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathcal{L}\{S(t)\} &= \frac{S_0}{s}+\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{\Lambda-\beta SI-\mu S\},\\
\mathcal{L}\{I(t)\} &= \frac{I_0}{s}+\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{(1-q_0)\beta SI-(q+\varepsilon+\mu)I\},\\
\mathcal{L}\{Q(t)\} &= \frac{Q_0}{s}+\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{qI+q_0\beta SI-(\gamma+\mu)Q\},\\
\mathcal{L}\{R(t)\} &= \frac{R_0}{s}+\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{\varepsilon I+\gamma Q-\mu R\}.
\end{aligned}$$

Now, using inverse Laplace transform implies that

$$\begin{aligned}
S(t) &= S_0+\mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{\Lambda-\beta SI-\mu S\}\right\},\\
I(t) &= I_0+\mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{(1-q_0)\beta SI-(q+\varepsilon+\mu)I\}\right\},\\
Q(t) &= Q_0+\mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{qI+q_0\beta SI-(\gamma+\mu)Q\}\right\},\\
R(t) &= R_0+\mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{\varepsilon I+\gamma Q-\mu R\}\right\}.
\end{aligned}$$

Based on the above process, we can define the following recurrent formula

$$\begin{aligned}
S_{j+1}(t) &= S_0 + \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{\Lambda - \beta S_j I_j - \mu S_j\}\right\},\\
I_{j+1}(t) &= I_0 + \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{(1-q_0)\beta S_j I_j - (q+\varepsilon+\mu) I_j\}\right\},\\
Q_{j+1}(t) &= Q_0 + \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{q I_j + q_0\beta S_j I_j - (\gamma+\mu) Q_j\}\right\},\\
R_{j+1}(t) &= R_0 + \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{\varepsilon I_j + \gamma Q_j - \mu R_j\}\right\}, \qquad (4.2)
\end{aligned}$$

for $j = 0, 1, 2, \cdots$. Now, we define vector $\mathbf{u}_j = (S_j, I_j, Q_j, R_j)^T$ and

$$F(\mathbf{u}) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}(\mathbf{u}) = \begin{pmatrix} \Lambda - \beta S_j I_j - \mu S_j \\ (1-q_0)\beta S_j I_j - (q+\varepsilon+\mu) I_j \\ q I_j + q_0 \beta S_j I_j - (\gamma+\mu) Q_j \\ \varepsilon I_j + \gamma Q_j - \mu R_j \end{pmatrix}. \qquad (4.3)$$

Hence, the self map $\mathfrak{T}$, define by

$$\mathfrak{T}(\mathbf{u}_j) = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}(\mathbf{u}_j) = \mathbf{u}_0 + \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}F(\mathbf{u}_j)\right\}. \qquad (4.4)$$

Since the SIQR model present population in the real world, then $S(t)$, $I(t)$, $Q(t)$ and $R(t)$ are non-negative and bounded. Therefore, there exist positive constants $\zeta_i, \quad i = 1, 2, 3, 4$ such that for all t

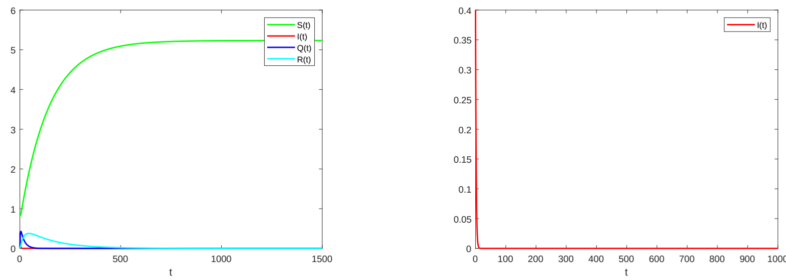
$$\|S(t)\| \le \zeta_1, \quad \|I(t)\| \le \zeta_2, \quad \|Q(t)\| \le \zeta_3, \quad \|R(t)\| \le \zeta_4. \qquad (4.5)$$

We also define

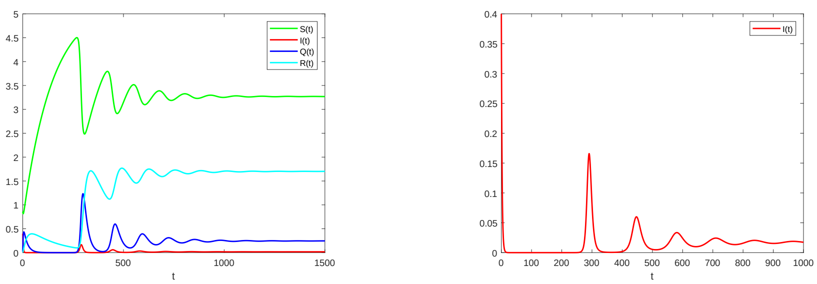
$$\|\mathbf{u}\| = \max\{\|S\|, \|I\|, \|Q\|, \|R\|\}.$$

Considering equations (4.4) and (4.5), for $i, j \in \mathbb{N}$, we have

$$\begin{aligned}
\|T_1(\mathbf{u}_j) - T_1(\mathbf{u}_i)\| &\le \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{-\beta S_j I_j - \mu S_j + \beta S_i I_i - \mu S_i\}\right\}\\
&\le \mathcal{L}^{-1}\left\{\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\{(\beta\zeta_2 - \mu)\|S_j - S_i\|\}\right\}\\
&\le \Phi(\alpha)(\beta\zeta_2 - \mu)\|\mathbf{u}_j - \mathbf{u}_i\|,
\end{aligned}$$



(a) The DFE E_0 is stable for $q_0 = 0.65$.



(b) The endemic equilibrium E^* is stable for $q_0 = 0.25$.

FIGURE 3. Solution of system (3.1) for fractional order $\alpha = 0.95$ and $\Lambda = 0.0382$, $\beta = 0.25177, \mu = 0.0073, q = 0.6, \varepsilon = 0.01, \gamma = 0.05$.

where, $\Phi(\alpha) = \mathcal{L}^{-1}\{\mathcal{L}\left(\frac{s+\alpha(1-s)}{s}\right)\}$. With similar calculation, one can obtain

$$
\begin{aligned}
\|T_2(\mathbf{u}_j) - T_2(\mathbf{u}_i)\| &\leq \Phi(\alpha)(\beta\zeta_1(1-q_0) - (q+\mu+\varepsilon))\|\mathbf{u}_j - \mathbf{u}_i\|, \\
\|T_3(\mathbf{u}_j) - T_3(\mathbf{u}_i)\| &\leq \Phi(\alpha)(q + q_0\beta\zeta_1 - (\gamma+\mu))\|\mathbf{u}_j - \mathbf{u}_i\|, \\
\|T_4(\mathbf{u}_j) - T_4(\mathbf{u}_i)\| &\leq \Phi(\alpha)(\varepsilon + \gamma - \mu)\|\mathbf{u}_j - \mathbf{u}_i\|,
\end{aligned}
$$

Here, we can summarize above results in the following theorem.

Theorem 4.1. *If the following conditions hold*

(C1) $\Phi(\alpha)(\beta\zeta_2 - \mu) < 1$;
(C2) $\Phi(\alpha)(\beta\zeta_1(1-q_0) - (q+\mu+\varepsilon)) < 1$;
(C3) $\Phi(\alpha)(q + q_0\beta\zeta_1 - (\gamma+\mu)) < 1$;
(C4) $\Phi(\alpha)(\varepsilon + \gamma - \mu) < 1$;

then recurrent procedure (4.2) *is* T*-stable and system* (3.1) *with initial conditions* (4.1) *has a unique solution.*

5. Numerical solutions

In this section, we first present the three-step fractional Adams-Bashforth method and then apply it to our model. First, Consider the Caputo–Fabrizio fractional differential equation

$$^{CF}D_t^{\alpha}u(t) = f(t,u(t)), \quad 0 < \alpha \le 1. \tag{5.1}$$

Applying the following fractional integral (2.2) to both sides of (5.1) implies

$$\begin{aligned} u(t) - u(0) &= ^{CF}I_t^{\alpha}\big(f\big(t,u(t)\big)\big) \\ &= \frac{(1-\alpha)}{M(\alpha)} f\big(t,u(t)\big) + \frac{\alpha}{M(\alpha)}\int_0^t f\big(s,u(s)\big)\,ds. \end{aligned}$$

Now, we discritize the above equation on $[0,T]$ with step size h and approximate function $f(t,u)$ by Lagrange interpolating polynomial of degree two, we obtain following formula

$$\begin{aligned} u_{k+1} = u_k &+ \frac{1}{M(\alpha)}\left[(1-\alpha) + \frac{23h\alpha}{12}\right] f(t_k,u_k) \\ &- \frac{1}{M(\alpha)}\left[(1-\alpha) + \frac{4}{3}h\alpha\right] f(t_{k-1},u_{k-1}) + \frac{5h\alpha}{12M(\alpha)} f(t_{k-2},u_{k-2}), \end{aligned} \tag{5.2}$$

where $u_k = u(t_k)$. For details and obtaining the truncation error of formula (5.2), see [15].

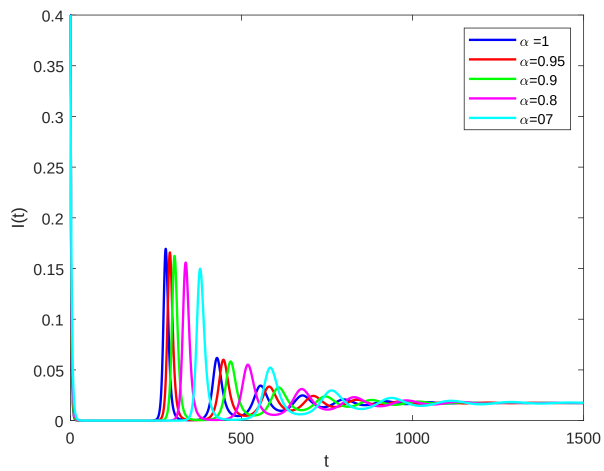
In this section, by using the numerical methods, we obtain some results. Indeed, we are present three scenarios.

Case 1.

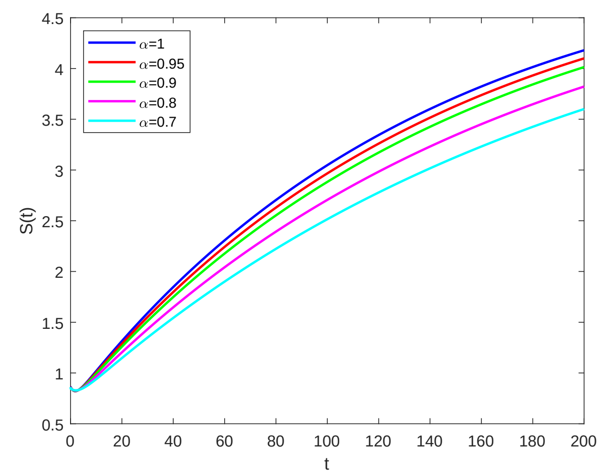
Here, we simulate the solution of system (3.1) for fractional order $\alpha = 0.95$. Moreover, we choose some values for parameters as $\Lambda = 0.0382, \beta = 0.25177, \mu = 0.0073, q = 0.6, \varepsilon = 0.01, \gamma = 0.05$. As we mentioned in Table 1, q_0 is per capita rate of quarantine of the new patients. Therefore, this parameters show the power of quarantine in the model. Here, by varying some different values of q_0, the behaviour of the model will be obtained. First, let $q_0 = 0.65$, therefore $\mathcal{R}_0 = 0.7470$ and system (3.1) has stable DFE $E_0 = (5.2329, 0, 0, 0)$. In this case, system has no any endemic equilibrium. See figure 3 (a).

Second, by choosing $q_0 = 0.25$, we have $\mathcal{R}_0 = 1.6007$ and system (3.1) has unstable DFE $E_0 = (5.2329, 0, 0, 0)$ and stable endemic $E^* = (3.2681, 0.0175, 0.2457, 1.7015)$. See figure 3 (b).

Figure 3, shows that by controlling rate of quarantine q_0 (for new patient) we can control the disease. In the other words, we can find a critical value $q_0^c = 1 - \frac{\mu(q+\mu+\varepsilon)}{\Lambda\beta}$ as a threshold. Indeed, $q_0 < q_0^c$, implies



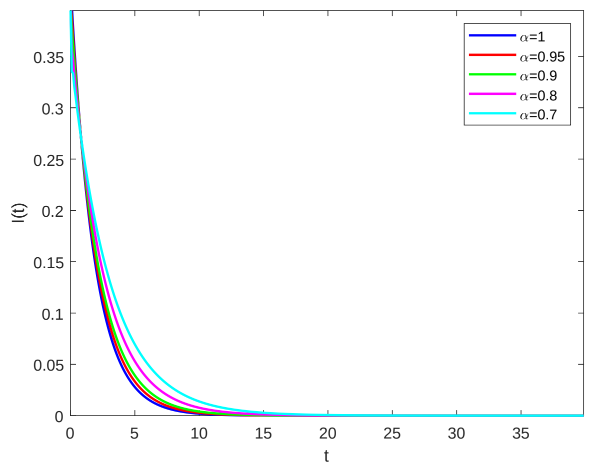
(a)



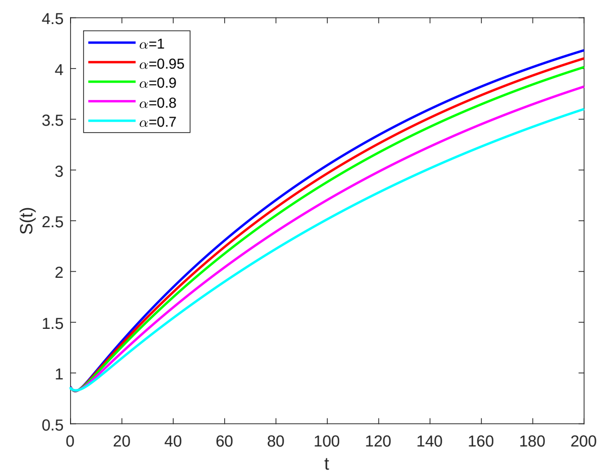
(b)

FIGURE 4. Stability of endemic equilibrium for fractional orders $\alpha = 1, 0.95, 0.9, 0.8, 0.7$ and parameters values $q_0 = 0.25, \Lambda = 0.0382,\ \beta = 0.25177, \mu = 0.0073, q = 0.6, \varepsilon = 0.01, \gamma = 0.05$.

$\mathcal{R}_0 > 1$ and the disease spreads in the community, and for $q_0 > q_0^c$, implies $\mathcal{R}_0 < 1$, then the disease will be eradicated. For the above values of parameters we have $q_0^c = 0.5315$. Figure 3 verifies this subject.



(a)



(b)

FIGURE 5. Stability of DFE for fractional orders $\alpha = 1, 0.95, 0.9, 0.8, 0.7$ and parameters values $q_0 = 0.65, \Lambda = 0.0382,\ \beta = 0.25177, \mu = 0.0073, q = 0.6, \varepsilon = 0.01, \gamma = 0.05$.

As a final result, we can find the threshold for quarantine rate q_0^c in each community (if we have other parameters), which for values greater than the q_0^c, the disease will be decrease and finally will be eradicated.

Case 2.

In this case, we fix parameters $\Lambda = 0.0382,\ \beta = 0.25177, \mu = 0.0073, q =$

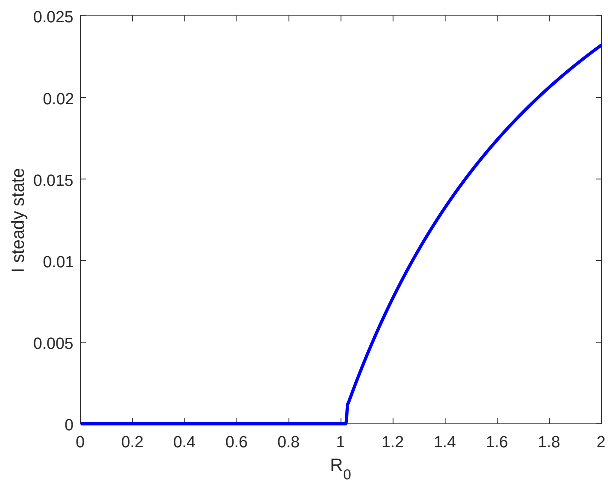


FIGURE 6. The forward bifurcation occurs in $\mathcal{R}_0 = 1$.

$0.6, \varepsilon = 0.01, \gamma = 0.05$ and illustrate the solutions for different values of fractional orders. Corresponding to the Case 1, we choose two values for q_0.

By choosing $q_0 = 0.25$, system (3.1) has an stable equilibrium. Fig. 4, shows the stability of endemic equilibrium for fractional orders $\alpha = 1, 0.95, 0.9, 0.8, 0.7$. One can see that when the value of α decreases, solutions of system (3.1) converge to the endemic equilibrium. But the peak of the disease occurs later and with a lower amount. Similarly, this happens for subsequent peaks until the disease converges to the endemic point.

Choosing $q_0 = 0.65$, system (3.1) has an stable disease-free equilibrium. Fig. 5, shows the stability of DFE for fractional orders $\alpha = 1, 0.95, 0.9, 0.8, 0.7$. In this situation, when the value of α decreases, solutions of system (3.1) converge slowly to the DFE.

Case 3.

In this situation, we illustrate the bifurcation diagram of system (3.1) with respect to basic reproduction number $\mathcal{R}_0$. Fig. 6 shows that before $\mathcal{R}_0 = 1$, the system has only stable DFE and for values $\mathcal{R}_0 > 1$, an stable endemic equilibrium appears and the DFE becomes unstable. Therefore, the forward bifurcation occurs in $\mathcal{R}_0 = 1$.

6. Conclusions

In this study, a new fractional SIQR model was proposed to investigate the dynamics of epidemics. The Caputo-Fabrizio derivative was utilized to analyze the model, with a focus on the role of quarantining newly infected individuals. The model assumes that patients could be quarantined immediately after diagnosis, before displaying symptoms, and the rate of quarantine for new patients was represented by a new parameter, q_0. Equilibria were calculated, and their dynamics were analyzed, revealing a forward bifurcation phenomenon. The existence of a solution was examined using an analytical method, and a three-step fractional Adams-Bashforth method was developed to solve fractional systems with Caputo-Fabrizio derivatives. The proposed numerical method was employed to obtain the model's solution, and the quarantine threshold, q_0^c, was determined for each community. When q_0^c is above a certain value, the disease gradually declines and is eventually eliminated.

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