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Common Fixed Point Theorems Concerning F-Contraction map in Complete Cone b-Metric Space

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Abstract. Some common fixed point theorems relating to F-contraction mapping in complete cone b-metric space are presented in this study. We generalize the results of Popescu and Stan [18] in complete metric space by extending them to complete cone b-metric space. The findings of this research are supported by an appropriate example.

Keywords: Cone metric spaces, cone b-metric spaces, F-contraction mappings, common fixed point.

2000 Mathematics subject classification: 47H10, 54H25, 46T99.

1. Introduction

As an abstract description of Picard's method of repeated approximations, S. Banach [4] established the renowned Banach contraction principle. It states that, if M is a complete metric space and T is a self mapping on M which satisfies

$$\varrho(T\nu, T\mu) \leq k\varrho(\nu, \mu), \quad \text{for all } \nu, \mu \in M \text{ and } k \in (0, 1),$$

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then T has a unique fixed point and for every $\nu_0 \in X$, a sequence $\{T^n \nu_0\}$ is converging to the fixed point. Boyd and Wong [5] generalized Banach contraction principle by replacing right hand term by a function as follows:

$$\varrho(T\nu, T\mu) \leq \psi(\varrho(\nu, \mu))$$

where ψ is some function defined on the closure of the range of ϱ. By removing the continuity assumption, relaxing the compatibility condition to the coincidentally commuting property, and substituting a set of four conditions for the completeness of metric space, Imdad *et al.* [14] enhanced a common fixed point theorem by Popa [17]. A few examples were discussed in regard to the findings. The expansion of the Banach contraction principle has led to the development of numerous fixed point theorems, many of which are not listed above.

In 2012 Wardowski [24] introduced a new type of contraction called F- contraction by using the mapping $F : R^+ \to R$ and proved a new fixed point theorem concerning F-contraction. Some examples of F-contractions which show the significance of obtained results are given in [24]. Popescu and Stan [18], generalized the results of Wardowski [24] in a complete metric space concerning F-contraction mapping. Kumar and Sholastica [16, 21] extened Wardowski results in ordered partial metric spaces. Kumar and Asim [15] further extended these results to Ordered F-contraction mappings in Ordered Metric Spaces. Recently, Wangwe and Kumar extended Wardowski results in different directions [22, 23] In this paper, we present an extension of the result of Popescu and Stan [18] to complete cone b-metric space. Some fixed point theorems and examples are provided to support the obtained results.
Throughout the article, $\mathbb{R}$ represent a set of real numbers, $\mathbb{R}^+$ reprents a set of all positive real numbers and $\mathbb{N}$ denotes a set of natural numbers. Following definitions, lemmas, and theorems are used in this article:

Huang and Zhang [9] introduced a concept of cone as a subset of Banach space by defining it as follows.

Definition 1.1. [9] A subset P of a Banach space E is called a cone if and only if it satisfies the following conditions:

- **(P1):** P is closed, nonempty and $P \neq \{0\}$.
- **(P2):** If $m, n \in \mathbb{R}$ such that $m, n \geq 0$ and $\nu, \mu \in P$ then $m\nu + n\mu \in P$.
- **(P3):** If $\nu \in P$ and $-\nu \in P \Rightarrow \nu = 0$.

Given a cone $P \subset E$, then we define a partial ordering c for P by $\nu \preceq \mu$ if and only if $\mu - \nu \in P$ and we write $\nu \ll \mu$ if $\nu - \mu \in intP$.

Huang and Zhang [9] introduced a concept of cone metric ϱ and cone metric space (X, ϱ). The following definition was given.

Definition 1.2. [9] Let X be a nonempty set, E be a real Banach space and assume that the mapping $\varrho : X \times X \to E$ satisfies the following conditions:

(d1): $0 \leq \varrho(\nu, \mu)$ and $\varrho(\nu, \mu) = 0$ if and only if $\nu = \mu$ for all ν and μ in X.

(d2): $\varrho(\nu, \mu) = \varrho(\mu, \nu)$ for all ν and μ in X.

(d3): $\varrho(\nu, \mu) \leq \varrho(\nu, \xi) + \varrho(\xi, \mu)$ for all ν, μ and ξ in X.

Then ϱ is called cone metric and (X, ϱ) is called a cone metric space.

Also, Huang and Zhang [9] introduced the concept of normal in a cone metric space. The following definition is given below:

Definition 1.3. [9] A cone P is said to be normal if there exists $M > 0$ such that $0 \leq \nu \leq \mu$ implies $\|\nu\| \leq M\|\mu\|$, $\forall \nu, \mu \in P$. The least value of M is called the normal constant.

The following definition of b-metric space was introduced by Bakhtin [3] as follows.

Definition 1.4. [3] Let X be a non empty set and $s \geq 1$ be a given real number. A function $\varrho : X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties for each $\nu, \mu, \xi \in X$

(**b1**): $\varrho(\nu, \mu) = 0 \Leftrightarrow \nu = \mu$

(**b2**): $\varrho(\nu, \mu) = \varrho(\nu, \mu)$

(**b3**): $\varrho(\nu, \xi) \leq s[\varrho(\nu, \mu) + \varrho(\mu, \xi)]$.

The pair (X, ϱ) is called b-metric space.

Hussain and Shah [13] introduced a concept of cone b-metric ϱ and cone b-metric space (X, ϱ) as follows.

Definition 1.5. [13] Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $\varrho : X \times X \to E$ is said to be cone b-metric if and only if, for all $\nu, \mu, \xi \in X$, the following conditions are satisfied:

(i) $0 \leq \varrho(\nu, \mu)$ with $\nu \neq \mu$ and $\varrho(\nu, \mu) = 0$ if and only if $\nu = \mu$;

(ii) $\varrho(\nu, \mu) = \varrho(\mu, \nu)$;

(iii) $\varrho(\nu, \mu) \leq s[\varrho(\nu, \xi) + \varrho(\xi, \mu)]$.

The pair (X, ϱ) is called a cone b-metric space.

Huang and Zhang [9] introduced the concept of Cauchy sequence in cone metric space as follows:

Definition 1.6. [9] The sequence $\{\nu_n\}$ in a cone metric space (X, ϱ) is said to be Cauchy if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $\varrho(\nu_m, \nu_n) \ll c$ for all $n, m \geq N$.

Completeness of a cone metric space was introduced by Huang and Zhang [9] which is useful for our main results.

Definition 1.7. [9] The cone metric space (X, ϱ) is said to be complete if every Cauchy sequence in X is convergent.

The concept of F-contraction mapping was introduced by Wardowski [24]. Following conditions were given to define F-contraction in metric space.

Definition 1.8. [24] Let (X, ϱ) be a metric space. A mapping $T : X \longrightarrow X$ is called an F-contraction if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(\varrho(T\nu, T\mu)) \leq F(\varrho(\nu, \mu)) \tag{1.1}$$

holds for any $\nu, \mu \in X$ with $\varrho(T\nu, T\mu) > 0$, where F is the set of all functions $F : R_+ \longrightarrow R$ satisfying the following conditions:
(F1) F is strictly increasing: $\nu < \mu \Rightarrow F(\nu) < F(\mu)$;
(F2) For each sequence $\{\alpha_n\}, n \in N$ in $\mathbb{R}^+$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$ if and only if $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$,
(**F3**) There exists $k \in (0, 1)$ such that $\lim_{n\longrightarrow\infty} (\alpha_n)^k F(\alpha_n) = 0$.

Following examples (1.1)-(1.4) satisfies the above conditions.

Example 1.9. [24] Let $F : R^+ \longrightarrow R$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies $(F1) - (F3)$, $((F3) \quad for \quad k \in (0, 1))$. Each mapping $T : X \longrightarrow X$ satisfying the above definition is F- contraction such that

$$\varrho(T\nu, T\mu) \leq e^{-t} \varrho(\nu, \mu)$$

for all $\nu, \mu \in X, T\nu \neq T\mu$. It is clear that for $\nu, \mu \in X$ such that $T\nu = T\mu$ the inequality,

$$\varrho(T\nu, T\mu) \leq e^{-t} \varrho(\nu, \mu)$$

also holds, hence T is a Banach contraction.

Example 1.10. [24] If $F(\alpha) = \ln \alpha + \alpha, \alpha > 0$ then F satisfies $(F1) - (F3)$ and the condition above is of the form

$$\frac{\varrho(T\nu, T\mu)}{\varrho(\nu, \mu)} e^{\varrho(T\nu, T\mu) - \varrho(\nu, \mu)} \leq e^{-t}$$

for all $\nu, \mu \in X, T\nu \neq T\mu$.

Example 1.11. [24] Consider $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$, $\alpha > 0$. Then, F satisfies $(F1) - (F3)$, $((F3) \quad for \quad k \in (\frac{1}{2}, 1))$. In this case, each F-contraction T satisfies

$$\varrho(T\nu, T\mu) \leq \frac{1}{(1+\tau\sqrt{\varrho(\nu, \mu)}\ )^2} \varrho(\nu, \mu) \text{ for all } \nu, \mu \in X, T\nu \neq T\mu.$$

Here, we obtain ϱ, a special case of nonlinear contraction of the type

$$\varrho(T\nu, T\mu) \leq \alpha(\varrho(\nu, \mu))\varrho(\nu, \mu).$$

Example 1.12. [24] Let $F(\alpha) = \ln(\alpha^2+\alpha), \alpha > 0$. Obviously F satisfies $(F1)-(F3)$ and for F- contraction T, the following conditon hold,

$$\frac{\varrho(T\nu, T\mu)(\varrho(T\nu, T\mu)+1)}{\varrho(\nu, \mu)(\varrho(\nu, \mu)+1)} \leq e^{-t}$$

for all $\nu, \mu \in X, T\nu \neq T\mu$.
As observed above from Example 1.1-1.4 the contraction conditions are satisfied for all $\nu, \mu \in X$ such that $T\nu = T\mu$.

The following remarks were introduced by Wardowski [24] in relaltion to F-contraction mapping.

Remark 1.13. [24] From $(F1)$ and $(F2)$ it is easy to conclude that every F-contraction T is a contractive mapping, i.e $\varrho(T\nu, T\mu) < \varrho(\nu, \mu)$, for all $\nu, \mu \in X, T\nu = T\mu$. Thus every F-contraction is a continuous mapping. From $(F1)$ and $(F2)$ it is easy to conclude that every F-contraction T is a contractive mapping.

Remark 1.14. [24] Let F_1, F_2 be the mappings satisfying $(F1)-(F3)$. If $F_1(\alpha) < F_2(\alpha)$ for all $\alpha > 0$ and a mappings $G = F_2 - F_1$ is non decreasing, then every F_1-contraction T is F_2-contraction.
Indeed, from Remark (1.1) we have

$$G(\varrho(T\nu, T\mu)) \leq G(\varrho(\nu, \mu))$$

for all $\nu, \mu \in X, T\nu \neq T\mu$. Thus, we obtain

$$\begin{aligned}\tau + F_2(\varrho(T\nu, T\mu)) &= \tau + F_1(\varrho(T\nu, T\mu)) + G(\varrho(T\nu, T\mu)) \\ &< F_1(\varrho(\nu, \mu)) + G(\varrho(\nu, \mu)).\end{aligned}$$

Cosentino [6] defined F-contraction of Hardy-Rogers type of contraction as follows.

Definition 1.15. [6] Let (X, ϱ) be a metric space. A mapping $T : X \longrightarrow X$ is called an F-contraction of Hardy-Rogers-type if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that:

$$\tau+F(\varrho(T\nu, T\mu)) \leq F(\alpha\varrho(\nu, \mu)+\beta\varrho(\nu, T\nu)+\gamma\varrho(\mu, T\mu)+\delta\varrho(\nu, T\mu)+L\varrho(\mu, T\nu))$$

holds for any $\nu, \mu \in X$ with $\varrho(T\nu, T\mu) > 0$, where $\alpha, \beta, \gamma, \delta, L$ are non-negative numbers,$\gamma \neq 1$ and $\alpha+\beta+\gamma+2\delta = 1$.

The following theorems are due to Huang and Zhang [9] where by the existence and uniqueness of a fixed point in a cone metric space is determined. These theorems state as follows.

Theorem 1.16. [9] *Let* (X, ϱ) *be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping* $T : X \to X$ *satisfies the contractive condition*

$$\varrho(T\nu, T\mu) \preceq k\varrho(\nu, \mu), \forall \nu, \mu \in X,$$

where $k \in [0, 1)$ *is a constant. Then, T has a unique fixed point in X. Moreover, for any* $\nu \in X$*, iterative sequence* $\{T^n\nu\}$ *converges to the fixed point.*

Theorem 1.17. [9] *Let* (X, ϱ) *be a complete cone metric space, P a normal cone with normal constant K. Suppose the mapping* $T : X \to X$ *satisfies the contractive condition*

$$\varrho(T\nu, T\mu) \quad \preceq \quad k\{\varrho(T\nu, \nu) + \varrho(T\mu, \mu)\}, \forall \nu, \mu \in X,$$

where $k \in [0, 1/2)$ *is a constant. Then T has a unique fixed point in X. Moreover, for any* $\nu \in X$*, iterative sequence* $\{T^n\nu\}$ *converges to the fixed point.*

Huang *et al.* [12] introduced the following lemma in cone metric space to prove that the given sequence is a Cauchy sequence in a cone metric space.

Lemma 1.18. [12] *Let* (X, ϱ) *be a cone metric space, P be a normal cone with a normal constant K. Let* $\{\nu_n\}$ *be a sequence in X. Then* $\{\nu_n\}$ *is a Cauchy sequence if and only if* $\varrho(\nu_n, \nu_m) \to 0$ *for* $n, m \to \infty$.

The following theorem is due to Popescu and Stan [18] in metric space.

Theorem 1.19. *Let T be a self mapping of a metric space into itself. Suppose there exist* $\tau > 0$ *such that for all* $\nu, \mu \in X$,

$$\begin{aligned} &\varrho(T\nu, T\mu) > 0 \Rightarrow \tau + F(\varrho(T\nu, T\mu)) \\ &\leq F(\alpha\varrho(\nu, \mu) + \beta\varrho(\nu, T\nu) + \gamma\varrho(\mu, T\mu) + \delta\varrho(\nu, T\mu) + L\varrho(\mu, T\nu)) \end{aligned}$$

where $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is an increasing map,* $\alpha, \beta, \gamma, \delta, L$ *are non negative numbers,*

$$\delta < \frac{1}{2}, \gamma < 1, \alpha + \beta + \gamma + 2\delta = 1, 0 < \alpha + \gamma + L \leq 1.$$

Then T has unique fixed point $\nu^\star \in X$ *and for all* $\nu \in X$ *the sequence* $\{T^n\nu\}_{n\in N}$ *converge to* $\nu^\star$.

Motivated by Popescu and Stan [18], we present our results by extending Theorem 1.19 to complete cone b-metric space. An example is given to support the results obtained.

Theorem 1.20. *Let $(G,.)$ be an Abelian group with identity and let $f, m : G \to \mathbb{C}$ be functions such that there exist functions $M_1, M_2 :\to [0,\infty)$ with*

$$\|f(x.y) - f(x)m(y)\| \leq \min\{M_1(x), M_2(y)\}$$

for all $x, y \in G$. Then either f is bounded or m is an exponential and $f(x) = f(1)g(x)$ for all $x \in G$.

For the reader's convenience and explicit later use, we will recall a fundamental results in fixed point theory.

Definition 1.21. The pair (X, d) is called a generalized complete metric space if X is a nonempty set and $d : X^2 \to [0,\infty]$ satisfies the following conditions:

(1) $d(x, y) \geq 0$ and the equality holds if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$;
(4) every d-Cauchy sequence in X is d-convergent.

for all $x, y \in X$.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

2. Main result

We present an extension of result due to Popescu and Stan *et al.* [18] to complete cone b-metric space and prove common fixed point theorems concerning F - contraction condition in complete cone *b*-metric space.

Theorem 2.1. *Let T be a self mapping of complete cone b-metric space with $S \geq 1$, $S^2\ell < 1$ and $S\ell < 1$. Suppose there exists $\tau > 0$ such that for all $\nu, \mu \in X$,*

$$\begin{aligned} &\varrho(T\nu, T\mu) > 0 \Rightarrow \tau + F(\varrho(T\nu, T\mu)) \\ &\leq F(\ell_1\varrho(\nu,\mu) + \ell_2\varrho(\nu, T\nu) + \ell_3\varrho(\mu, T\mu) + \ell_4\varrho(\nu, T\mu) + \ell_5\varrho(\mu, T\nu)). \end{aligned} \tag{2.1}$$

where $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is an increasing map, $\ell_1, \ell_2, , \ell_3, \ell_4, \ell_5$ are non negative numbers,

$$\ell_4 < \frac{1}{2}, \quad \ell_3 < 1, \quad \ell_1 + \ell_2 + \ell_3 + 2\ell_4 = 1, \quad 0 < \ell_1 + \ell_3 + \ell_5 \leq 1.$$

Then T has unique fixed point $\nu^\star \in X$ and for all $\nu \in X$ the sequence $\{T^n\nu\}_{n\in N}$ converge to $\nu^\star$.

Proof. Suppose $\nu_0 \in X$ is an arbitrary point in X and let us construct a sequence $\{\nu_n\} \in X$ such that,

$$\begin{array}{rcl} \nu_1 & = & T\nu_0 \\ \nu_2 & = & T\nu_1 = T^2\nu_0, \\ .... & ..... & .... \\ \nu_n & = & T\nu_{n-1} = T^n\nu_0, \end{array}$$

for all $n \in \mathbb{N}$. If there exist $n \in \mathbb{N}$ such that $\varrho(\nu_n, T\nu_n) = 0$, then ν_n is a fixed point of T. Suppose,

$$0 < \varrho(\nu_n, T\nu_n) = \varrho(T\nu_{n-1}, T\nu_n)$$

for all $n \in \mathbb{N}$. Also let $d_n = \varrho(\nu_n, \nu_{n+1})$, hence by using the hypothesis of monotony of f, we have,

$$\begin{array}{rcl} \tau + F(d_n) & = & \tau + F(\varrho(\nu_n, \nu_{n+1})) \\ & = & \tau + F(\varrho(T\nu_{n-1}, T\nu_n)) \\ & \leq & F(\ell_1\varrho(\nu_{n-1}, \nu_n) + \ell_2\varrho(\nu_{n-1}, T\nu_{n-1}) + \ell_3\varrho(\nu_n, T\nu_n) \\ & & +\ell_4\varrho(\nu_{n-1}, T\nu_n) + \ell_5\varrho(\nu_n, T\nu_{n-1})) \\ & = & F(\ell_1\varrho(\nu_{n-1}, \nu_n) + \ell_2\varrho(\nu_{n-1}, \nu_n) + \ell_3\varrho(\nu_n, \nu_{n+1}) \\ & & +\ell_4\varrho(\nu_{n-1}, \nu_{n+1}) + \ell_5\varrho(\nu_n, \nu_n)). \end{array}$$

Applying condition (iii) of Definition (1.5)

$$\begin{array}{rl} = & F(\ell_1\varrho(\nu_{n-1}, \nu_n) + \ell_2\varrho(\nu_{n-1}, \nu_n) + \ell_3\varrho(\nu_n, \nu_{n+1}) \\ & +S\{\ell_4\varrho(\nu_{n-1}, \nu_n) + \ell_4\varrho(\nu_n, \nu_{n+1})\}) \\ = & F(\ell_1 d_{n-1} + \ell_2 d_{n-1} + \ell_3 d_n + S\ell_4\{d_{n-1} + d_n\}) \\ \leq & F(\ell_1 d_{n-1} + \ell_2 d_{n-1} + \ell_3 d_n + S\ell_4 d_{n-1} + S\ell_4 d_n) \\ = & F((\ell_1 + \ell_2 + S\ell_4)d_{n-1} + (\ell_3 + S\ell_4)d_n). \end{array}$$

which follows that:

$$\begin{array}{rcl} F(d_n) & \leq & F((\ell_1 + \ell_2 + S\ell_4)d_{n-1} + (\ell_3 + S\ell_4)d_n) - \tau \\ & < & F((\ell_1 + \ell_2 + S\ell_4)d_{n-1} + (\ell_3 + S\ell_4)d_n). \end{array} \tag{2.2}$$

By the monotony of F we have,

$$d_n < (\ell_1 + \ell_2 + S\ell_4)d_{n-1} + (\ell_3 + S\ell_4)d_n$$

which gives,

$$(1 - \ell_3 - S\ell_4)d_n < (\ell_1 + \ell_2 + S\ell_4)d_{n-1}$$

for all $n \in N$, since $\ell_3 \neq 1$ and $\ell_1 + \ell_2 + \ell_3 + 2\ell_4 = 1$. We deduce the following

$$1 - \ell_3 - S\ell_4 > 0.$$

Hence,

$$d_n < \frac{(\ell_1 + \ell_1 + S\ell_4)}{(1 - \ell_3 - S\ell_4)} d_{n-1}$$

for all $n \in \mathbb{N}$.

From the above, we deduce that $\{d_n\}_{n \in N}$ is strictly decreasing, hence there exists $\lim\limits_{n \longrightarrow \infty} d_n = d$. Let $d > 0$. Given that F is an increasing mapping then,

$$\lim_{x \longrightarrow d^+} F(\nu_n) = F(d). \tag{2.3}$$

Taking $n \longrightarrow \infty$ in (2.2) we get,

$$F(d) < F(d) - \tau,$$

which is a contradiction.

Thus we have,

$$\lim_{n \longrightarrow \infty} d_n = 0.$$

We claim that $\{\nu_n\}$ is a Cauchy sequence. Hence by contradiction, we assume that there exists $\epsilon > 0$ and a sequence $\{p(n)\}$ and $\{q(n)\}$ of the natural numbers so that,

$$p(n) > q(n) > n, \varrho(\nu_{p(n)}, \nu_{q(n)}) > \epsilon, \varrho(\nu_{p(n)-1}, \nu_{q(n)}) \leq \epsilon$$

for all $n \in N$.

Hence,

$$\epsilon < \varrho(\nu_{p(n)}, \nu_{q(n)}) \leq \varrho(\nu_{p(n)}, \nu_{p(n)-1}) + \varrho(\nu_{p(n)-1}, \nu_{q(n)}) \leq \varrho(\nu_{p(n)-1}, \nu_{p(n)}). \tag{2.4}$$

Using 2.4 above, we get

$$\lim_{n \longrightarrow \infty} \varrho(\nu_{p(n)}, \nu_{q(n)}) = \epsilon.$$

But from $\varrho(\nu_{p(n)}, \nu_{q(n)}) > \epsilon > 0$, then using the monotony of F we get,

$$\begin{aligned} \tau + \varrho(\nu_{p(n)}, \nu_{q(n)}) &\leq F(\ell_1 \varrho(\nu_{p(n)-1}, \nu_{q(n)-1}) + \ell_2 \varrho(\nu_{p(n)-1}, T\nu_{p(n)-1}) \\ &\quad + \ell_3 \varrho(\nu_{q(n)-1}, T\nu_{q(n)-1}) + \ell_4 \varrho(\nu_{p(n)-1}, T\nu_{q(n)-1}) \\ &\quad + \ell_5 \varrho(\nu_{q(n)-1}, T\nu_{p(n)-1})) \\ &= F(\ell_1 \varrho(\nu_{p(n)-1}, \nu_{q(n)-1}) + \ell_2 \varrho(\nu_{p(n)-1}, \nu_{p(n)}) \\ &\quad + \ell_3 \varrho(\nu_{q(n)-1}, \nu_{q(n)}) + \ell_4 \varrho(\nu_{p(n)-1}, \nu_{q(n)}) \\ &\quad + \ell_5 \varrho(\nu_{q(n)-1}, \nu_{p(n)})). \end{aligned}$$

Applying condition (iii) of Definition (1.5) gives,

$$\begin{aligned}
&\leq F(S\ell_1[\varrho(\nu_{p(n)-1},\nu_{p(n)})+\varrho(\nu_{p(n)},\nu_{q(n)-1})]+\ell_2\varrho(\nu_{p(n)-1},\nu_{p(n)})\\
&\quad +\ell_3\varrho(\nu_{q(n)-1},\nu_{q(n)})+\ell_4\varrho(\nu_{p(n)},\nu_{q(n)})+\ell_5\varrho(\nu_{q(n)-1},\nu_{p(n)}))\\
&\leq F(S\ell_1\varrho(\nu_{p(n)-1},\nu_{p(n)})+S\ell_1\varrho(\nu_{p(n)},\nu_{q(n)-1})+\ell_2\varrho(\nu_{p(n)-1},\nu_{p(n)})\\
&\quad +\ell_3\varrho(\nu_{q(n)-1},\nu_{q(n)})+\ell_4\varrho(\nu_{p(n)},\nu_{q(n)})+\ell_5\varrho(\nu_{q(n)-1},\nu_{p(n)}))\\
&\leq F(S\ell_1\varrho(\nu_{p(n)-1},\nu_{p(n)})+S[S\{\ell_1\varrho(\nu_{p(n)},\nu_{q(n)})+\ell_1\varrho(\nu_{q(n)},\nu_{q(n)-1})\}]\\
&\quad +\ell_2\varrho(\nu_{p(n)-1},\nu_{p(n)})+\ell_3\varrho(\nu_{q(n)-1},\nu_{q(n)})+\ell_4\varrho(\nu_{p(n)},\nu_{q(n)})+\ell_5\varrho(\nu_{q(n)-1},\nu_{p(n)}))\\
&\leq F(S\ell_1\varrho(\nu_{p(n)-1},\nu_{p(n)})+S^2\ell_1\varrho(\nu_{p(n)},\nu_{q(n)})+S^2\ell_1\varrho(\nu_{q(n)},\nu_{q(n)-1})\\
&\quad +\ell_2\varrho(\nu_{p(n)-1},\nu_{p(n)})+\ell_3\varrho(\nu_{q(n)-1},\nu_{q(n)})+S[\ell_4\varrho(\nu_{p(n)-1},\nu_{p(n)})+\ell_4\varrho(\nu_{p(n)},\nu_{q(n)})]\\
&\quad +S[\ell_5\varrho(\nu_{q(n)-1},\nu_{q(n)})+\ell_5\varrho(\nu_{q(n)},\nu_{p(n)})])\\
&\leq F(S\ell_1\varrho(\nu_{p(n)-1},\nu_{p(n)})+S^2\ell_1\varrho(\nu_{p(n)},\nu_{q(n)})+S^2\ell_1\varrho(\nu_{q(n)},\nu_{q(n)-1})\\
&\quad +\ell_2\varrho(\nu_{p(n)-1},\nu_{p(n)})+\ell_3\varrho(\nu_{q(n)-1},\nu_{q(n)})+S\ell_4\varrho(\nu_{p(n)-1},\nu_{p(n)})+S\ell_4\varrho(\nu_{p(n)},\nu_{q(n)})\\
&\quad +S\ell_5\varrho(\nu_{q(n)-1},\nu_{q(n)})+S\ell_5\varrho(\nu_{q(n)},\nu_{p(n)}))
\end{aligned}$$

Applying the limit as $n \longrightarrow \infty$ we get,

$$\tau+F(\epsilon)<F[(S^2\ell_1+S\ell_4+S\ell_5)\epsilon],$$

which is a contradiction.
Hence $\{\nu_n\}$ is a Cauchy sequence in X.
Since X is complete cone b-metric space, then the sequence $\{\nu_n\}$ converge to $\nu^\star \in X$.
Suppose that there exists a sequence $\{p(n)\}$ of natural numbers N such that,

$$\nu_{p(n)+1}=T\nu_{p(n)}=T\nu^\star$$

then we get $\lim_{n\longrightarrow\infty}\nu_{p(n)+1}=\nu^\star$, which gives $T\nu^\star=\nu^\star$ or there exists $n\in\mathbb{N}$ such that $\nu_{n+1}=T\nu_n\neq T\nu^\star$ for all $n\in\mathbb{N}$.
Now let $T\nu^\star\neq\nu^\star$ hence,

$$\begin{aligned}
\tau+F(\varrho(T\nu_n,T\nu^\star)) &\leq F(\ell_1\varrho(\nu_n,\nu^\star)+\ell_2\varrho(\nu_n,T\nu_n)+\ell_3\varrho(\nu^\star,T\nu^\star)\\
&\quad +\ell_4\varrho(\nu_n,T\nu^\star)+\ell_5\varrho(\nu^\star,T\nu_n)),\\
&= F(\ell_1\varrho(\nu_n,\nu^\star)+\ell_2\varrho(\nu_n,\nu_{n+1})+\ell_3\varrho(\nu^\star,T\nu^\star)+\ell_4\varrho(\nu_n,T\nu^\star)\\
&\quad +\ell_5\varrho(\nu^\star,\nu_{n+1})).
\end{aligned}$$

But F is an increasing function, so we deduce that:

$$\begin{aligned}
\varrho(T\nu_n,T\nu^\star) &< \ell_1\varrho(\nu_n,\nu^\star)+\ell_2\varrho(\nu_n,\nu_{n+1})+\ell_3\varrho(\nu^\star,T\nu^\star)\\
&\quad +\ell_4\varrho(\nu_n,T\nu^\star)+\ell_5\varrho(\nu_{n+1},\nu^\star)
\end{aligned}$$

as $n\longrightarrow\infty$, we have,

$$\varrho(\nu^\star,T\nu^\star)<\ell_3\varrho(\nu^\star,T\nu^\star)+\ell_4\varrho(\nu^\star,T\nu^\star)<\ell_3\varrho(\nu^\star,T\nu^\star).$$

This is a contradiction, hence $T\nu^\star = \nu^\star$ is a fixed point.
Now we will show that T has a unique fixed point in X.
Suppose that $\nu, \mu \in X$ are two fixed points of $T \in X$ such that $T\nu = \nu$ and $T\mu = \mu$.
We have,

$$\varrho(T\nu, T\mu) = \varrho(\nu, \mu) > 0.$$

From the condition of Theorem (2.1) we have,

$$\begin{aligned} \tau + F(\varrho(\nu,\mu)) &= \tau + F(\varrho(T\nu, T\mu)) \\ &\leq F(\ell_1\varrho(\nu,\mu) + \ell_2\varrho(\nu, T\nu) + \ell_3\varrho(\mu, T\mu) \\ &\quad +\ell_4\varrho(\nu, T\mu) + \ell_5\varrho(\mu, T\nu)), \\ &= F(\ell_1\varrho(\nu,\mu) + \ell_4\varrho(\nu,\mu) + \ell_5\varrho(\nu,\mu)) \\ &\leq F[(\ell_1 + \ell_4 + \ell_5)\varrho(\nu,\mu)]. \end{aligned}$$

But $0 < \ell_1 + \ell_4 + \ell_5 \leq 1$. Thus,

$$\tau + F(\varrho(T\nu, T\mu) \leq F(\varrho(\nu,\mu))$$

which is a contradiction.
Therefore, T has a unique fixed point in X. □

Following is an alternative technique for proving the existence and uniqueness of fixed point as per Theorem 2.1 by considering the continuous mapping T in a complete cone b-metric space.

Theorem 2.2. *Let (X, ϱ) be a complete cone b-metric space and let $T : X \longrightarrow X$ be an F-contraction mapping with $S \geq 1$. If T is a continuous mapping then it has has a unique fixed point $\nu \in X$ and for all $\nu_0 \in X$ the sequence $\{T^n\nu_0\}$ converge to ν.*

Proof. Suppose T has a unique fixed point in X. If $\nu_1, \nu_2 \in X$ such that,

$$T\nu_1 = \nu_1 \neq \nu_2 = T\nu_2$$

we observe that,

$$\tau \leq F(\varrho(\nu_1, \nu_2)) - F(\varrho(T\nu_1, T\nu_2)) = 0$$

which contradict the assumption. Thus, to show that T has a unique fixed point. Let ν_0 be an arbitrary fixed point in the space X. By defining a sequence $\{\nu_n\} \subset X$ as follows:

$$\begin{aligned} &\nu_1 = T\nu_0, \\ &\nu_2 = T\nu_1, \\ &\nu_3 = T\nu_2, \\ &\ldots\ldots\ldots. \\ &\nu_{n+1} = T\nu_n. \end{aligned}$$

Assuming that there exists $n \in \mathbb{N}$ such that,

$$\nu_{n+1} = \nu_n$$

implies that,

$$T\nu_n = \nu_n.$$

Hence, T has a unique fixed point.
Now, let $\nu_{n+1} \neq \nu_n$ and for every $n \in N$ then

$$\varrho(\nu_{n+1}, \nu_n) > 0$$

for all $n \in N$. Considering (1.1) then

$$\begin{aligned} F(\varrho(\nu_{n+1}, \nu_n)) &< F(\varrho(\nu_n, \nu_{n-1})) - \tau \\ &\leq F(\varrho(\nu_{n-1}, \nu_{n-2})) - 2\tau \\ &\leq .... \\ &\leq F(\varrho(\nu_n, \nu_0)) - n\tau. \end{aligned}$$

From (2.5) above we get,

$$\lim_{n \longrightarrow \infty} F(\varrho(\nu_{n+1}, \nu_n)) = -\infty.$$

Hence, considering Definition(1.8) we have

$$\lim_{n \longrightarrow \infty} \varrho(\nu_{n+1}, \nu_n) = 0. \tag{2.5}$$

Therefore, there exists $k \in (0,1)$ such that

$$\lim_{n \longrightarrow \infty} (\varrho(\nu_{n+1}, \nu_n))^k F(\varrho(\nu_{n+1}, \nu_n)) = 0. \tag{2.6}$$

Then,

$$\begin{aligned} &[\varrho(\nu_{n+1}, \nu_n)]^k F(\varrho(\nu_{n+1}, \nu_n)) - [\varrho(\nu_{n+1}, \nu_n)]^k F(\varrho(\nu_n, \nu_0)) \\ &- n\tau - [\varrho(\nu_{n+1}, \nu_n)]^k F(\varrho(\nu_n, \nu_0)) = -[\varrho(\nu_{n+1}, \nu_n)]^k n\tau \leq 0 \end{aligned}$$

as $n \longrightarrow \infty$.
Thus,

$$\lim_{n \longrightarrow \infty} [\varrho(\nu_{n+1}, \nu_n)]^k = 0. \tag{2.7}$$

From the above Equation (2.7) we can see that there exists $n \in N$ such that,

$$[n\varrho(\nu_{n+1}, \nu_n)]^k \leq 1$$

for all $n \geq n_1$. Hence,

$$\varrho(\nu_{n+1}, \nu_n) \leq \frac{1}{n^{\frac{1}{k}}}, \tag{2.8}$$

for all $n \geq n_1$.

Therefore, to show that $\{\nu_n\}$ is a Cauchy sequence, let us consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Thus from the definition of cone b-metric space and (2.8) we have,

$$\begin{aligned} \varrho(\nu_m, \nu_n) &\leq S[\varrho(\nu_m, \nu_{m-1}) + \varrho(\nu_{m-1}, \nu_{m-2}) + ... + \varrho(\nu_{m-n}, \nu_n)] \\ &\leq S\sum_{i=n}^{\infty} \varrho(\nu_{i+1}, \nu_i) \\ &\leq S\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Hence from the definition of convergence of series,

$$S\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Then the sequence $\{\nu_n\}$ is a Cauchy sequence in X. But X is a complete space, then there exists $\nu \in X$ such that,

$$\lim_{n \longrightarrow \infty} \nu_n = \nu.$$

From the continuity of T we get the following,

$$\varrho(T\nu, \nu) = \lim_{n \longrightarrow \infty} \varrho(T\nu_m, T\nu_n) = \lim_{n \longrightarrow \infty} \varrho(\nu_{n+1}, \nu_n) = 0,$$

hence $T\nu = \nu$. Therefore, T has a unique fixed point in a complete cone b-metric space.

□

From Theorem 2.1 we now use pair of maps to prove the existance and uniqueness of fixed point in a complete cone b-metric space concerning F-contraction mapping.
The following theorem use pair of maps to prove that there exists a common fixed point in a complete cone b-metric space.

Theorem 2.3. *Let (S, T) be a pair of maps in a complete cone b- metric space (X, ϱ) with $\alpha \geq 1$, $\alpha^2\ell < 1$ and $\alpha\ell < 1$. Suppose there exists $\tau > 0$ for all $\nu, \mu \in X$,*

$$\begin{aligned} \varrho(TS\nu, TS\mu) &> 0 \Rightarrow \tau + F(\varrho(TS\nu, TS\mu)) \\ &\leq F(\ell_1\varrho(S\nu, S\mu) + \ell_2\varrho(S\nu, TS\nu) + \ell_3\varrho(S\mu, TS\mu) \\ &\quad +\ell_4\varrho(S\nu, TS\mu) + \ell_5\varrho(S\mu, TS\nu)), \end{aligned}$$

where $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is an increasing map and $\ell_1, \ell_2, , \ell_3, \ell_4, \ell_5$ are non negative numbers,

$$\ell_4 < \frac{1}{2}, \quad \ell_3 < 1, \quad \ell_1 + \ell_2 + \ell_3 + 2\ell_4 = 1, \quad 0 < \ell_1 + \ell_3 + \ell_5 \leq 1.$$

Then S and T have unique fixed point $\nu^\star \in X$.

Proof. Let $\nu_0 \in X$ be arbitrary and a sequence $\{\nu_n\}_{n\in\mathbb{N}} \in X$ satisfies:

$$\begin{aligned}
&TS\nu_0 = S\nu_1,\\
&TS\nu_1 = S\nu_2,\\
&..........,\\
&TS\nu_n = S\nu_{n+1}.
\end{aligned}$$

Now, let $TS\nu_n = \nu_n = S\nu_n$, for all $n \in \mathbb{N}$. Hence, there exists $n \in \mathbb{N}$ such that;

$$\varrho(\nu_n, TS\nu_n) = \varrho(\nu_n, S\nu_n) = 0.$$

Therefore, this proves that ν_n is a fixed point of T and S.
Now, assume that,

$$0 < \varrho(S\nu_n, TS\nu_n) = \varrho(TS\nu_{n-1}, TS\nu_n), \forall n \in \mathbb{N}.$$

Now, let $d_n = \varrho(S\nu_n, S\nu_{n+1})$. By the monotony of F, we have;

$$\begin{aligned}
\tau + F(\varrho(S\nu_n, S\nu_{n+1}) &= \tau + F(\varrho(TS\nu_{n-1}, TS\nu_n))\\
&\leq F(\ell_1\varrho(S\nu_{n-1}, S\nu_n) + \ell_2\varrho(S\nu_{n-1}, TS\nu_{n-1}) + \ell_3\varrho(S\nu_n, TS\nu_n)\\
&\quad +\ell_4\varrho(S\nu_{n-1}, TS\nu_n) + \ell_5\varrho(S\nu_n, TS\nu_{n-1}))\\
&= F(\ell_1\varrho(S\nu_{n-1}, S\nu_n) + \ell_2\varrho(S\nu_{n-1}, S\nu_n) + \ell_3\varrho(S\nu_n, S\nu_{n+1})\\
&\quad +\ell_4\varrho(S\nu_{n-1}, S\nu_{n+1}) + \ell_5\varrho(S\nu_n, S\nu_n))
\end{aligned}$$

$$\begin{aligned}
&\leq F(\ell_1\varrho(S\nu_{n-1}, S\nu_n) + \ell_2\varrho(S\nu_{n-1}, S\nu_n) + \alpha[\ell_3\varrho(S\nu_n, S\nu_{n-1})\\
&\quad +\ell_3\varrho(S\nu_{n-1}, S\nu_{n+1})] + \ell_4\varrho(S\nu_{n-1}, S\nu_{n+1}))\\
&\leq F((\ell_1\varrho(S\nu_{n-1}, S\nu_n) + \ell_2\varrho(S\nu_{n-1}, S\nu_n) + \alpha[\ell_3\varrho(S\nu_n, S\nu_{n-1})\\
&\quad +\alpha[\ell_3\varrho(S\nu_{n-1}, S\nu_n) + \ell_3\varrho(S\nu_n, S\nu_{n+1})]] + \alpha[\ell_4\varrho(S\nu_{n-1}, S\nu_n)\\
&\quad +\ell_4\varrho(S\nu_n, S\nu_{n+1})])\\
&= F(\ell_1\varrho(S\nu_{n-1}, S\nu_n) + \ell_2\varrho(S\nu_{n-1}, S\nu_n) + \alpha\ell_3\varrho(S\nu_{n-1}, S\nu_n)\\
&\quad +\alpha^2\ell_3\varrho(S\nu_{n-1}, S\nu_n) + \alpha^2\ell_3\varrho(S\nu_n, S\nu_{n+1}) + \alpha\ell_4\varrho(S\nu_{n-1}, S\nu_n)\\
&\quad +\alpha\ell_4\varrho(S\nu_n, S\nu_{n+1}))\\
&= F((\ell_1 + \ell_2 + \alpha\ell_3 + \alpha^2\ell_3 + \alpha\ell_4)\varrho(S\nu_{n-1}, S\nu_n)\\
&\quad +(\alpha^2\ell_3 + \alpha\ell_4)\varrho(S\nu_n, S\nu_{n+1}))\\
&= F((\ell_1 + \ell_2 + \alpha\ell_3 + \alpha^2\ell_3 + \alpha\ell_4)d_{n-1} + (\alpha^2\ell_3 + \alpha\ell_4)d_n).
\end{aligned}$$

Using the assumption that $d_n = \varrho(S\nu_n, S\nu_{n+1})$ we get,

$$\begin{aligned}
F(d_n) &\leq F((\ell_1 + \ell_2 + \alpha\ell_3 + \alpha^2\ell_3 + \alpha\ell_4)d_{n-1} + (\alpha^2\ell_3 + \alpha\ell_4)d_n) - \tau\\
&< F((\ell_1 + \ell_2 + \alpha\ell_3 + \alpha^2\ell_3 + \alpha\ell_4)d_{n-1} + (\alpha^2\ell_3 + \alpha\ell_4)d_n)
\end{aligned}$$

By the monotony of F we get,

$$d_n < (\ell_1 + \ell_2 + \alpha\ell_3 + \alpha^2\ell_3 + \alpha\ell_4)d_{n-1} + (\alpha^2\ell_3 + \alpha\ell_4)d_n$$

Thus,

$$(1-\alpha^2\ell_3-\alpha\ell_4)d_n < (\ell_1+\ell_2+\alpha\ell_3+\alpha^2\ell_3+\alpha\ell_4)d_{n-1}$$

for all $n\in\mathbb{N}$.
But $\ell_3\neq 1$ and $\ell_4\neq 1$ therefore we obtain,

$$1-\alpha^2\ell_3-\alpha\ell_4>0$$

and so,

$$d_n < \frac{\ell_1+\ell_2+\alpha\ell_3+\alpha^2\ell_3+\alpha\ell_4}{1-\alpha^2\ell_3-\alpha\ell_4}$$

for all $n\in\mathbb{N}$.
As observed above we deduce that the sequence $\{d_n\}$ is an increasing, so there exists $\lim\limits_{n\longrightarrow\infty} d_n = d$. Suppose that $d>0$ then there exists $\lim\limits_{\nu\longrightarrow d^+} F(\nu)=F(d+0)$. Taking $n\longrightarrow\infty$, we get

$$F(d+0)\leq F(d+0)-\tau.$$

Which is a contradiction. Hence $\lim\limits_{n\longrightarrow\infty} d_n = 0$.
Claiming that a sequence $\{\nu_n\}$ is a Cauchy sequence, let us assume that there exists $\epsilon>0$ and sequances $\{p(n)\}$ and $\{q(n)\}$ of natural numbers such that

$$\begin{aligned}&p(n)>q(n)>n,\\&\varrho(\nu_{p(n)},\nu_{q(n)})>\epsilon,\quad \varrho(\nu_{p(n)-1},\nu_{q(n)})\leq\epsilon\end{aligned}$$

for all $n\in\mathbb{N}$.
Hence,

$$\begin{aligned}\epsilon &< \varrho(\nu_{p(n)},\nu_{q(n)})\\ &\leq \varrho(\nu_{p(n)-1},\nu_{q(n)})+\varrho(\nu_{p(n)-1},\nu_{q(n)})\\ &\leq \varrho(\nu_{p(n)-1},\nu_{p(n)})-\epsilon,\end{aligned}$$

which follows that,

$$\lim_{n\longrightarrow\infty}\varrho(\nu_{p(n)},\nu_{q(n)})=\epsilon,$$

from

$$\varrho(\nu_{p(n)},\nu_{q(n)})>\epsilon>0.$$

Hence, by considering the monotony of F, we obtain the following;

$$\begin{aligned}\tau+F(\varrho(S\nu_{p(n)},S\nu_{q(n)})) &= \tau+F(TS\nu_{p(n)-1},TS\nu_{q(n)-1})\\ &\leq F(\ell_1\varrho(S\nu_{p(n)-1},S\nu_{q(n)-1})+\ell_2\varrho(S\nu_{p(n)-1},TS\nu_{p(n)-1})\\ &\quad+\ell_3\varrho(S\nu_{q(n)-1},TS\nu_{q(n)-1})+\ell_4\varrho(S\nu_{p(n)-1},TS\nu_{q(n)-1})\\ &\quad+\ell_5\varrho(S\nu_{q(n)-1},TS\nu_{p(n)-1}))\\ &= F(\ell_1\varrho(S\nu_{p(n)-1},S\nu_{q(n)-1})+\ell_2\varrho(S\nu_{p(n)-1},S\nu_{p(n)})\\ &\quad+\ell_3\varrho(S\nu_{q(n)-1},S\nu_{q(n)})+\ell_4\varrho(S\nu_{p(n)-1},S\nu_{q(n)})\\ &\quad+\ell_5\varrho(S\nu_{q(n)-1},S\nu_{p(n)}))\\ &< F(\alpha\ell_1[\varrho(S\nu_{p(n)-1},S\nu_{q(n)})+\varrho(S\nu_{q(n)},S\nu_{q(n)-1})]\\ &\quad+\ell_2\varrho(S\nu_{p(n)-1},S\nu_{p(n)})+\ell_3\varrho(S\nu_{q(n)-1},S\nu_{q(n)})\\ &\quad+\ell_4\varrho(S\nu_{p(n)-1},S\nu_{q(n)})+\ell_5\varrho(S\nu_{q(n)-1},S\nu_{p(n)}))\\ &= F((\alpha\ell_1+\alpha\ell_4)\varrho(S\nu_{p(n)-1},S\nu_{p(n)})+(\alpha\ell_1+\ell_3)\varrho(S\nu_{q(n)},S\nu_{q(n)-1})\\ &\quad+\ell_2\varrho(S\nu_{p(n)-1},S\nu_{p(n)})+\ell_5\varrho(S\nu_{q(n)-1},S\nu_{p(n)}).\end{aligned}$$

By applying the limit as $n\longrightarrow\infty$ we obtain,

$$\tau+F(\epsilon)\leq F(\epsilon)$$

which contradict the assumption.
Hence, $\{\nu_n\}$ is a Cauchy sequence in X. But since (X,ϱ) is a complete cone b-metric space, then $\{\nu_n\}$ converge to $\nu\in X$.
Let $\{p(n)\}$ be a sequence of natural numbers such that,

$$S\nu_{p(n)+1}=TS\nu_{p(n)}=TS\nu,\ \lim_{n\longrightarrow\infty}S\nu_{p(n)+1}=S\nu$$

so that, $TS\nu=S\nu=\nu$. Suppose $TS\nu\neq S\nu$ and $S\nu_{p(n)+1}=TS\nu_{p(n)}\neq TS\nu$ for all $n\in\mathbb{N}$.
Then,

$$\begin{aligned}\tau+F(\varrho(TS\nu_{p(n)},TS\nu)) &\leq F(\ell_1\varrho(S\nu_{p(n)},S\nu)+\ell_2\varrho(S\nu_{p(n)},TS\nu_{p(n)})+\ell_3\varrho(S\nu,TS\nu)\\ &\quad+\ell_4\varrho(S\nu_{p(n)},TS\nu)+\ell_5\varrho(S\nu,TS\nu_{p(n)}))\\ &= F(\ell_1\varrho(S\nu_{p(n)},S\nu)+\ell_2\varrho(S\nu_{p(n)},S\nu_{p(n)+1})+\ell_3\varrho(S\nu,S\nu)\\ &\quad+\ell_4\varrho(S\nu_{p(n)},S\nu)+\ell_5\varrho(S\nu,S\nu_{p(n)+1})).\end{aligned}$$

But F is increasing, so we have,

$$\begin{aligned}\varrho(TS\nu_{p(n)},TS\nu) &\leq \ell_1\varrho(S\nu_{p(n)},S\nu)+\ell_2\varrho(S\nu_{p(n)},S\nu_{p(n)+1})+\ell_4\varrho(S\nu_{p(n)},S\nu)\\ &\quad+\ell_5\varrho(S\nu,S\nu_{p(n)+1}).\end{aligned}$$

Taking $n\longrightarrow\infty$ we get,

$$0\leq 0,$$

which is a contradiction, hence

$$T\nu=S\nu=\nu.$$

We show that (S,T) have a unique fixed point in X.
Suppose that $\nu,\mu \in X$ be two fixed points of (S,T) such that $\nu \neq \mu$.
Hence, $S\nu = T\nu \neq S\mu = T\mu$,
which gives

$$\varrho(T\nu, T\mu) = \varrho(S\nu, S\mu) > 0.$$

Then, we have

$$\begin{aligned}
\tau + F(\varrho(S\nu, S\nu)) &= \tau + F(\varrho(T\nu, T\mu)) \\
&\leq F(\ell_1\varrho(S\nu, S\mu) + \ell_2\varrho(S\nu, T\nu) + \ell_3\varrho(S\mu, T\mu) \\
&\quad +\ell_4\varrho(S\nu, T\mu) + \ell_5\varrho(S\mu, T\nu)) \\
&= F(\ell_1\varrho(S\nu, S\mu) + \ell_4\varrho(S\nu, T\mu) + \ell_5\varrho(S\nu, S\mu)) \\
&\leq F((\ell_1 + \ell_4 + \ell_5)\varrho(S\nu, S\mu)) \\
&\leq F(\varrho(S\nu, S\mu))
\end{aligned}$$

which is a contradiction. Therefore, (S,T) have a unique fixed point. □

Corollary 2.4. *Let (X,ϱ) be a complete Cone b-metric space and let T be a self mapping on X with $S \geq 1$ and $S\ell < 1$. Assume that there exists $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ an increasing mapping and $\tau > 0$ such that:*

$$\tau + F(\varrho(T\nu, T\mu)) \leq F(\ell_1\varrho(\nu, \mu) + \ell_2\varrho(\nu, T\nu) + \ell_3\varrho(\mu, T\mu))$$

for all $\nu, \mu \in X, T\nu \neq T\mu$ where $\ell_i > 0$. Then T has a unique fixed point.

Example 2.5. Let $X = \{p_n : n \in N\} \cup \{q\}$ and $\varrho : X \times X \longrightarrow E$ such that $\varrho(p_n, p_n) = \varrho(q, q) = 0$ for every $n \in \mathbb{N}$,

$$\varrho(p_n, p_{n+b}) = \varrho(p_{n+b}, p_n) = \varrho(q, p_n) = \frac{1}{n}$$

for all $n, b \in \mathbb{N}$.
Certainly (X,ϱ) is a complete cone b-metric space.
Let $T : X \longrightarrow X$ such that

$$Tp_n = p_{n+1} \quad \textit{and} \quad Tq = q.$$

Suppoese that there exists $F : R^+ \longrightarrow R$ an increasing mapping and $\tau > 0$ such that

$$\tau + F(\varrho(T\nu, T\mu)) \leq F(\varrho(\nu, \mu)).$$

Let $\nu = p_{n+i}$ and $\mu = p_{n+i+1}$ for every $n \geq 1$ and $i \geq 0$
so,

$$\tau + F(\varrho(Tp_{n+i}, Tp_{n+i+1})) = \tau + F\left(\frac{1}{n+i+1}\right) \leq F\left(\frac{1}{n+i}\right).$$

Hence, we obtain

$$\sum_{i=0}^{b-1}\left(\tau + F\left(\frac{1}{n+i+1}\right)\right) \leq \sum_{i=0}^{b-1} F\left(\frac{1}{n+i}\right)$$

which gives,

$$b\tau + F\left(\frac{1}{n+b}\right) \leq F\left(\frac{1}{n}\right).$$

Let b integer such that $b \leq n^k < b+1$.
By using (F3) we have,

$$\lim_{n\longrightarrow\infty} \frac{n^k}{b} = 0,$$

So,

$$\lim_{n\longrightarrow\infty} \frac{1}{b}F\left(\frac{1}{n}\right) = \lim_{n\longrightarrow\infty} \frac{n^k}{b} = \frac{1}{n^k}F\left(\frac{1}{n}\right) = 0,$$

and

$$\lim_{n\longrightarrow\infty} \frac{1}{b}F\left(\frac{1}{n+b}\right) = \lim_{n\longrightarrow\infty} \frac{(n+b)^k}{n^k}.\frac{n^k}{b}.\frac{1}{(n+b)^k}F\left(\frac{1}{n+b}\right) = 0. \tag{2.9}$$

Thus, as $n \longrightarrow \infty$ in (2.9) above we get, $\tau \leq 0$ which is a contradiction. Hence, there exists $F : R^+ \longrightarrow R$ an increasing mapping and $\tau > 0$ such that

$$\tau + F(\varrho(T\nu, T\mu)) \leq F(\varrho(\nu, \mu))$$

for all $\nu, \mu \in X, T\nu \neq T\mu$.
Then we have,

$$\begin{aligned} \tau + F(\varrho(Tp_n, Tp_{n+b})) &= \tau + F\left(\frac{1}{n+1}\right) \\ &= \tau - n - 1 \leq -n \\ &= F\left(\frac{1}{n}\right) = F(\varrho(p_n, p_{n+b})). \end{aligned}$$

So

$$\begin{aligned} &\tau + F(\varrho(Tp_n, Tq)) + \tau + F\left(\frac{1}{n+1}\right) \leq F\left(\frac{1}{n}\right) \\ &= F(\varrho(p_n, q)) \end{aligned}$$

for all $\tau \leq 1$.

3. An Application

We shall present a fixed point theorem as an application for an asymptotically regular map in a complete cone b-metric space.
We define an asymptotically regular map which we are going to apply to show the existence and uniqueness of a fixed point in a complete cone b-metric space as follows.

Definition 3.1. A self map T in a complete cone b-metric space is said to be asymptotically regular at some $\nu \in X$, if

$$\lim_{n \longrightarrow \infty} \varrho(T^n \nu, T^{n+1} \nu) = 0,$$

where T^n denotes the nth iterate of T at ν.

Theorem 3.2. *Let (X, ϱ) be a complete cone b-metric space with a constant $s \geq 1$. Suppose that there exists $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ an increasin mapping with $\tau > 0$. Also let $T \in X$ be a self mapping satisfying:*

$$\begin{aligned}&\varrho(T\nu, T\mu) > 0 \Rightarrow \tau + F(\varrho(T\nu, T\mu)) \\ &\leq F(\ell_1 \varrho(\nu, \mu) + \ell_2 \varrho(\nu, T\nu) + \ell_3 \varrho(\mu, T\mu) + \ell_4 \varrho(\nu, T\mu) + \ell_5 \varrho(\mu, T\nu))\end{aligned}$$

for all $\nu, \mu \in X$ where $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 > 0$. If T is asymptotically regular at some fixed point $\nu \in X$, then T has a unique fixed point.

Proof. Suppose T is an asymptotically regular at $\nu_0 \in X$. Therefore considering the sequence $\{T^n \nu_0\}$ for all $m, n \geq 1$.

$$\begin{aligned}\varrho(T^m \nu_0, T^n \nu_0) \leq\ & F(\ell_1 \varrho(T^{m-1} \nu_0, T^{n-1} \nu_0) + \ell_2 \varrho(T^{m-1} \nu_0, T^m \nu_0) \\ & + \ell_3 \varrho(T^{n-1} \nu_0, T^n \nu_0) + \ell_4 \varrho(T^{m-1} \nu_0, T^n \nu_0) + \ell_5 \varrho(T^{n-1} \nu_0, T^m \nu_0))\end{aligned}$$

Applying Definition(1.5) we get,

$$\begin{aligned}\leq\ & F(S\ell_1 [\varrho(T^{m-1} \nu_0, T^m \nu_0) + \varrho(T^m \nu_0, T^{n-1} \nu_0)] \\ & + \ell_2 \varrho(T^{m-1} \nu_0, T^m \nu_0) + \ell_3 \varrho(T^{n-1} \nu_0, T^n \nu_0) + S[\ell_4 \varrho(T^{m-1} \nu_0, T^m \nu_0) \\ & + \varrho(T^m \nu_0, T^n \nu_0)] + S[\ell_5 \varrho(T^{n-1} \nu_0), T^n \nu_0 + \ell_5 \varrho(T^n \nu_0, T^m \nu_0)]) \\ \leq\ & F(S\ell_1 [\varrho(T^{m-1} \nu_0, T^m \nu_0) + S[\varrho(T^m \nu_0, T^n \nu_0) + \varrho(T^n \nu_0, T^{n-1} \nu_0)]] \\ & + \ell_2 \varrho(T^{m-1} \nu_0, T^m \nu_0) + \ell_3 \varrho(T^{n-1} \nu_0, T^n \nu_0) + S[\ell_4 \varrho(T^{m-1} \nu_0, T^m \nu_0) \\ & + \varrho(T^m \nu_0, T^n \nu_0)] + S[\ell_5 \varrho(T^{n-1} \nu_0), T^n \nu_0 + \ell_5 \varrho(T^n \nu_0, T^m \nu_0)]) \\ =\ & F(S\ell_1 \varrho(T^{m-1} \nu_0, T^m \nu_0) + S^2 \ell_1 \varrho(T^m \nu_0, T^n \nu_0) + S^2 \ell_1 \varrho(T^n \nu_0, T^{n-1} \nu_0) \\ & + \ell_2 \varrho(T^{m-1} \nu_0, T^m \nu_0) + \ell_3 \varrho(T^{n-1} \nu_0, T^n \nu_0) + S\ell_4 \varrho(T^{m-1} \nu_0, T^m \nu_0) \\ & + S\ell_4 \varrho(T^m \nu_0, T^n \nu_0) + S[\ell_5 \varrho(T^{n-1} \nu_0), T^n \nu_0 + \ell_5 \varrho(T^n \nu_0, T^m \nu_0)]).\end{aligned}$$

By the monotony of F we have,

$$\begin{aligned}(1 - S^2 \ell_1 - S\ell_4 - S\ell_5) \varrho(T^m \nu_0, T^n \nu_0) \leq\ & (S\ell_1 + \ell_2 + S\ell_4) \varrho(T^{m-1} \nu_0, T^m \nu_0) \\ & + (S^2 \ell_1 + \ell_3 + S\ell_5) \varrho(T^{n-1} \nu_0, T^n \nu_0)\end{aligned}$$

and hence,

$$\begin{aligned}\|\varrho(T^m \nu_0, T^n \nu_0)\| \leq\ & \Big(\frac{S\ell_1 + \ell_2 + S\ell_4}{1 - S^2 \ell_1 - S\ell_4 - S\ell_5}\Big) \|\varrho(T^{m-1} \nu_0, T^m \nu_0)\| \\ & + \Big(\frac{S^2 \ell_1 + \ell_3 + S\ell_5}{1 - S^2 \ell_1 - S\ell_4 - S\ell_5}\Big) \|\varrho(T^{n-1} \nu_0, T^n \nu_0)\|.\end{aligned}$$

But T is an asymptotically regular at x_0 therefore $\varrho(T^{m-1}\nu_0, T^m\nu_0) \longrightarrow 0$ and $\varrho(T^{n-1}\nu_0, T^n\nu_0) \longrightarrow 0$ as $m, n \longrightarrow \infty$. Also implies that $\varrho(T^m\nu_0, T^n\nu_0) \longrightarrow 0$ as $m, n \longrightarrow \infty$. Therefore, $\{T^n x_0\}$ is a Cauchy sequence in a complete cone b-metric space.
Since X is a complete space, then there exists $\nu \in X$ such that $T^n\nu_0 \longrightarrow \nu$ as $n \longrightarrow \infty$.
Hence,

$$\lim_{n\longrightarrow\infty} \varrho(T^n\nu_0, \nu) = \varrho(\nu, \nu) = \lim_{n\longrightarrow\infty} \varrho(T^n\nu_0, T^n\nu_0) = 0. \tag{3.1}$$

Then $\{T^n\nu_0\} \longrightarrow \nu \in X$.
Now, we claim that ν is a fixed point in X.
Hence we have,

$$\begin{aligned}
\varrho(T\nu, \nu) &\leq S[\varrho(T\nu, T^n\nu_0) + \varrho(T^n\nu_0, \nu)] \\
&\leq S[\ell_1\varrho(\nu, T^{n-1}\nu_0) + \ell_2\varrho(\nu, T\nu) + \ell_3\varrho(T^{n-1}\nu_0, T^n\nu_0) + \ell_4\varrho(\nu, T^n\nu_0) \\
&\quad +\ell_5\varrho(T^{n-1}\nu_0, T\nu)] + S\varrho(T^n\nu_0, \nu) \\
&= S\ell_1\varrho(\nu, T^{n-1}\nu_0) + S\ell_2\varrho(\nu, T\nu) + S\ell_3\varrho(T^{n-1}\nu_0, T^n\nu_0) + S\ell_4\varrho(\nu, T^n\nu_0) \\
&\quad +S\ell_5\varrho(T^{n-1}\nu_0, T\nu) + S\varrho(T^n\nu_0, \nu).
\end{aligned}$$

Using (3.2) we get,

$$\begin{aligned}
\varrho(T\nu, \nu) &\leq S\ell_1\varrho(T^{n-1}\nu_0, T^n\nu_0) + S\ell_2\varrho(\nu, T\nu) + S\ell_3\nu\varrho(T^{n-1}\nu_0, T^n\nu_0) \\
&\quad S\ell_5\varrho(\nu, T\nu)
\end{aligned}$$

Collecting like terms gives,

$$\begin{aligned}
(1 - S\ell_2 - S\ell_5)\varrho(T\nu, \nu) &\leq (S\ell_1 + S\ell_3)\varrho(T^{n-1}\nu_0, T^n\nu_0) \\
\varrho(T\nu, \nu) &\leq \Big(\frac{S\ell_1 + S\ell_3}{1 - S\ell_2 - S\ell_5}\Big)\varrho(T^{n-1}\nu_0, T^n\nu_0).
\end{aligned}$$

Since T is asymptotically regular at ν_0, therefore $\varrho(T^{n-1}\nu_0, T^n\nu_0) \longrightarrow 0$ as $n \longrightarrow \infty$.
Therefore this proves that $T\nu = \nu$ is a fixed point in X.
We shall show that ν is a unique fixed point in X.
Let μ be another fixed point of T such that $T\mu = \mu$.
Hence,

$$\begin{aligned}
\varrho(\nu, \mu) &= \varrho(T\nu, T\mu) \\
&\leq \ell_1\varrho(\nu, \mu) + \ell_2\varrho(\nu, T\nu) + \ell_3\varrho(\mu, T\mu) + \ell_4\varrho(\nu, T\mu) + \ell_5\varrho(\mu, T\nu).
\end{aligned}$$

Therefore, this proves that T has a unique fixed point in a complete cone b-metric space. □

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