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On generalized $\pounds-$ cotorsion LCA groups

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Abstract. A locally compact abelian group G is called a generalized $\pounds-$cotorsion group if G contains an open $\pounds-$cotorsion subgroup H such that G/H is a cotorsion group. In this paper, we determine the generalized $\pounds-$ cotorsion groups.

Keywords: Generalized $\pounds-$cotorsion, $\pounds-$cotorsion, Cotorsion, LCA groups.

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1. Introduction

Let $\pounds$ be the category of all locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image and a short exact sequence $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ in $\pounds$ is said to be an extension of A by C if ϕ and ψ are proper morphisms. We let $Ext(C, A)$ denote the group extensions of A by C [7]. A discrete group A is called cotorsion if $Ext(X, A) = 0$ for all discrete torsion-free groups X. The theory of cotorsion groups was developed by Harrison for the first time [9]. For more on cotorsion groups, see [5]. In [6], Fulp generalized the concept of cotorsion groups to LCA groups. A group $G \in \pounds$ is called an $\pounds-$cotorsion group if $Ext(X, G) = 0$ for all torsion-free groups $X \in \pounds$ [6]. Fulp studied the $\pounds-$cotorsion LCA groups and determined the discrete or compact $\pounds-$cotorsion groups [6].

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In this paper, we generalize the concept of $\pounds-$ cotorsion groups. A group $G \in \pounds$ will be called a generalized $\pounds-$cotorsion group if G contains an open $\pounds-$cotorsion subgroup H such that G/H is a cotorsion group. We determine the discrete or compact generalized $\pounds-$cotorsion groups (see Lemma 2.5 and 2.6). We also determine non discrete, divisible, generalized $\pounds-$cotorsion groups (see Theorem 2.11). This paper is a part of an investigation which answers the following question:
For groups $A, C \in \pounds$, under what conditions on A and C, an extension of A by C splits? In [1, 2, 3, 12, 13, 14, 16, 17] we have been able to answer the above question by defining a new subset or subgroup of $Ext(C, A)$. The concept of splitting of extensions is very important in LCA groups. By this concept, we determined the structure of an LCA group G such that tG is closed [15]. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals with the discrete topology, $\mathbb{Z}$ is the group of integers and $\mathbb{Z}(n)$ is the cyclic group of order n. For any group G and H, G_0 and tG are the identity component and the maximal torsion subgroups of G respectively, and $Hom(G, H)$, the group of all continuous homomorphisms from G to H, endowed with the compact open topology. The dual group of G is $\hat{G} = Hom(G, \mathbb{R}/\mathbb{Z})$ and $(\hat{G}, S)$ denotes the annihilator of $S \subseteq G$ in $\hat{G}$. For more on locally compact abelian groups, see [10].

2. Generalized $\pounds-$ cotorsion LCA groups

Recall that a discrete group A is called cotorsion if $Ext(X, A) = 0$ for all discrete torsion-free groups X. The concept of cotorsion groups was developed by Harrison for the first time [9]. Fulp generalized the concept of cotorsion groups to $\pounds-$cotorsion groups. A group $G \in \pounds$ is called an $\pounds-$cotorsion group if $Ext(X, G) = 0$ for all torsion-free groups $X \in \pounds$ [6]. In this section, we define the concept of a generalized $\pounds-$cotorsion group.

Definition 2.1. An LCA group G is called a generalized $\pounds-$cotorsion group if G contains an open $\pounds-$cotorsion subgroup H such that G/H is a cotorsion group.

Example 2.2. Discrete cotorsion and $\pounds-$cotorsion groups are generalized $\pounds-$cotorsion.

Fulp determined the discrete or compact $\pounds-$cotorsion groups which plays a fundamental role in obtaining our results.

Theorem 2.3. *(*[6, Corollary 10]*) A discrete group is an £−cotorsion group if and only if it is a divisible torsion group.*

Theorem 2.4. *(*[6, Corollary 9]*) A compact group is an £−cotorsion group if and only if it is a connected group.*

Lemma 2.5. *A discrete group is a generalized £−cotorsion group if and only if it is a cotorsion group.*

Proof. Let G be a discrete generalized £−cotorsion group. Then, there exists an £− cotorsion subgroup H of G such that G/H is a cotorsion group. By Theorem 2.3, H is a divisible torsion group. So, $0 \to H \hookrightarrow G \to G/H \to 0$ splits. Hence

$$G \cong H \oplus G/H$$

It follows that G is a cotorsion group. Conversely is clear. □

Lemma 2.6. *A compact group G is a generalized £−cotorsion group if and only if $G \cong M \oplus \mathbb{Z}(n)$ which M is a connected group and n a positive integer number.*

Proof. Let G be a compact generalized £−cotorsion group. Then, there exists an open £− cotorsion subgroup H of G such that G/H is a cotorsion group. By Theorem 2.4, H is connected. Since H is open, $H = G_0$. So, $0 \to H \hookrightarrow G \to G/H \to 0$ splits. Hence, $G \cong H \oplus G/H$. Since G/H is a compact and discrete group, G/H is a finite cotorsion group. By Corollary 54.4 of [5], $G/H \cong \mathbb{Z}(n)$ for some positive integers n.

Conversely, let $G \cong M \oplus \mathbb{Z}(n)$ which M is a connected group and n a positive integer number. Set $H = M$. Then $H \oplus 0$ is an open £−cotorsion subgroup of G such that $G/(H \oplus 0) \cong \mathbb{Z}(n)$ is a cotorsion group. □

Our next goal is to determine the divisible generalized £−cotorsion groups. To do this, we first present a classification of divisible LCA groups. Recall that a group $G \in$ £ is said to be torsion-closed if tG is closed in G [15]. For the classification of divisible LCA groups, we need the following Theorem:

Theorem 2.7. *(*[15, Theorem 3.5]*) A group $G \in$ £ is torsion-closed if and only if $G \cong \mathbb{R}^n \oplus C \oplus L$ where C is a compact, connected, torsion-free group and L contains a compact open subgroup $H \cong \prod_{i \in I} \mathbb{Z}(p_i^{r_i}) \oplus \prod_p \Delta_p^{n_p}$ (Δ_p denotes the group of $p-$adic integers and $\mathbb{Z}(p_i^{r_i})$ is the cyclic group of order $p_i^{r_i}$) where only finitely many distinct primes p_i and positive integers r_i occur and n_p are cardinal number.*

Lemma 2.8. *Let $G \in \pounds$ be a divisible group. Then, $G \cong \mathbb{R}^n \oplus \hat{C} \oplus N$ where C is a compact, connected, torsion-free group and N contains a compact open subgroup K such that N/K is a discrete divisible torsion group.*

Proof. Let $G \in \pounds$ be a divisible group. Then, $\hat{G}$ is torsion-free. It follows that $\hat{G} \cong \mathbb{R}^n \oplus C \oplus L$ where C and L are according to what was mentioned in Theorem 2.7. Set $N = \hat{L}$ and $K = (\hat{L}, H)$. By Theorem 24.11 of [10], $N/K \cong \hat{H}$. Since H is a compact, totally disconnected, torsion-free group, N/K is a discrete divisible torsion group. □

The exact sequences (1) and (2) of the following Theorem establish a closed connection between *Hom* and *Ext* in $\pounds$.

Theorem 2.9. *(*[8, Corollary 2.10]*) Let $G \in \pounds$ and $0 \to A \to B \to C \to 0$ be an extension in $\pounds$. Then, the following sequences are exact:*

(1) $0 \to Hom(C,G) \to Hom(B,G) \to Hom(A,G) \to Ext(C,G) \to Ext(B,G) \to Ext(A,G) \to 0$

(2) $0 \to Hom(G,A) \to Hom(G,B) \to Hom(G,C) \to Ext(G,A) \to Ext(G,B) \to Ext(G,C) \to 0$

The dual of an extension $E : 0 \to A \to B \to C \to 0$ is defined by $\hat{E} : 0 \to \hat{C} \to \hat{B} \to \hat{A} \to 0$. The following Lemma establishes a connection between discrete torsion groups and *Ext*.

Lemma 2.10. *A discrete group G is torsion if and only if $Ext(\hat{\mathbb{Q}}, G) = 0$.*

Proof. First, let G be a discrete group such that $Ext(\hat{\mathbb{Q}}, G) = 0$. Consider the two exact sequences $0 \to tG \hookrightarrow G \to G/tG \to 0$ and $0 \to \widehat{\mathbb{Q}/\mathbb{Z}} \to \hat{\mathbb{Q}} \to \hat{\mathbb{Z}} \to 0$. By Theorem 2.9, we have the two following exact sequences

$$... \to Ext(\hat{\mathbb{Q}}, G) \to Ext(\hat{\mathbb{Q}}, G/tG) \to 0 \tag{2.1}$$

$$... \to Hom(\widehat{\mathbb{Q}/\mathbb{Z}}, G/tG) \to Ext(\hat{\mathbb{Z}}, G/tG) \to Ext(\hat{\mathbb{Q}}, G/tG) \tag{2.2}$$

By (2.1), $Ext(\hat{\mathbb{Q}}, G/tG) = 0$. By Theorem 24.25 of [10] and Corollary 2, p. 377 of [11], $Hom(\widehat{\mathbb{Q}/\mathbb{Z}}, G/tG) \cong Hom(\widehat{G/tG}, \mathbb{Q}/\mathbb{Z}) = 0$. It follows from (2.2) that $Ext(\hat{\mathbb{Z}}, G/tG) = 0$. Hence, by Proposition 2.17 of [7], $G/tG \cong Ext(\hat{\mathbb{Z}}, G/tG) = 0$ and G is a torsion group. Conversely is clear by Theorem 3.1 of [8]. □

Theorem 2.11. *A non discrete, divisible group G in $\pounds$ is generalized $\pounds-$cotorsion if and only if G be $\pounds-$cotorsion.*

Proof. Let G be a non discrete, divisible and generalized $\pounds-$cotorsion group. By Lemma 2.8, $G \cong \mathbb{R}^n \oplus N$ where N contains a compact open subgroup K such that N/K is a discrete divisible torsion group. First, we show that $Ext(\hat{\mathbb{Q}}, G) = 0$. It is sufficient to show that $Ext(\hat{\mathbb{Q}}, N) = 0$. Consider the exact sequence $0 \to K \hookrightarrow N \to N/K \to 0$. By Theorem 2.9, we have the following exact sequence

$$... \to Ext(\hat{\mathbb{Q}}, K) \to Ext(\hat{\mathbb{Q}}, N) \to Ext(\hat{\mathbb{Q}}, N/K) \to 0 \qquad (2.3)$$

By Theorem 2.12 and Proposition 2.17 of [7], $Ext(\hat{\mathbb{Q}}, K) = 0$. Also, by Theorem 2.3, $Ext(\hat{\mathbb{Q}}, N/K) = 0$. It follows from (2.3) that $Ext(\hat{\mathbb{Q}}, N) = 0$. Now, let H be an open $\pounds-$ cotorsion subgroup of G. By Lemma 2.10, G/H is a torsion group. Consider the following exact sequence

$$... \to Ext(X, H) \to Ext(X, G) \to Ext(X, G/H) \to 0 \qquad (2.4)$$

Where X is a torsion-free group in $\pounds$. Since H is $\pounds-$cotorsion, $Ext(X, H) = 0$. By Theorem 2.3, $Ext(X, G/H) = 0$. It follows from (2.4) that $Ext(X, G) = 0$ for all torsion-free groups $X \in \pounds$. Hence, G is an $\pounds-$cotorsion group. Conversely is clear. □

Let $G \in \pounds$. We denote by G^*, the minimal divisible extension of G. By [10, 4.18.h], G^* is an LCA group containing G as an open subgroup.

Corollary 2.12. *Every generalized $\pounds-$cotorsion group can be imbedded in an $\pounds-$cotorsion group.*

Proof. Let $G \in \pounds$ be a generalized $\pounds-$cotorsion group. Then, G contains an open $\pounds-$cotorsion subgroup H. Clearly, H is an open, $\pounds-$cotorsion subgroup of G^* such that G^*/H is a cotorsion group. By Theorem 2.11, G^* is $\pounds-$cotorsion. □

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