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Oscillation and asymptotic behavior for fourth order neutral differential equations with mixed deviating arguments

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Abstract. This paper aims to study the oscillatory nature of solutions for fourth-order neutral differential equations with mixed deviating arguments and improved oscillation conditions obtained in various cases. We reduced the problem to first-order differential inequality by using suitable substitutions that enabled us to use comparison theorems. Further, we discuss the asymptotic nature of solutions, and in the end, an example is given to validate the results.

Keywords: Fourth order, Neutral differential equations, Mixed deviating arguments, Oscillation, Asymptotic behavior.

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1. Introduction

The differential equations in which the highest-order derivatives appear with and without delay are called neutral differential equations. Neutral

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differential equations are used in the modeling of many mathematical phenomena in the field of natural science and technology. Initially, the existence, uniqueness, and stability of solutions for different types of neutral equations have been studied, and in recent years, a lot of attention has been paid to the oscillatory and asymptotic behavior of such equations, see [4, 5, 10, 11, 12, 15, 16, 18, 19, 20, 21, 24] and the references therein. In the last few decades, the number of research activities has been increased to obtain necessary and sufficient conditions on oscillation theory for higher order neutral type equations on time scales [14, 17, 23]. Moreover, the papers [6, 7, 8, 9] provide a motivational background for the present paper.

Basic definitions and a few approaches to investigating the oscillatory and asymptotic properties of the solutions of neutral equations were given in the book [13].

For similar results on various classes of fourth-order delay differential/dynamic equations, we refer the reader to the papers [1, 2, 23]. In paper [14], authors established the oscillation criterion for third-order linear dynamics equations on time scales.

Fourth-order neutral differential equations can be used to model various mathematical phenomena in biological and chemical science; see, for instance, the paper [10]. Due to the wide applicability of these equations in various fields of science and engineering, there is a great interest in obtaining new oscillation criteria for higher-order neutral differential equations; see, for instance, [3, 10, 12, 20].

Oscillation theorems for third-order delay equations were discussed by Tiryaki et al. [20]. Arul and Shobha et al. [4] generalized neutral differential equations of order two, and by using the Riccati transformation, they presented some new oscillation criteria under some conditions. Improved sufficient conditions for the oscillation and asymptotic stability have been obtained in the paper [22].

Motivated by all the above works, we obtain improved oscillation results for fourth-order neutral differential equations with mixed delays. We reduced (1.1) into the first-order differential inequality using suitable substitutions. Further, some sufficient conditions for the oscillation of solutions are obtained in various cases using comparison results, and their asymptotic nature is also discussed.

Here, we start with the following model of fourth order neutral differential equations with mixed delay terms:

$$\Big(k_2(t)\Big(k_1(t)v^{(2)}(t)\Big)^{(1)}\Big)^{(1)} + k_3(t)u(\eta_3(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $\upsilon(t) = u(t) + au(\eta_1(t)) + bu(\eta_2(t))$, $a > 0$, $b > 0$, $\eta_1(t) \leq t$, $\eta_2(t) \geq t$, $\eta_3(t) \leq t$, $t \geq t_0$, and $\upsilon^{(i)}(t)$, denote the derivative of order i with respect to t.

The remaining part of this paper is designed in such a way that section 2 consists of some assumptions and definitions with a lemma. In section 3, the statement and proof of the main results are provided; in section 4, an example is presented to validate the results.

2. Preliminaries and Assumptions

Throughout the paper, we consider the following assumptions:

(C1) The functions $k_r : (t_0, \infty) \to \mathbb{R}^+$, $r = 1, 2, 3$ are continuous.

(C2) $\eta_r : (t_0, \infty) \to \mathbb{R}$, $r = 1, 2, 3$ are continuous functions with the following conditions:

- (i) $\eta_1(t) \leq t$, $\eta_2(t) \geq t$, $\eta_3(t) \leq t$,
- (ii) $\eta_r^{(1)}(t) = 1$,
- (iii) $\lim_{t\to\infty} \eta_r(t) = \infty$.

Definition 2.1. A function $u \in C([T_u, \infty))$, $T_u \geq t_0$ is a solution of (1.1) if corresponding function υ has two properties:

- (i) $k_1\upsilon^{(2)}(t) \in C^1([T_u, \infty))$, and
- (ii) $k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \in C^1([T_u, \infty))$,

and $u(t)$ satisfies (1.1).

Definition 2.2. A solution u of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

Lemma 2.3. [23] *Let $u(t) > 0$ be an eventually positive solution of (1.1). Then for sufficiently large $\varrho \geq t_0$ such that for $t \geq \varrho$, there are only four possibilities:*

- (i) $\upsilon(t) > 0$, $\upsilon^{(1)}(t) < 0$, $\upsilon^{(2)}(t) > 0$, $(k_1(t)\upsilon^{(2)}(t))^{(1)} < 0$,
- (ii) $\upsilon(t) > 0$, $\upsilon^{(1)}(t) > 0$, $\upsilon^{(2)}(t) > 0$, $(k_1(t)\upsilon^{(2)}(t))^{(1)} < 0$,
- (iii) $\upsilon(t) > 0$, $\upsilon^{(1)}(t) > 0$, $\upsilon^{(2)}(t) > 0$, $(k_1(t)\upsilon^{(2)}(t))^{(1)} > 0$,
- (iv) $\upsilon(t) > 0$, $\upsilon^{(1)}(t) > 0$, $\upsilon^{(2)}(t) < 0$, $(k_1(t)\upsilon^{(2)}(t))^{(1)} > 0$.

We define

$$K_1(t) = \int_t^\infty \frac{1}{k_2(\nu)} d\nu, \; K_2(t) = \int_t^\infty \frac{K_1(\nu)}{k_1(\nu)} d\nu, \; K_3(t) = \int_t^\infty K_2(\nu) d\nu,$$

$$K_1^\varrho(t) = \int_\varrho^t k_3(\nu) d\nu, \; K_2^\varrho(t) = \int_\varrho^t \frac{K_1^\varrho(\nu)}{k_2(\nu)} d\nu$$

and

$$R_1(t) = \int_t^\infty k_3(\nu)d\nu, \; R_2(t) = \int_t^\infty \frac{R_1(\nu)}{k_2(\nu)}d\nu.$$

3. Main Results

Theorem 3.1. *Suppose conditions (C1) and (C2) hold. Further, we assume that*

$$\int_{t_0}^\infty \Big[A_0 k_3(\nu)K_3(\nu) - \frac{K_2(\nu)}{4K_3(\nu)}\Big]d\nu = \infty, \tag{3.1}$$

$$\int_{t_0}^\infty \left[A_0 K_1(\nu)k_3(\nu)\int_\varrho^{\eta_3(\nu)}\int_\varrho^{\nu_2}\frac{1}{k_1(\nu_1)}d\nu_1 d\nu_2 - \frac{1}{4k_2(\nu)K_1(\nu)}\right]d\nu = \infty, \tag{3.2}$$

and there exist positive continuously differentiable functions c, d defined on $[t_0,\infty)$ such that

$$\int_{t_0}^\infty \left[\frac{A_0 c(\nu)k_3(\nu)}{K_2^\varrho(\eta_3(\nu))} - \frac{\big(c^{(1)}(\nu)\big)^2}{4\big(c(\nu)\big)^3}k_2(\nu)\right]d\nu = \infty, \tag{3.3}$$

and

$$\int_{t_0}^\infty \left[\frac{A_0 d(\nu)R_2(\nu)}{k_1(\nu)} - \frac{\big(d^{(1)}(\nu)\big)^2}{4d(\nu)}\right]d\nu = \infty, \tag{3.4}$$

where $A_0 = 1-a-b$. Then each non zero solution of (1.1) is oscillatory.

Proof. Suppose that $u(t)$ is an eventually positive solution of (1.1). Then for some $t \geq \varrho$, $u(\eta_r(t)) > 0$, $r = 1,2,3$. Now according to Lemma 2.3, there are four possibilities. Suppose case (i) holds. Since $\big(k_2(t)(k_1(t)v^{(2)}(t))^{(1)}\big)^{(1)} < 0$, $k_2(t)(k_1(t)v^{(2)}(t))^{(1)}$ is decreasing for $t \geq \varrho$. Therefore, we have

$$k_2(\nu)\Big(k_1(\nu)v^{(2)}(\nu)\Big)^{(1)} \leq k_2(t)\Big(k_1(t)v^{(2)}(t)\Big)^{(1)}, \quad \nu \in [t,\infty).$$

Integrating from t to ∞ after dividing by $k_2(\nu)$, we have

$$-\Big(k_1(t)v^{(2)}(t)\Big) \leq k_2(t)\Big(k_1(t)v^{(2)}(t)\Big)^{(1)}\int_t^\infty \frac{1}{k_2(\nu)}d\nu,$$

which implies that

$$v^{(2)}(t) \geq -\frac{k_2(t)\Big(k_1(t)v^{(2)}(t)\Big)^{(1)}}{k_1(t)}K_1(t).$$

Integrating from t to ∞, we have

$$\int_t^\infty \upsilon^{(2)}(\nu)d\nu \geq -\int_t^\infty \frac{k_2(\nu)\Big(k_1(\nu)\upsilon^{(2)}(\nu)\Big)^{(1)}}{k_1(\nu)}K_1(\nu)d\nu,$$

which implies that

$$-\upsilon^{(1)}(t) \geq -k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}K_2(t). \tag{3.5}$$

Again integrating the latter inequality from t to ∞, we obtain

$$\begin{aligned} \upsilon(t) &\geq -\int_t^\infty k_2(\nu)\Big(k_1(\nu)\upsilon^{(2)}(\nu)\Big)^{(1)}K_2(\nu)d\nu \\ &\geq -k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}K_3(t). \end{aligned} \tag{3.6}$$

If we set

$$\chi_1(t) = \frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{\upsilon(t)}, \tag{3.7}$$

then

$$\chi_1(t)K_3(t) \geq -1. \tag{3.8}$$

Since $u(t) = \upsilon(t) - au(\eta_1(t)) - bu(\eta_2(t))$, we have $u(t) \geq (1-a-b)\upsilon(t) = A_0\upsilon(t)$. Using it in (1.1), we get

$$\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)} + A_0k_3(t)\upsilon(\eta_3(t)) \leq 0, \quad t \geq t_0. \tag{3.9}$$

Differentiating (3.7) with respect to t, we obtain

$$\chi_1^{(1)}(t) = \frac{\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)}}{\upsilon(t)} - \frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{(\upsilon(t))^2}\upsilon^{(1)}(t).$$

Using (3.9), we get

$$\chi_1^{(1)}(t) \leq -A_0k_3(t) - \frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{(\upsilon(t))^2}\upsilon^{(1)}(t).$$

Using (3.5), we have

$$\chi_1^{(1)}(t) \leq -A_0k_3(t) - \frac{\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^2}{(\upsilon(t))^2}K_2(t).$$

Using (3.7), we have

$$\chi_1^{(1)}(t) \leq -A_0k_3(t) - \chi_1^2(t)K_2(t). \tag{3.10}$$

Integrating the resulting inequality from ϱ_1 to t after multiplying by $K_3(t)$, we obtain

$$\chi_1(t)K_3(t) - \chi_1(\varrho_1)K_3(\varrho_1) + \int_{\varrho_1}^{t} \chi_1(\nu)K_2(\nu)d\nu + A_0\int_{\varrho_1}^{t} k_3(\nu)K_3(\nu)d\nu + \int_{\varrho_1}^{t} \chi_1^2(\nu)K_2(\nu)K_3(\nu)d\nu \leq 0.$$

Using (3.8), we get

$$\int_{\varrho_1}^{t} \Big\{\chi_1(\nu)K_2(\nu) + \chi_1^2(\nu)K_2(\nu)K_3(\nu)\Big\}d\nu + A_0\int_{\varrho_1}^{t} k_3(\nu)K_3(\nu)d\nu \leq 1 + \chi_1(\varrho_1)K_3(\varrho_1).$$

Using the inequality $Q\psi^2 - P\psi \geq -\frac{P^2}{4Q}$, we get

$$\int_{\varrho_1}^{t} \Big[A_0 k_3(\nu)K_3(\nu) - \frac{K_2(\nu)}{4K_3(\nu)}\Big]d\nu \leq 1 + \chi_1(\varrho_1)K_3(\varrho_1).$$

On taking limit as $t \to \infty$, we get a contradiction to the condition (3.1). Suppose case (ii) holds. Define

$$\chi_2(t) = \frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{k_1(t)\upsilon^{(2)}(t)}. \tag{3.11}$$

From (3.5), we have

$$-1 \leq \chi_2(t)K_1(t). \tag{3.12}$$

Since $\upsilon^{(1)}(t) \geq 0$, we have

$$\begin{aligned} \upsilon^{(1)}(t) &\geq \int_{\varrho}^{t} \upsilon^{(2)}(\nu_1)d\nu_1 \\ &= \int_{\varrho}^{t} \frac{1}{k_1(\nu_1)}\Big(k_1(\nu_1)\upsilon^{(2)}(\nu_1)\Big)d\nu_1. \end{aligned}$$

As $k_1(t)\upsilon^{(2)}(t)$ is nonincreasing, we have

$$\upsilon^{(1)}(t) \geq k_1(t)\upsilon^{(2)}(t)\int_{\varrho}^{t} \frac{1}{k_1(\nu_1)}d\nu_1.$$

Integrating from ϱ to t, we obtain

$$\upsilon(t) \geq k_1(t)\upsilon^{(2)}(t)\int_{\varrho}^{t}\int_{\varrho}^{\nu_2} \frac{1}{k_1(\nu_1)}d\nu_1 d\nu_2,$$

which implies that

$$\frac{\upsilon(t)}{k_1(t)\upsilon^{(2)}(t)} \geq \int_{\varrho}^{t}\int_{\varrho}^{\nu_2} \frac{1}{k_1(\nu_1)}d\nu_1 d\nu_2. \tag{3.13}$$

Differentiating (3.11) with respect to t, we obtain

$$\begin{aligned}\chi_2^{(1)}(t) &= \frac{\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)}}{k_1(t)\upsilon^{(2)}(t)} - \frac{1}{k_2(t)}\left[\frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{k_1(t)\upsilon^{(2)}(t)}\right]^2\\ &= -\frac{k_3(t)u(\eta_3(t))}{k_1(t)\upsilon^{(2)}(t)} - \frac{\chi_2^2(t)}{k_2(t)}\\ &\leq -A_0\frac{k_3(t)\upsilon(\eta_3(t))}{k_1(\eta_3(t))\upsilon^{(2)}(\eta_3(t))} - \frac{\chi_2^2(t)}{k_2(t)}.\end{aligned}$$

Using (3.13), we get

$$\chi_2^{(1)}(t) \leq -A_0k_3(t)\int_{\varrho}^{\eta_3(t)}\int_{\varrho}^{\nu_2}\frac{1}{k_1(\nu_1)}d\nu_1d\nu_2 - \frac{\chi_2^2(t)}{k_2(t)}. \tag{3.14}$$

Integrating above inequality from ϱ_2 to t after multiplying by $K_1(t)$,

$$\begin{aligned}&K_1(t)\chi_2(t) - K_1(\varrho_2)\chi_2(\varrho_2) + \int_{\varrho_2}^{t}\frac{\chi_2(\nu)}{k_2(\nu)}d\nu + \int_{\varrho_2}^{t}\frac{K_1(\nu)\chi_2^2(\nu)}{k_2(\nu)}d\nu\\ &+A_0\int_{\varrho_2}^{t}K_1(\nu)k_3(\nu)\int_{\varrho}^{\eta_3(\nu)}\int_{\varrho}^{\nu_2}\frac{1}{k_1(\nu_1)}d\nu_1d\nu_2d\nu \leq 0.\end{aligned}$$

Using (3.12), we get

$$\begin{aligned}&\int_{\varrho_2}^{t}\left[\frac{\chi_2(\nu)}{k_2(\nu)} + \frac{K_1(\nu)\chi_2^2(\nu)}{k_2(\nu)} + A_0K_1(\nu)k_3(\nu)\int_{\varrho}^{\eta_3(\nu)}\int_{\varrho}^{\nu_2}\frac{1}{k_1(\nu_1)}d\nu_1d\nu_2\right]d\nu\\ &\qquad \leq 1 + K_1(\varrho_2)\chi_2(\varrho_2).\end{aligned}$$

Using the inequality $Q\psi^2 - P\psi \geq -\frac{P^2}{4Q}$, we get

$$\begin{aligned}&\int_{\varrho_2}^{t}\left[A_0K_1(\nu)k_3(\nu)\int_{\varrho}^{\eta_3(\nu)}\int_{\varrho}^{\nu_2}\frac{1}{k_1(\nu_1)}d\nu_1d\nu_2 - \frac{1}{4k_2(\nu)K_1(\nu)}\right]d\nu\\ &\qquad \leq 1 + K_1(\varrho_2)\chi_2(\varrho_2).\end{aligned}$$

On taking limit as $t \to \infty$, we get a contradiction to the condition (3.2).
Suppose case (iii) holds. Since $u(\eta_3(t)) \leq \upsilon(t)$, we have from (1.1)

$$\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)} \geq -k_3(t)\upsilon(t). \tag{3.15}$$

Integrating from ϱ to t, we get

$$k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \geq -\int_{\varrho}^{t}k_3(\nu)\upsilon(\nu)d\nu.$$

As $\upsilon(t)$ is increasing, we get

$$\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \geq -\frac{\upsilon(t)}{k_2(t)}K_1^{\varrho}(t).$$

Integrating from ϱ to t, we get

$$\upsilon^{(2)}(t) \geq -\frac{\upsilon(t)}{k_1(t)}\int_{\varrho}^{t}\frac{K_1^{\varrho}(\nu)}{k_2(\nu)}d\nu = -\frac{\upsilon(t)}{k_1(t)}K_2^{\varrho}(t),$$

which implies that

$$\frac{\upsilon(t)}{\upsilon^{(2)}(t)} \geq -\frac{k_1(t)}{K_2^{\varrho}(t)}. \tag{3.16}$$

Define

$$\chi_3(t) = c(t)\frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{k_1(\eta_3(t))\upsilon^{(2)}(\eta_3(t))}. \tag{3.17}$$

Differentiating (3.17) with respect to t, we have

$$\chi_3^{(1)}(t) = c^{(1)}(t)\frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{k_1(\eta_3(t))\upsilon^{(2)}(\eta_3(t))} + c(t)\left[\frac{k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}}{k_1(\eta_3(t))\upsilon^{(2)}(\eta_3(t))}\right]^{(1)}.$$

Using the fact that $\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}$ is decreasing and (3.17), we have

$$\chi_3^{(1)}(t) \leq \frac{c^{(1)}(t)}{c(t)}\chi_3(t) + c(t)\frac{\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)}}{k_1(\eta_3(t))\upsilon^{(2)}(\eta_3(t))} - \frac{c(t)}{k_2(t)}\chi_3^2(t).$$

Using (1.1) and inequality $u(t) \geq A_0\upsilon(t)$, we get

$$\chi_3^{(1)}(t) \leq \frac{c^{(1)}(t)}{c(t)}\chi_3(t) - A_0c(t)\frac{k_3(t)\upsilon(\eta_3(t))}{k_1(\eta_3(t))\upsilon^{(2)}(\eta_3(t))} - c(t)\frac{\chi_3^2(t)}{k_2(t)}.$$

Using (3.16), we have

$$\chi_3^{(1)}(t) \leq \frac{c^{(1)}(t)}{c(t)}\chi_3(t) - \frac{A_0c(t)k_3(t)}{K_2^{\varrho}(\eta_3(t))} - c(t)\frac{\chi_3^2(t)}{k_2(t)}.$$

Using the inequality $-P\psi - Q\psi^2 \leq \frac{P^2}{4Q}$, we get

$$\chi_3^{(1)}(t) \leq \frac{\big(c^{(1)}(t)\big)^2}{4\big(c(t)\big)^3}k_2(t) - \frac{A_0c(t)k_3(t)}{K_2^{\varrho}(\eta_3(t))}. \tag{3.18}$$

Integrating from ϱ_3 to t, we get

$$\chi_3(t) - \chi_3(\varrho_3) \leq \int_{\varrho_3}^{t}\left[\frac{\big(c^{(1)}(\nu)\big)^2}{4\big(c(\nu)\big)^3}k_2(\nu) - \frac{A_0c(\nu)k_3(\nu)}{K_2^{\varrho}(\eta_3(\nu))}\right]d\nu,$$

which implies that

$$\int_{\varrho_3}^{t}\left[\frac{A_0c(\nu)k_3(\nu)}{K_2^{\varrho}(\eta_3(\nu))} - \frac{\big(c^{(1)}(\nu)\big)^2}{4\big(c(\nu)\big)^3}k_2(\nu)\right]d\nu \leq \chi_3(\varrho_3).$$

On taking limit as $t \to \infty$, we get a contradiction to the condition (3.3).
Suppose case (iv) holds.
Define

$$\chi_4(t) = d(t)\frac{\upsilon^{(1)}(t)}{\upsilon(\eta_3(t))}.$$

Differentiating with respect to t, we get

$$\begin{aligned}\chi_4^{(1)}(t) &= d^{(1)}(t)\frac{\upsilon^{(1)}(t)}{\upsilon(\eta_3(t))} + d(t)\left(\frac{\upsilon^{(1)}(t)}{\upsilon(\eta_3(t))}\right)^{(1)} \\ &\leq d^{(1)}(t)\frac{\upsilon^{(1)}(t)}{\upsilon(\eta_3(t))} + d(t)\frac{\upsilon^{(2)}(t)}{\upsilon(\eta_3(t))} - d(t)\left(\frac{\upsilon^{(1)}(t)}{\upsilon(\eta_3(t))}\right)^2 \\ &= \frac{d^{(1)}(t)}{d(t)}\chi_4(t) + d(t)\frac{\upsilon^{(2)}(t)}{\upsilon(\eta_3(t))} - \frac{\chi_4^2(t)}{d(t)}. \qquad (3.19)\end{aligned}$$

Using the inequality $u(t) \geq A_0\upsilon(t)$ in (1.1), we get

$$\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)} \leq -A_0k_3(t)\upsilon(\eta_3(t)).$$

Integrating the above inequality from t to ∞, we get

$$-k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \leq -A_0\int_t^\infty k_3(\nu)\upsilon(\eta_3(\nu))d\nu.$$

As $\upsilon(t)$ is increasing, we have

$$-\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \leq -\frac{A_0\upsilon(\eta_3(t))}{k_2(t)}R_1(t).$$

Now integrating the resulting inequality from t to ∞, we get

$$k_1(t)\upsilon^{(2)}(t) \leq -A_0\upsilon(\eta_3(t))\int_t^\infty \frac{R_1(\nu)}{k_2(\nu)}d\nu,$$

which implies that

$$\frac{\upsilon^{(2)}(t)}{\upsilon(\eta_3(t))} \leq -\frac{A_0}{k_1(t)}R_2(t).$$

Using the above inequality in (3.19), we get

$$\chi_4^{(1)}(t) \leq \frac{d^{(1)}(t)}{d(t)}\chi_4(t) - \frac{A_0d(t)R_2(t)}{k_1(t)} - \frac{\chi_4^2(t)}{d(t)}.$$

Using the inequality $P\psi - Q\psi^2 \leq \frac{P^2}{4Q}$, we get

$$\chi_4^{(1)}(t) \leq \frac{[d^{(1)}(t)]^2}{4d(t)} - \frac{A_0d(t)R_2(t)}{k_1(t)}.$$

Integrating the above inequality from ϱ_4 to t, we get

$$\chi_4(t) - \chi_4(\varrho_4) \leq \int_{\varrho_4}^{t} \left[\frac{\left(d^{(1)}(\nu)\right)^2}{4d(\nu)} - \frac{A_0 d(\nu) R_2(\nu)}{k_1(\nu)} \right] d\nu,$$

which implies that

$$\int_{\varrho_4}^{t} \left[\frac{A_0 d(\nu) R_2(\nu)}{k_1(\nu)} - \frac{\left(d^{(1)}(\nu)\right)^2}{4d(\nu)} \right] d\nu \leq \chi_4(\varrho_4).$$

Condition (3.4) contradicted on taking limit as $t \to \infty$. This completes the proof. □

Corollary 3.2. *Suppose $u(t)$ be an eventually positive solution of (1.1) such that case (i) of Lemma 2.3 holds. Further, if for $\varrho \geq t_0$*

$$\int_{\varrho}^{\infty} \int_{\nu_4}^{\infty} \frac{1}{k_1(\nu_3)} \int_{\nu_3}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3 d\nu_4 = \infty, \tag{3.20}$$

then $\lim_{t\to\infty} u(t) = 0$.

Proof. Suppose that $u(t)$ is an eventually positive solution of (1.1). Then for some $t \geq \varrho$, $u(\eta_r(t)) > 0,\ r = 1, 2, 3$. Since case (i) of Lemma 2.3 holds.
So if $\lim_{t\to\infty} \upsilon(t) = L$, then $L \geq 0$. Claim $L = 0$, otherwise $L > 0$. Since $u(t) \geq A_0 \upsilon(t)$, from (1.1), we have

$$\left(k_2(t)\left(k_1(t)\upsilon^{(2)}(t)\right)^{(1)}\right)^{(1)} \leq -A_0 k_3(t) \upsilon(\eta_3(t)), \quad t \geq t_0.$$

Integrating the above inequality from ϱ to t, we get

$$k_2(t)\left(k_1(t)\upsilon^{(2)}(t)\right)^{(1)} \leq -A_0 \int_{\varrho}^{t} k_3(\nu) \upsilon(\eta_3(\nu)) d\nu.$$

As $\upsilon(t)$ is decreasing, we have

$$\left(k_1(t)\upsilon^{(2)}(t)\right)^{(1)} \leq -A_0 \frac{\upsilon(t)}{k_2(t)} \int_{\varrho}^{t} k_3(\nu_1) d\nu_1.$$

Integrating the above inequality from t to ∞, we get

$$k_1(t)\upsilon^{(2)}(t) \geq A_0 \int_{t}^{\infty} \frac{\upsilon(\nu_2)}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2.$$

As $\upsilon(t) \geq L$ for $t \geq \varrho$, we have

$$k_1(t)\upsilon^{(2)}(t) \geq A_0 L \int_{t}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2.$$

Integrating the resulting inequality from t to ∞ after dividing by $k_1(t)$, we get

$$-\upsilon^{(1)}(t) \geq A_0 L \int_t^{\infty} \frac{1}{k_1(\nu_3)} \int_{\nu_3}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3.$$

Integrating the above inequality from ϱ to t, we get

$$\upsilon(\varrho) \geq A_0 L \int_{\varrho}^{t} \int_{\nu_4}^{\infty} \frac{1}{k_1(\nu_3)} \int_{\nu_3}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3 d\nu_4.$$

On taking limit as $t \to \infty$, we get a contradiction to the condition (3.20). Hence, $L = 0$. Since $u(t) \leq \upsilon(t)$, we have $\lim_{t\to\infty} u(t) = 0$. Now the proof is completed. □

Corollary 3.3. *Let $u(t)$ be an eventually positive solution of (1.1) such that the case (ii) of Lemma 2.3 holds. Further if for $\varrho \geq t_0$*

$$\int_{\varrho}^{\infty} \int_{\varrho}^{\nu_4} \frac{1}{k_1(\nu_3)} \int_{\nu_3}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\nu_2}^{\infty} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3 d\nu_4 = -\infty, \quad (3.21)$$

then $\lim_{t\to\infty} u(t) = \infty$.

Proof. Assume $\lim_{t\to\infty} \upsilon(t) = L$. Since case (ii) of Lemma 2.3 holds, we have $L \leq \infty$. Claim $L = \infty$, otherwise $L < \infty$. Since $u(\eta_3(t)) \leq \upsilon(t)$, from (1.1), we have

$$\Big(k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)}\Big)^{(1)} \geq -k_3(t)\upsilon(t).$$

Integrating the above inequality from t to ∞, we get

$$-k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \geq -\int_t^{\infty} k_3(\nu_1)\upsilon(\nu_1) d\nu_1.$$

As $\upsilon(t) \leq L$, we have

$$-k_2(t)\Big(k_1(t)\upsilon^{(2)}(t)\Big)^{(1)} \geq -L \int_t^{\infty} k_3(\nu_1) d\nu_1.$$

Integrating the resulting inequality from t to ∞ after dividing by $k_2(t)$, we get

$$k_1(t)\upsilon^{(2)}(t) \geq -L \int_t^{\infty} \frac{1}{k_2(\nu_2)} \int_{\nu_2}^{\infty} k_3(\nu_1) d\nu_1 d\nu_2.$$

Integrating the resulting inequality twice from ϱ to t after dividing by $k_1(t)$, we get

$$\upsilon(t) \geq -L \int_{\varrho}^{t} \int_{\varrho}^{\nu_4} \frac{1}{k_1(\nu_3)} \int_{\nu_3}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\nu_2}^{\infty} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3 d\nu_4.$$

Condition (3.21) contradicted on taking limit as $t \to \infty$. Hence, $L = \infty$. Since $u(t) \geq A_0 v(t)$, we have $\lim_{t\to\infty} u(t) = \infty$. This completes the proof. □

Following the proof of above corollary, we can prove the following two results:

Corollary 3.4. *Let $u(t)$ be an eventually positive solution of (1.1) such that case (iii) of Lemma 2.3 holds. Further if for $\varrho \geq t_0$*

$$\int_{\varrho}^{\infty} \int_{\varrho}^{\nu_4} \frac{1}{k_1(\nu_3)} \int_{\varrho}^{\nu_3} \frac{1}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3 d\nu_4 = -\infty,$$

then $\lim_{t\to\infty} u(t) = \infty$.

Corollary 3.5. *Let $u(t)$ be an eventually positive solution of (1.1) such that case (iv) of Lemma 2.3 holds. Further if for $\varrho \geq t_0$*

$$\int_{\varrho}^{\infty} \int_{\nu_4}^{\infty} \frac{1}{k_1(\nu_3)} \int_{\nu_3}^{\infty} \frac{1}{k_2(\nu_2)} \int_{\varrho}^{\nu_2} k_3(\nu_1) d\nu_1 d\nu_2 d\nu_3 d\nu_4 = -\infty,$$

then $\lim_{t\to\infty} u(t) = \infty$.

4. Application

Example 4.1. Suppose the following model of neutral differential equation:

$$\Big(e^{2t}\Big(e^{-t}\Big(u(t) + au(t-2\pi) + bu(t+\pi)\Big)^{(2)}\Big)^{(1)}\Big)^{(1)} + 2e^t u\Big(t - \frac{\pi}{6}\Big) = 0, \quad t \geq t_0, \quad (4.1)$$

$u^{(i)}(t)$, denote the derivative of order i with respect to t.

Here $k_1(t) = e^{-t}$, $k_2(t) = e^{2t}$, $k_3(t) = 2e^t$, $c(t) = e^{3t}$, $d(t) = t^3$, $\eta_3(t) = t - \frac{\pi}{6}$.
We can easily calculate the following.
$K_1(t) = \int_t^{\infty} \frac{1}{k_2(\nu)} d\nu = \int_t^{\infty} e^{-2\nu} d\nu = \frac{e^{-2t}}{2}$, $K_2(t) = \int_t^{\infty} \frac{K_1(\nu)}{k_1(\nu)} d\nu = \frac{1}{2}e^{-t}$, $K_3(t) = \int_t^{\infty} K_2(\nu) d\nu = \frac{1}{2}e^{-t}$, $K_1^{\varrho}(t) = \int_{\varrho}^{t} k_3(\nu) d\nu = 2(e^t - e^{\varrho})$, $K_2^{\varrho}(t) = \int_{\varrho}^{t} \frac{K_1^{\varrho}(\nu)}{k_2(\nu)} d\nu = -2e^{-t} + e^{\varrho}e^{-2t} + e^{-\varrho}$, $R_1(t) = \infty$ and $R_2(t) = \infty$.
Next we check the conditions (3.1), (3.2), (3.3), and (3.4).
Clearly

$$\int_{t_0}^{\infty} \Big[A_0 k_3(\nu) K_3(\nu) - \frac{K_2(\nu)}{4K_3(\nu)}\Big] d\nu = \infty.$$

Thus the condition (3.1) is satisfied.

$$\int_{t_0}^{\infty}\left[A_0K_1(\nu)k_3(\nu)\int_{\varrho}^{\eta_3(\nu)}\int_{\varrho}^{\nu_2}\frac{1}{k_1(\nu_1)}d\nu_1 d\nu_2-\frac{1}{4k_2(\nu)S_1(\nu)}\right]d\nu$$
$$=\int_{t_0}^{\infty}\Big[A_0e^{-\nu}\int_{\varrho}^{\nu-\frac{\pi}{6}}\int_{\varrho}^{\nu_2}e^{\nu_1}d\nu_1 d\nu_2-\frac{1}{2}\Big]d\nu=\infty.$$

Thus, the condition (3.2) is satisfied.

$$\int_{t_0}^{\infty}\left[\frac{A_0c(\nu)k_3(\nu)}{K_2^{\varrho}(\eta_3(\nu))}-\frac{\big(c^{(1)}(\nu)\big)^2}{4\big(c(\nu)\big)^3}k_2(\nu)\right]d\nu$$
$$=\int_{t_0}^{\infty}\Big[\frac{2A_0e^{4\nu}}{-2e^{-(\nu-\pi)}+e^{\varrho+2\pi}e^{-2\nu}+e^{-\varrho}}-\frac{9}{4}e^{-\nu}\Big]d\nu=\infty.$$

Thus the condition (3.3) is satisfied. Similarly, we can show that condition (3.4) is satisfied. As all the conditions of Theorem 3.1 are fulfilled, therefore by applying Theorem 3.1, we conclude that each non zero solution of the problem (4.1) is oscillatory.

5. Conclusion

The goal of this paper was to provide an investigation of the oscillatory and asymptotic behavior for fourth-order neutral differential equations with mixed delay terms. We reduced the fourth-order neutral differential equation using suitable substitutions into the first-order differential inequality. We used the comparison theorem to ensure that every solution of the studied equation oscillates. In Corollaries, we investigated the asymptotic nature of the solution.

It would be of interest to discuss the problem (1.1) with different neutral coefficients (where the neutral coefficients are not constants). Studying the problem (1.1) with a nonlinear neutral term would also be interesting.

References

[1] S.R. Grace, J. Džurina, I. Judlovska, and T. Li, On the oscillation of fourth-order delay differential equations, *Adv. Difference Equ.,* **2019***(118), (2019), 1-15.*

[2] C.H. Zhange, R.P. Agarwal, M. Bohner, T. Li, Oscillation of fourth-order delay dynamic equations, *Sci. China Math.,* **58***(1), (2015), 143-160*

[3] T. Li and Y.V. Rogovchenko, Oscillation criteria for even-order neutral differential equations, *Appl. Math. Lett.,* **61***, (2016), 35-41*

[4] Arul, R., Shobha, V.S.: Oscillation of second order nonlinear neutral differential equations with mixed neutral term, *J. Appl. Math. Phy.* **3***, 1080-1089 (2015)*

[5] S.H. Abdallah, Oscillatory and non-oscillatory behaviour of second-order neutral delay differential equations, *Appl. Math. Comput.,* **135***, (2003), 333-344*
[6] O. Bazighifan and H. Ramos, On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term, *Appl. Math. Lett.,* **107***, (2020), 1-9*
[7] O. Bazighifan, Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations, *Adv. Difference Equ.,* **2020***(201), (2020), 1-12*
[8] O. Bazighifan, On the oscillation of certain fourth-order neutral differential equations with p-Laplacian like operator, *Appl. Math. Comput.,* **386***, (2020), 1-8*
[9] O. Bazighifan, Oscillatory applications of some fourth-order differential equations, *Math. Methods Appl. Sci., https://doi.org/10.1002/mma.6694*
[10] B. Baculíková and J. Džurina, Oscillation of third-order neutral differential equations, *Math. Comput. Model. Dyn. Syst.,* **52***, (2010), 215-226*
[11] B. Baculíková and J. Džurina, Oscillation theorems for second order nonlinear neutral differential equations, *Comput. Math. Appl.,* **62***, (2011), 4472-4478*
[12] M. Bartušek and Z. Došlá, Oscillation of fourth order sub-linear differential equations, *Appl. Math. Lett.* **36***, (2014), 36-39*
[13] D.D. Bainov and D.P. Mishev, Oscillation Theory for Neutral Differential Equations with Delay, *IOP Publishing Ltd, New York, 1991*
[14] L. Erbe, A. Peterson and S.H. Saker, Hille and Nehar type criteria for third order dynamics equation, *J. Math. Anal. Appl.,* **329***, (2007), 112-131*
[15] A. Garab and I.P. Stavroulakis, Oscillation criteria for first order linear delay differential equations with several variable delays, *Appl. Math. Lett.,* **106***, (2020), 1063-1066*
[16] Z. Han, T. Li, S. Sun and W. Chen, On the oscillation of second order neutral delay differential equation, *Advan. Difference Equ.,* **2010***, (2010), 1-8*
[17] T. Li, C. Zhang and E. Thandapani, Asymptotic Behavior of fourth-order neutral dynamics equations with noncanonical operators, *Taiwanese J. Math.,* **18***(4), (2014), 1003-1019*
[18] X. Lina and X.H. Tangb, Oscillation of solutions of neutral differential equations with a superlinear neutral term, *Appl. Math. Lett.,* **20***, (2007), 1016-1022*
[19] S. Ruan, Oscillations of second order neutral differential equations, *Canad. Math. Bull.,* **36** *(4), (1993), 485-496*
[20] A. Tiryaki and M.F. Aktas, Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping, *J. Math. Anal. Appl.,* **325***, (2007), 54-68*
[21] R. Xu and F. Meng, New Kamenev-type oscillation criteria for second order neutral nonlinear differential equations, *Appl. Math. Comput.,* **188***, (2007), 1364-1370*
[22] J. Yan, Oscillation of nonlinear delay impulsive differential equations and inequalities,*J. Math. Anal. Appl.,* **265***, (2002), 332-342*
[23] C. Zhang, T. Li, R.P. Agarwal and M. Bohner, Oscillation results for fourth-order non linear dynamic equations, *Appl. Math. Lett.,* **25***, (2012), 2058-2065*

[24] R.K. Zhuang and W.T. Li, Interval oscillation criteria for second order neutral nonlinear differential equations, *Appl. Math. Comput.* **157***, (2004), 39-51*