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A brief on some equalities for continuous g-frames in Hilbert spaces

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Abstract. In this work, we generalize the equalities from g-frame and g-Bessel sequences to continuous g-frame and continuous g-Bessel sequences in Hilbert spaces. This generalization enables us to obtain the relationship between continuous g-frames and their alternate dual on a measure space.

Keywords: Hilbert spaces, continuous g-Bessel sequences, continuous g-frames, Pseudo-inverse operator.

2000 Mathematics subject classification: 42C15; Secondary 42C99.

1. Introduction

Frames for Hilbert space were first formally defined by Duffin and Schaeffer [4] in 1952 for studying non-harmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossman, and Meyer [3]. Frames are a redundant set of vectors that yield a representation for each vector in the space. One significant application of frames is in wireless sensor networks and signal preprocessing. Weaving frames, introduced by Li, are powerful tools in these fields due to their ability

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to enable distributed processing under different frames and signal preprocessing using Gabor frames [6]. The concept of g-frames, presented by W. Sun, is an extension that includes bounded invertible operators and all the above-mentioned extensions of discrete frames [9]. Continuous g-frames, a generalization of discrete g-frames, were introduced by Abdollahpour and Faroughi. For a comprehensive understanding of continuous frames, please refer to the following sources [2, 5, 8]. Throughout this work,(Ω, μ) is a measure space and $\mathcal{H}$ and $\mathcal{K}$ are two complex Hilbert spaces, $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ is a sequence of closed subspaces of $\mathcal{K}$. $\mathcal{B}(\mathcal{H}, \mathcal{K}_\omega)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}_\omega$.

Definition 1.1. $\{\Lambda_\omega \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\omega) : \quad \omega \in \Omega\}$ is called a continuous generalized frame or simply a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ if

(1) For each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,
(2) There are two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2. \tag{1.1}$$

A and B are called lower and upper continuous g-frame bounds, respectively. A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a continuous g-Bessel family if the right hand inequality in (1.1) holds. In this case, B is called the continuous g-Bessel constant.

Definition 1.2. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ be two continuous g-frames for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ such that

$$\langle f, g\rangle = \int_\Omega \langle f, \Gamma_\omega^* \Lambda_\omega g\rangle d\mu(\omega). \tag{1.2}$$

Then $\{\Gamma_\omega\}_{\omega\in\Omega}$ is called an alternate dual continuous g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

Based on their definition, if the measure space is set $\Omega := \mathbb{N}$ and μ be the counting measure, then the continuous g-frame is a discrete g-frame and so it is expected that some equalities and inequalities of g-frames can be satisfied in continuous g-frames. Xiuge Zhua and Guochang Wub [10] generalized the equalities to a more general form which did not involve the real parts of the complex numbers.

Jian-Zhen Li and Yu-Can Zhu [7] introduced an operator L and established some general equalities and inequalities for g-Bessel sequences in Hilbert spaces with pseudo-inverse operator $L^\dagger$ [7].

Lemma 1.3. [7] *Suppose that $L, P, Q \in \mathcal{B}(\mathcal{H})$, such that $P + Q = L$, and that the range of L is closed. Then*

$$L^* L^\dagger P + Q^* L^\dagger Q = Q^* L^\dagger L + P^* L^\dagger P, \tag{1.3}$$

where $L^\dagger$ is the pseudo-inverse of L.

Lemma 1.4. [10] *Let P and Q be two linear bounded operators on $\mathcal{H}$ such that $P+Q=I$; then*

$$P - P^*P = Q^* - Q^*Q,$$

where I denotes the identity operator on $\mathcal{H}$.

In this work, we will extend these equalities for continuous g-frames and continuous g-Bessel sequences in Hilbert spaces. We also show that generalization of Theorem 2.2 in [10] is a special case of generalization of Theorem 2.1 in [7].

2. The main results and their proofs

Proposition 2.1. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$. For any measurable subset $K\subset\Omega$, we define the operator L_K as follows:*

$$L_K : \mathcal{H} \longrightarrow \mathcal{H} \qquad \langle L_K f, g\rangle = \int_K \langle f, \Gamma_\omega^* \Lambda_\omega g\rangle d\mu(\omega), \tag{2.1}$$

then L_K is a bounded linear operator.

Proof. Let A,B be continuous g-Bessel bounds for $\{\Lambda_\omega\}$ and $\{\Gamma_\omega\}$ respectively; then for any $f\in\mathcal{H}$

$$\begin{aligned}\|L_K f\| = \sup_{\|g\|=1} |\langle L_K f, g\rangle| &= \sup_{\|g\|=1} \Big|\int_K \langle f, \Gamma_\omega^*\Lambda_\omega g\rangle d\mu(\omega)\Big| \\ &\le \sup_{\|g\|=1} \Big(\int_K \|\Gamma_\omega f\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} \Big(\int_K \|\Lambda_\omega g\|^2 d\mu(\omega)\Big)^{\frac{1}{2}} \\ &\le \sqrt{A}\Big(\int_K \|\Gamma_\omega f\|^2 d\mu(\omega)\Big)^{\frac{1}{2}}.\end{aligned}$$

Hence, $\|L_K f\| \le \sqrt{AB}\|f\|$. It is easy to see that L_K is linear. Therefore L_K is a bounded linear operator. □

Theorem 2.2. *Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$, $\{\Gamma_\omega\}_{\omega\in\Omega}$ and $\{\Theta_\omega\}_{\omega\in\Omega}$ are continuous g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ and the operator $L := L_\Omega$ defined by (2.1) has a closed range; then for all $f\in\mathcal{H}$:*

$$\begin{aligned}&\int_\Omega \langle \Theta_\omega f, \Lambda_\omega (L^\dagger)^* L f\rangle d\mu(\omega) + \int_\Omega\int_\Omega \langle L^\dagger \Lambda_\omega^* \Delta_\omega f, \Lambda_\upsilon^* \Delta_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon) \\ &= \int_\Omega \langle \Lambda_\omega L^\dagger L f, \Delta_\omega f\rangle d\mu(\omega) + \int_\Omega\int_\Omega \langle L^\dagger \Lambda_\omega^* \Theta_\omega f, \Lambda_\upsilon^* \Theta_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon),\end{aligned} \tag{2.2}$$

where $\Delta_\omega = \Gamma_\omega - \Theta_\omega$.
Moreover, if L *is positive, then for all* $f \in \mathcal{H}$:

$$Re\Big\{ \int_\Omega \langle \Theta_\omega f, \Lambda_\omega (L^\dagger)^* L f\rangle d\mu(\omega)\Big\}$$
$$+ Re\Big\{ \int_\Omega \int_\Omega \langle L^\dagger \Lambda_\omega^* \Delta_\omega f, \Lambda_\upsilon^* \Delta_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon)\Big\} \geq \frac{3}{4}\|L^{\frac{1}{2}} f\|^2, \qquad (2.3)$$

and the equality of (2.3) holds if and only if for all $f, g \in \mathcal{H}$,

$$\langle L^{\frac{1}{2}} f, g\rangle = 2\int_\Omega \langle \Lambda_\omega^* \Delta_\omega f, L^\dagger L^{\frac{1}{2}} g\rangle d\mu(\omega).$$

Proof. let $\langle Pf, g\rangle = \int_\Omega \langle f, \Theta_\omega^* \Lambda_\omega g\rangle d\mu(\omega)$ and $\langle Qf, g\rangle = \int_\Omega \langle f, \Delta_\omega^* \Lambda_\omega g\rangle d\mu(\omega)$. From Proposition 2.1, we can see that $P, Q \in \mathcal{B}(\mathcal{H})$. Therefore for all $f \in \mathcal{H}$,

$$\langle Pf + Qf, g\rangle = \int_\Omega \langle f, \Big(\Theta_\omega^* \Lambda_\omega + \Delta_\omega^* \Lambda_\omega\Big) g\rangle d\mu(\omega) = \int_\Omega \langle f, \Gamma_\omega^* \Lambda_\omega g\rangle d\mu(\omega) = \langle Lf, g\rangle.$$

By Lemma 1.3, we obtain

$$\begin{aligned}
&\int_\Omega \langle \Theta_\omega f, \Lambda_\omega (L^\dagger)^* L f\rangle d\mu(\omega) + \int_\Omega \int_\Omega \langle L^\dagger \Lambda_\omega^* \Delta_\omega f, \Lambda_\upsilon^* \Delta_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon) \\
&= \int_\Omega \langle f, \Theta_\omega^* \Lambda_\omega (L^\dagger)^* L f\rangle d\mu(\omega) + \int_\Omega \int_\Omega \langle \Lambda_\omega^* \Delta_\omega f, (L^\dagger)^* \Lambda_\upsilon^* \Delta_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon) \\
&= \langle Pf, (L^\dagger)^* L f\rangle + \int_\Omega \langle Qf, (L^\dagger)^* \Lambda_\upsilon^* \Delta_\upsilon f\rangle d\mu(\upsilon) \\
&= \langle L^* L^\dagger P f, f\rangle + \langle Q^* L^\dagger Q f, f\rangle \\
&= \langle Q^* L^\dagger L f, f\rangle + \langle P^* L^\dagger P f, f\rangle \\
&= \int_\Omega \langle L^\dagger L f, \Lambda_\omega^* \Delta_\omega f\rangle d\mu(\omega) + \int_\Omega \langle L^\dagger P f, \Lambda_\upsilon^* \Theta_\upsilon f\rangle d\mu(\upsilon) \\
&= \int_\Omega \langle \Lambda_\omega L^\dagger L f, \Delta_\omega f\rangle d\mu(\omega) + \int_\Omega \int_\Omega \langle L^\dagger \Lambda_\omega^* \Theta_\omega f, \Lambda_\upsilon^* \Theta_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon).
\end{aligned}$$

Hence, the equality (2.2) holds. For the next part we set $\langle Tf, g\rangle := \langle f, Q^* L^\dagger L^{\frac{1}{2}} g\rangle$ and the rest of the proof is the same as proof of second part of Theorem 2.1 in [7]. □

Example 2.3. Suppose $\mathcal{H}$ and $\mathcal{K}_\omega$ are defined as follows:

$$\mathcal{H} = \left\{ A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\},$$

and

$$\mathcal{K}_\omega = \left\{ B = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \;\middle|\; x, y \in \mathbb{C} \right\}.$$

With the inner product defined as $\langle A, B\rangle = \mathrm{tr}(A\overline{B}^T)$, both $\mathcal{H}$ and $\mathcal{K}_\omega$ are Hilbert spaces. Additionally, consider a measure space $(\Omega = [0,1], d\mu)$, where $d\mu$ is the Lebesgue measure restricted to $[0,1]$. Now, suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a sequence of operators from $\mathcal{H}$ to $\mathcal{K}_\omega$ defined by $\Lambda_\omega(A) = \begin{bmatrix} \omega a & 0 \\ 0 & \frac{\omega}{2}b \end{bmatrix}$. In this case, since $\|\Lambda_\omega\| \le \omega$, the sequence $\{\Lambda_\omega\}_{\omega\in\Omega}$ is bounded, and therefore, it is continuous. Furthermore, the sequence $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence, as

$$\begin{aligned}
\int_{[0,1]} \langle \Lambda_\omega(A), \Lambda_\omega(A)\rangle d\mu(\omega) &= \int_{[0,1]} \langle \begin{bmatrix} \omega a & 0 \\ 0 & \frac{\omega}{2}b \end{bmatrix}, \begin{bmatrix} \omega a & 0 \\ 0 & \frac{\omega}{2}b \end{bmatrix} \rangle d\mu(\omega) \\
&= \frac{1}{3}|a|^2 + \frac{1}{12}|b|^2 \\
&\le \frac{1}{3}|a|^2 + \frac{1}{3}|b|^2 \\
&= \frac{1}{3}\|A\|^2.
\end{aligned}$$

We also define $\Gamma_\omega = \Lambda_\omega$ and $\Theta_\omega(A) = \begin{bmatrix} \frac{\omega}{2}a & 0 \\ 0 & \omega b \end{bmatrix}$, which forms another continuous g-Bessel sequence.

The difference $\Delta_\omega = \Gamma_\omega - \Theta_\omega$ is given by the following matrix expression:

$$\Delta_\omega(A) = \begin{bmatrix} \frac{\omega}{2}a & 0 \\ 0 & -\frac{\omega}{2}b \end{bmatrix}$$

Additionally, the following matrix expressions are provided:

$$\Lambda_\omega^*\Theta_\omega(A) = \begin{bmatrix} \frac{\omega^2}{2}a & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{2}b \end{bmatrix},$$

and

$$\Lambda_\omega^*\Delta_\omega(A) = \begin{bmatrix} \frac{\omega^2}{2}a & 0 & 0 \\ 0 & 0 & -\frac{\omega^2}{4}b \end{bmatrix}.$$

The continuous g-frame operator, denoted as S_Λ, is defined as:

$$\langle S_\Lambda A, B\rangle = \int_{[0,1]} \langle \Lambda_\omega A, \Lambda_\omega B\rangle d\mu(\omega) = \frac{1}{3}a\,\overline{c} + \frac{1}{12}b\,\overline{d},$$

Thus,

$$\langle S_\Lambda^{-1}A, B\rangle = 3a\,\overline{c} + 12b\,\overline{d}.$$

Since $\Gamma_\omega = \Lambda_\omega$, we can conclude that $L = S_\Lambda$, and therefore, $L^\dagger = S_\Lambda^{-1}$.

To verify the validity of the equation (2.2) in this example, we will calculate both the left-hand side and the right-hand side of the equation and compare them.

$$\int_\Omega \langle \Theta_\omega A, \Lambda_\omega (L^\dagger)^* LA\rangle d\mu(\omega) + \int_\Omega\int_\Omega \langle L^\dagger \Lambda_\omega^* \Delta_\omega A, \Lambda_\upsilon^* \Delta_\upsilon A\rangle d\mu(\omega)d\mu(\upsilon)$$

$$= \int_{[0,1]} \langle S_\Lambda^{-1} \begin{bmatrix} \frac{\omega^2}{2}a & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{2}b \end{bmatrix}, S_\Lambda \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \end{bmatrix}\rangle d\mu(\omega)$$

$$+ \int_{[0,1]}\int_{[0,1]} \langle S_\Lambda^{-1} \begin{bmatrix} \frac{\omega^2}{2}a & 0 & 0 \\ 0 & 0 & -\frac{\omega^2}{4}b \end{bmatrix}, \begin{bmatrix} \frac{\upsilon^2}{2}a & 0 & 0 \\ 0 & 0 & -\frac{\upsilon^2}{4}b \end{bmatrix}\rangle d\mu(\omega)d\mu(\upsilon)$$

$$= \int_{[0,1]} \frac{\omega^2}{2}(|a|^2 + |b|^2)d\mu(\omega) + \int_{[0,1]} (\frac{1}{4}\upsilon^2|a|^2 + \frac{1}{4}\upsilon^2|b|^2)d\mu(\upsilon)$$

$$= \frac{1}{6}(|a|^2 + |b|^2) + \frac{1}{12}(|a|^2 + |b|^2)$$

$$= \frac{1}{4}(|a|^2 + |b|^2).$$

Similarly,

$$\int_\Omega \langle \Lambda_\omega L^\dagger LA, \Delta_\omega A\rangle d\mu(\omega) + \int_\Omega\int_\Omega \langle L^\dagger \Lambda_\omega^* \Theta_\omega A, \Lambda_\upsilon^* \Theta_\upsilon A\rangle d\mu(\omega)d\mu(\upsilon)$$

$$= \int_{[0,1]} \langle \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \end{bmatrix}, \begin{bmatrix} \frac{\omega^2}{2}a & 0 & 0 \\ 0 & 0 & -\frac{\omega^2}{4}b \end{bmatrix}\rangle d\mu(\omega)$$

$$+ \int_{[0,1]}\int_{[0,1]} \langle S_\Lambda^{-1} \begin{bmatrix} \frac{\omega^2}{2}a & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{2}b \end{bmatrix}, \begin{bmatrix} \frac{\upsilon^2}{2}a & 0 & 0 \\ 0 & 0 & \frac{\upsilon^2}{2}b \end{bmatrix}\rangle d\mu(\omega)d\mu(\upsilon)$$

$$= \int_{[0,1]} (\frac{\omega^2}{2}|a|^2 - \frac{\omega^2}{4}|b|^2)d\mu(\omega) + \int_{[0,1]} (\frac{1}{4}\upsilon^2|a|^2 + \upsilon^2|b|^2)d\mu(\upsilon)$$

$$= (\frac{1}{6}|a|^2 - \frac{1}{12}|b|^2) + (\frac{1}{12}|a|^2 + \frac{1}{3}|b|^2)$$

$$= \frac{1}{4}(|a|^2 + |b|^2).$$

Therefore, the left-hand side and the right-hand side of the equation are equal. Hence, the equation (2.2) is valid for this example.

Theorem 2.4. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ is an alternate dual continuous g-frame for $\{\Lambda_\omega\}_{\omega\in\Omega}$; then for any measurable subset $K\subset\Omega$ and for all $f\in\mathcal{H}$,*

$$\begin{aligned}&\int_K \langle \Gamma_\omega f, \Lambda_\omega f\rangle d\mu(\omega) \ - \int_K\int_K \langle \Lambda_\omega^*\Gamma_\omega f, \Lambda_\upsilon^*\Gamma_\upsilon f\rangle d\mu(\omega)d\mu(\upsilon)\\ =&\int_{K^c} \langle \Lambda_\omega f, \Gamma_\omega f\rangle d\mu(\omega) \ - \int_{K^c}\int_{K^c} \langle \Lambda_\omega^*\Gamma_\omega f, \Lambda_\upsilon^*\Gamma_\upsilon f\rangle d\mu(\omega)d\mu(\upsilon).\end{aligned} \tag{2.4}$$

Proof. For a measurable subset $K\subset\Omega$, by Proposition 2.1 the operator

$$\langle L_K(f), g\rangle = \int_K \langle f, \Gamma_\omega^*\Lambda_\omega g\rangle d\mu(\omega),$$

is well defined and bounded linear operator, so the adjoint operator is given by

$$\langle L_K^*(f), g\rangle = \int_K \langle f, \Lambda_\omega^*\Gamma_\omega g\rangle d\mu(\omega),$$

and by (1.2), $L_K + L_{K^c} = I$. Thus, by Lemma 1.4

$$\begin{aligned}\int_K \langle f, \Gamma_\omega^*\Lambda_\omega f\rangle d\mu(\omega) - &\int_K\int_K \langle \Lambda_\omega^*\Gamma_\omega f, \Lambda_\omega^*\Gamma_\omega f\rangle d\mu(\omega)d\mu(\omega)\\ &= \langle L_K(f), f\rangle - \langle L_K^* L_K(f), f\rangle\\ &= \langle L_{K^c}^*(f), f\rangle - \langle L_{K^c}^* L_{K^c}(f), f\rangle\\ &= \int_{K^c} \langle f, \Lambda_\omega^*\Gamma_\omega f\rangle d\mu(\omega)\\ &- \int_{K^c}\int_{K^c} \langle \Lambda_\omega^*\Gamma_\omega f, \Lambda_\omega^*\Gamma_\omega f\rangle d\mu(\omega)d\mu(\omega).\end{aligned}$$

□

Example 2.5. Let $\mathcal{H}$, $\mathcal{K}_\omega$, $(\Omega=[0,1], d\mu)$, and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be as defined in example 2.3. We have

$$\frac{1}{12}\|A\|^2 \le \int_{[0,1]} \|\Lambda_\omega\|^2 d\mu(\omega) \le \frac{1}{3}\|A\|^2.$$

This implies that $\{\Lambda_\omega\}_{\omega\in\Omega}$ forms a continuous g-frame. The alternate dual continuous g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$ is defined as

$$\Gamma_\omega(A) = \Lambda_\omega S_\Lambda^{-1}(A) = \begin{bmatrix} 3\omega a & 0\\ 0 & 6\omega b\end{bmatrix}.$$

Now, let's take $K=[0,\frac{1}{3}]$. To verify the validity of the equation (2.4) in this example, we will calculate both the left-hand side and the right-hand side of the equation and compare them.

$$\begin{aligned}
&\int_K \langle \Gamma_\omega f, \Lambda_\omega f\rangle d\mu(\omega) \ - \int_K \int_K \langle \Lambda_\omega^* \Gamma_\omega f, \Lambda_\upsilon^* \Gamma_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon) \\
&= \int_0^{\frac{1}{3}} 3\omega^2(|a|^2+|b|^2) d\mu(\omega) - \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} 9\omega^2\upsilon^2(|a|^2+|b|^2) d\mu(\omega) d\mu(\upsilon) \\
&= \frac{1}{27}(|a|^2+|b|^2) - \frac{1}{27} \times \frac{1}{27}(|a|^2+|b|^2) \\
&= \frac{26}{27\times 27}(|a|^2+|b|^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{K^c} \langle \Lambda_\omega f, \Gamma_\omega f\rangle d\mu(\omega) \ - \int_{K^c} \int_{K^c} \langle \Lambda_\omega^* \Gamma_\omega f, \Lambda_\upsilon^* \Gamma_\upsilon f\rangle d\mu(\omega) d\mu(\upsilon) \\
&= \int_{\frac{1}{3}}^1 3\omega^2(|a|^2+|b|^2) d\mu(\omega) - \int_{\frac{1}{3}}^1 \int_{\frac{1}{3}}^1 9\omega^2\upsilon^2(|a|^2+|b|^2) d\mu(\omega) d\mu(\upsilon) \\
&= \frac{26}{27}(|a|^2+|b|^2) - \frac{26}{27} \times \frac{26}{27}(|a|^2+|b|^2) \\
&= \frac{26}{27\times 27}(|a|^2+|b|^2).
\end{aligned}$$

Therefore, the left-hand side and the right-hand side of the equation are equal. Hence, the equation (2.4) is valid for this example.

Remark 2.6. The Theorem 2.4 is a special case of Theorem 2.2. Since $\{\Gamma_\omega\}_{\omega\in\Omega}$ is an alternate continuous dual g-frame for $\{\Lambda_\omega\}_{\omega\in\Omega}$, we have $L = I$ and $L^\dagger = I$. Now for every measurable subset $K \subset \Omega$ and $\omega \in \Omega$, taking $\Theta_\omega = \Gamma_\omega \chi_K(\omega)$ and $\Delta_\omega = \Gamma_\omega(1-\chi_K(\omega))$ in equality (2.2), we then obtain the equality (2.4).

Corollary 2.7. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$; then for all $f \in \mathcal{H}$*

$$\int_K \|\Lambda_\omega f\|^2 d\mu(\omega) - \int_\Omega \|\tilde{\Lambda}_\omega S_K f\|^2 d\mu(\omega) = \int_{K^c} \|\Lambda_\omega f\|^2 d\mu(\omega) - \int_\Omega \|\tilde{\Lambda}_\omega S_{K^c} f\|^2 d\mu(\omega), \tag{2.5}$$

where K is a measurable subset of Ω and $\langle S_K f, g\rangle = \int_K \langle f, \Lambda_\omega^* \Lambda_\omega g\rangle d\mu(\omega)$.

Proof. We consider $\Gamma_\omega := \Lambda_\omega$ for all $\omega \in \Omega$; then the operator L_Ω defined by (2.1) is the continuous g-frame operator S_Λ of $\{\Lambda_\omega\}_{\omega\in\Omega}$, and $L_\Omega^\dagger = S_\Lambda^{-1}$. For every $\omega \in \Omega$, let $\Theta_\omega = \Lambda_\omega \chi_K(\omega)$ then we can see that $\{\Theta_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$, and for any $\omega\in\Omega$, $\Delta_\omega=\Lambda_\omega-\Theta_\omega$ has the following form:

$$\Delta_\omega=\Lambda_\omega(1-\chi_K(\omega)),$$

by Theorem 2.2,

$$\begin{aligned}
&\int_\Omega\langle\Theta_\omega f,\Lambda_\omega(L^\dagger)^*Lf\rangle d\mu(\omega)-\int_\Omega\int_\Omega\langle L^\dagger\Lambda_\omega^*\Theta_\omega f,\Lambda_\upsilon^*\Theta_\upsilon f\rangle d\mu(\omega)d\mu(\upsilon)\\
&=\int_K\langle\Lambda_\omega f,\Lambda_\omega(S_\Lambda^{-1})^*S_\Lambda f\rangle d\mu(\omega)-\int_K\int_K\langle S_\Lambda^{-1}\Lambda_\omega^*\Lambda_\omega f,\Lambda_\upsilon^*\Lambda_\upsilon f\rangle d\mu(\omega)d\mu(\upsilon)\\
&=\int_K\langle\Lambda_\omega f,\Lambda_\omega f\rangle d\mu(\omega)-\int_K\langle S_\Lambda^{-1}S_K f,\Lambda_\upsilon^*\Lambda_\upsilon f\rangle d\mu(\upsilon)\\
&=\int_K\|\Lambda_\omega f\|^2d\mu(\omega)-\langle S_\Lambda^{-1}S_K f,S_K f\rangle\\
&=\int_K\|\Lambda_\omega f\|^2d\mu(\omega)-\langle S_\Lambda S_\Lambda^{-1}S_K f,S_\Lambda^{-1}S_K f\rangle\\
&=\int_K\|\Lambda_\omega f\|^2d\mu(\omega)-\int_\Omega\langle\Lambda_\omega S_\Lambda^{-1}S_K f,\Lambda_\omega S_\Lambda^{-1}S_K f\rangle d\mu(\omega)\\
&=\int_K\|\Lambda_\omega f\|^2d\mu(\omega)-\int_\Omega\|\tilde{\Lambda}_\omega S_K f\|^2d\mu(\omega).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega\langle\Lambda_\omega L^\dagger Lf,\Delta_\omega f\rangle d\mu(\omega)-\int_\Omega\int_\Omega\langle L^\dagger\Lambda_\omega^*\Delta_\omega f,\Lambda_\upsilon^*\Delta_\upsilon f\rangle d\mu(\omega)d\mu(\upsilon)\\
&=\int_{K^c}\|\Lambda_\omega f\|^2d\mu(\omega)-\int_\Omega\|\tilde{\Lambda}_\omega S_{K^c}f\|^2d\mu(\omega).
\end{aligned}$$

Hence, (2.5) holds. □

CONCLUSION

The study presented a generalization of the equalities from g-frame and g-Bessel sequences to continuous g-frame and continuous g-Bessel sequences in Hilbert spaces. This generalization enabled the establishment of the relationship between continuous g-frames and their alternate dual on a measure space. The work extended the equalities for continuous g-frames and continuous g-Bessel sequences in Hilbert spaces, and it also demonstrated that the generalization of certain theorems is a special case of a more general form. The article provided definitions, lemmas, propositions, and examples to support the generalization and presented the main results along with their proofs. The conclusion of the article highlights the significance of the generalization and its implications for the study of continuous g-frames in Hilbert spaces, providing a valuable contribution to this area of mathematical research.

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