Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **12**(2)(2023), 224-232

Wavelet adaptive algorithm for discrete pseudo-differential operators

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Abstract. In this study, we examine biorthogonal wavelets that are tailored to a specific discrete pseudo-differential equation of the form $T_\sigma u = f$, where T_σ is an invertible discrete pseudo-differential operator defined on the lattice $\mathbb{Z}^n$ for every $f \in \ell^2(\mathbb{Z}^n)$. Our focus is on computing Galerkin approximations of the solution to this problem using an adaptive algorithm.

Keywords: Wavelet, Discrete Pseudo-differential operators, Error bound, Galerkin method, Approximation algorithm.

1. Introduction and Preliminaries

Wavelets theory has been developing intensively in the last decades and has become a powerful tool to study mathematics, applied sciences and technology, like for example, the theory of the singular integral, singular integro-differential equations, and in applied sciences sound analysis, image compression, neural networks, mechanics, physics, see e.g., [8, 10, 18, 31] and references therein. Wavelets are a very powerful mathematical tool which enables to approximate functions by using both the concept of scale and translation so that we can easily and efficiently represent a function in terms of a set of basis functions, namely wavelets,

[1] Corresponding author: kadkhoda@buqaen.ac.ir
Received: 10 September 2023
Revised: 14 February 2024
Accepted: 17 February 2024

which are localized both in location and scale. The translated instances of a wavelet for all dilations form an unconditional orthonormal bases of $\ell^2(\mathbb{Z}^n)$ and the translates of a scaling function for all dilations form an unconditional orthonormal bases for $V_j \subset \ell^2(\mathbb{Z}^n)$ which is a great improvement over the standard polynomial basis or a trigonometric basis for the Galerkin Method.

The numerical methods for the solution of PDE's or integral boundary problems are usually based on the Galerkin, also known as Petrov-Galerkin, method which consists on the following steps:
1) finding a functional basis for the solution space of the equation,
2) projecting the solution on the functional basis, and at last
3) minimizing the residual with respect to the functional basis.

The majority of problems in science and engineering can be formulated as boundary integral equations which can be solved numerically by several methods like, in particular, wavelet-based adaptive algorithms for the numerical solution of elliptic equations. Boundary value problems in complex function theory [2, 3] and the method of boundary reduction for the oblique derivative problem in the plane, [5, 14] and references therein, lead to singular integro-differential [6, 17, 26], or more generally, to pseudo-differential equations on a closed curve [15, 27, 28]. Projection methods with trigonometric polynomials for the approximate solution of singular integral equations on the unit circle, also in the degenerate (i.e. non-elliptic) case, have been studied in detail in [16]. In [20], the authors investigated Galerkin methods with finite elements for an integral operator with logarithmic kernel which can be considered as a strongly elliptic pseudo-differential operator of order -1. Moreover, the authors in [29] dealt with finite element collocation methods for one-dimensional singular integral and pseudo-differential equations.

The singular integral operators arise naturally in the regularity study of elliptic and parabolic equations. In particular, the pseudo-differential operators are a special case of singular integral operator with Schwartz kernel which are characterized by

$$P(x,D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} p(x,\xi) f(y)dyd\xi$$

where $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is a suitable function so-called the symbol of pseudo-differential operator $P(x,D)$ [1, 24]. It is well-konwn that $P(x,D)$ has a representation by a kernel in the form

$$P(x,D)f(x) = \int_{\mathbb{R}^n} k(x,x-y)f(y)dy \qquad \forall x \notin Supp(f),$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ for a suitable locally integrable function $k : \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}) \to \mathbb{C}$, (see Theorem 5.12 in [1]). Pseudo-differential equations are a special case of singular integral equations corresponding to the pseudo-differential operators $P(x, D)$, for more details see Chapter 7 in [1] and [24]. The pseudo-differential operators on the lattice $\mathbb{Z}^n$ are suitable for solving difference equations on $\mathbb{Z}^n$. Such equations naturally appear in various problems of modelling and in the discretisation of continuous problems [13, 22, 23]. One can define (see Section 2) a general pseudo-differential operator with symbol $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ depending on a spatial discrete variable ξ by the similar formula

$$(T_\sigma f)(\xi) = \int_{\mathbb{T}^n} \sum_{\eta\in\mathbb{Z}^n} e^{2\pi i(\xi-\eta).x}\sigma(\xi, x)f(\eta)dx, \quad \xi \in \mathbb{Z}^n. \tag{1.1}$$

The main goal of this manuscript is to apply the Galerkin methods for the approximate solutions of discrete pseudo-differential equations of the type:

$$T_\sigma u = f \tag{1.2}$$

where T_σ is a pseudo-differential operator as 1.1, u is unknown function on $\mathbb{Z}^n \times \mathbb{T}^n$ and f is a given suitable function on $\mathbb{Z}^n$. In particular case, we study the $n-$dimensional discrete wave equation by the pseudo-differential operator $T_\sigma = \partial_t^2 - \Delta$ (see also [1, 4, 24])

In order to obtain approximate solutions to pseudo-differential equations, adaptive strategies have become very popular in recent years (see e.g. [11, 19, 22, 28]). The aim is to compute a numerical solution in such a way that the error i.e., the difference between the exact and the approximate solution, is measured under a suitable norm as given in the next section. In the following we will develop an adaptive refinement strategy and show that it will guarantee an improvement for the approximate solution after the refinement step. We will extend the adaptive strategy, already developed in [12, 19], to compute the inverse of discrete pseudo-differential operators 1.1 on lattice $\mathbb{Z}^n$. Our generalization is based on the discrete pseudo-differential calculus developed in [13, 24]. In fact, we will apply some results about the ellipticity and the concept of parametrix operators to compute the approximation solutions via Galerkin method. The introduced algorithm here can be applied to estimate for the approximation solutions and compare with the exact solution for many discrete system such as discrete wave system with the discrete Laplacian or the discrete Riesz operator (see Examples 3.2 and 3.3. The pseudo-differential operators on the lattice $\mathbb{Z}^n$ are suitable for solving difference equations on $\mathbb{Z}^n$. Such equations naturally appear in various problems of modelling and in the discretisation of continuous problems. Several attempts of developing a suitable theory of pseudo-differential operators on the lattice $\mathbb{Z}^n$ have been done in the literature [13, 24].

2. Wavelets and discrete multi-resolution analysis

In this section, we recall some definitions and preliminary results about the wavelets theory and discrete pseudo-differential operators theory from [10, 24]. A discrete multi-resolution analysis (DMRA) is a sequence of closed subspaces $\{V_j\}_{\mathbb{Z}_0}$ of $\ell^2(\mathbb{Z}^n)$ such that

$$V_j \subset V_{j+1}, \;\; j \in \mathbb{Z}, \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = \ell^2(\mathbb{Z}^n)$$

$$u \in V_j \Leftrightarrow \mathfrak{D}_2 u \in V_{j+1} \;\; j \in \mathbb{Z}, \qquad u_0 \in V_0 \Leftrightarrow \mathfrak{T}_{-k} u \in V_0 \;\; k \in \mathbb{Z}^n,$$

where $\mathfrak{D}_2$ and $\mathfrak{T}_{-k}$ are the dilation and the translation given, respectively, by $\mathfrak{D}_2 f(\xi) = f(2\xi)$ and $(\mathfrak{T}_{-k} f)(\xi) = f(\xi - k)$ for $\xi \in \mathbb{Z}^n$ and $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ and for all measurable functions f on $\mathbb{Z}$. Suppose that $\psi \in \ell^2(\mathbb{Z}^n)$, one can consider the translations and dilations $\psi_{j,k}$ of ψ defined as the following

$$\psi_{j,k}(\xi) = 2^{\frac{j}{2}} \psi(2^j \xi - k) \qquad \xi, k \in \mathbb{Z}^n, j \in \mathbb{Z}.$$

For any fixed $j \in \mathbb{Z}$, the sequence $\{\psi_{j,k} \;:\; k \in \mathbb{Z}^n\}$ is an orthonormal sequence for V_j for which the sequence is uniformly stable in the following sense

$$\Big\| \sum_{k\in\mathbb{Z}^n} C_{j,k}\psi_{j,k} \Big\|_2 \sim \Big(\sum_{k\in\mathbb{Z}^n} |C_{j,k}|^2 \Big)^{\frac{1}{2}}$$

uniformly with respect to $j \in \mathbb{Z}$, i.e., there exist positive constants M and M' such that

$$M\Big(\sum_{k\in\mathbb{Z}^n} |C_{j,k}|^2 \Big)^{\frac{1}{2}} \le \Big\| C_{j,k}\psi_{j,k} \Big\|_2^2 \le M' \Big(\sum_{k\in\mathbb{Z}^n} |C_{j,k}|^2 \Big)^{\frac{1}{2}}, \quad j \in \mathbb{Z}$$

then we call ψ a scaling function of the DMRA. For $j \in \mathbb{Z}$, it is denoted the orthogonal complement of V_{j-1} in V_j by W_j. The *raison d'étre* for W_{j-1} contains the details needed to pass from an approximation at level $j-1$ to an approximation at level j [9, 19]. Assume that $\varphi \in W_0$. Thus the translations and dilations $\varphi_{j,k}$ of φ are given as the following

$$\varphi_{j,k}(\xi) = 2^{\frac{j}{2}} \varphi(2^j \xi - k) \quad \xi, k \in \mathbb{Z}^n,$$

for all $j \in \mathbb{Z}$. For any fixed $j \in \mathbb{Z}$, the collection $\{\varphi_{j,k} \;:\; k \in \mathbb{Z}^n\}$ forms an orthonormal basis for W_j. Then, φ is said to be a mother wavelet and $\varphi_{j,k}$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, the wavelets for the DMRA. As the similar way in [9, 10], one can construct orthonormal bases for $\ell^2(\mathbb{Z}^n)$

consisting of compactly supported of wavelets that can be represented by polynomials of a fixed degree.

Hence, if it is denoted for convenience W_0 and V_0, then for all positive integer n, every element $v_n \in V_n$ given by $v_n = \sum_{k\in\mathbb{Z}^n} C_{n,k}\psi_{n,k}$, where every $C_{n,k}$ is complex number, has an alternative multiscale representation defined by the wavelets. In fact,

$$v_n = \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}^k} D_{j,k}\varphi_{j,k},$$

where every $D_{j,k}$ is a complex number. In other words, one can represent the subspace V_n as $V_n = \bigoplus_{j=0}^{n} W_j$, for every n.

Therefore, one can consider two biorthogonal DMRAs of $\ell^2(\mathbb{Z}^n)$. This means that $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ are DMRAs of $\ell^2(\mathbb{Z})$ for which the primal DMRA $\{V_j\}_{j\in\mathbb{Z}}$ and the dual DMRA $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ can be equipped with, respectively, Riesz bases $\Psi_j = \{\psi_{j,k} \ : \ k \in \mathbb{Z}^n\}$ and $\tilde{\Psi}_j = \{\tilde{\psi}_{j,k} \ : \ k \in \mathbb{Z}^n\}$ with the property of biorthogonality to the effect that $\langle \psi_{j,k}, \tilde{\psi}_{j,k'}\rangle_2 = \delta_{k,k'}$ for all $k, k^{'} \in \mathbb{Z}^n$, where $\langle .,. \rangle_2$ is the inner product in $\ell^2(\mathbb{Z}^n)$. Every primal scaling function ψ and dual scaling function $\tilde{\psi}$ is assumed to have compact support such that the measure of $\psi_{j,k}$ and $\tilde{\psi}_{j,k}$ are $\sim 2^{-j}$ for all $j \in \mathbb{Z}$. Moreover, these biorthogonal bases define the projection operators $P_j : \ell^2(\mathbb{Z}^n) \to V_j$ and $\tilde{P}_j : \ell^2(\mathbb{Z}^n) \to \tilde{V}_j$, such that are uniformly stable in $\ell^2(\mathbb{Z}^n)$. They are defined as the following

$$P_j v = \sum_{k\in\mathbb{Z}^n} \langle v, \tilde{\psi}_{j,k}\rangle_2 \psi_{j,k} \quad \textit{and} \quad \tilde{P}_j v = \sum_{k\in\mathbb{Z}^n} \langle v, \psi_{j,k}\rangle_2 \tilde{\psi}_{j,k}$$

for all $v \in \ell^2(\mathbb{Z}^n)$ and $j = 0, 1, 2, ...$. The nestedness of the DMRA spaces gives the properties that $P_j P_{j+1} = P_j \quad \textit{and} \quad \tilde{P}_j \tilde{P}_{j+1} = \tilde{P}_j$ for all $j \in \mathbb{Z}$. Thus for $j \in \mathbb{Z}$, again the operators Q_j and $\tilde{Q}_j$ defined by

$$Q_j = P_{j+1} - P_j \quad \textit{and} \quad \tilde{Q}_j = \tilde{P}_{j+1} - \tilde{P}_j$$

are projection operators. For $j \in \mathbb{Z}$, the wavelet spaces W_j and $\tilde{W}_j$ are defined as the following

$$W_j = V_{j+1} \cap \tilde{V}_j^{\perp} \qquad \tilde{W}_j = \tilde{V}_{j+1} \cap V_j^{\perp},$$

which are, respectively, the range $R(Q_j)$ of Q_j and the range $R(\tilde{Q}_j)$ of $\tilde{Q}_j$. The wavelet spaces $\{W_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{W}_j\}_{j\in\mathbb{Z}}$ induce two multiscale decompositions of $\ell^2(\mathbb{Z})$ via

$$v = P_1 v + \sum_{j=1}^{\infty} Q_j v = \sum_{j=0}^{\infty} Q_j v, \qquad v \in \ell^2(\mathbb{Z}^n),$$

where $Q_0 = P_1$ and

$$\tilde{v} = \tilde{P}_1 v + \sum_{j=1}^{\infty} \tilde{Q}_j v, \qquad v \in \ell^2(\mathbb{Z}^n).$$

Moreover, for $j \in \mathbb{Z}$, the wavelet spaces W_j and $\tilde{W}_j$ are equipped with compactly supported biorthogonal the Riesz bases is indicated , respectively, by $\Phi_j = \{\varphi_{j,k} \;:\; k \in \mathbb{Z}^n\}$ and $\tilde{\Phi}_j = \{\tilde{\varphi}_j \;:\; k \in \mathbb{Z}^n\}$. For every nonnegative integer n, one can introduce the canonical truncated projection operators Q_n and $Q_n^{'}$ which is defined by

$$Q_n v = \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}^n} \langle v, \tilde{\varphi}_{j,k}\rangle_2 \varphi_{j,k} \quad \textit{and} \quad Q_n^{'} v = \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}^n} \langle v, \varphi_{j,k}\rangle_2 \tilde{\varphi}_{j,k} \quad \forall v \in \ell^2(\mathbb{Z}^n).$$

3. Discrete pseudo-differential operators

Now, let us give some definitions and implications about the discrete calculus of pseudo-differential operators on the lattice $\mathbb{Z}^n$ [13]. The discrete Fourier transform $\hat{f}$ of a function f in $\ell^1(\mathbb{Z})$ is defined by

$$\hat{f}(x) = \sum_{\xi\in\mathbb{Z}^n} e^{-2\pi i\xi.x} f(\xi)$$

for all $x \in \mathbb{T}^n$ where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The discrete Fourier transform can be extended to $\ell^2(\mathbb{Z}^n)$ using the usual density arguments. We normalize the Haar measure on $\mathbb{Z}^n$ and $\mathbb{T}^n$ in such a way that the Plancherel formula to the effect that

$$\sum_{\xi\in\mathbb{Z}^n} |f(\xi)|^2 = \int_{\mathbb{T}^n} |\hat{f}(x)|^2 dx,$$

is valid. Then, the inverse discrete Fourier transform is defined as the following

$$f(\xi) = \int_{\mathbb{T}^n} e^{2\pi i\xi.x} \hat{f}(x) dx, \qquad \xi \in \mathbb{Z}^n.$$

We recall the discrete calculus developed in [13, 24]. Let f be a function on $\mathbb{Z}^n$ and $e_j \in \mathbb{N}^n$ be such that e_j has 1 in the j^{th} entry and zeros elsewhere. The difference operator Δ_{ξ_j} is given by

$$\Delta_{\xi_j} f(\xi) = f(\xi + e_j) - f(\xi)$$

and take $\Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1}\Delta_{\xi_2}^{\alpha_2}...\Delta_{\xi_n}^{\alpha_n}$ for all $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n = \mathbb{N}^n \cup \{0\}$ and $\xi = (\xi_1, ..., \xi_n) \in \mathbb{Z}^n$. We use the usual notations, $D_x^\alpha = D_{x_1}^{\alpha_1}...D_{x_n}^{\alpha_n}$, $D_{x_j} = \frac{1}{2\pi i}\frac{\partial}{\partial_{x_j}}$ and

$$D_x^{(\alpha)} = D_{x_1}^{(\alpha_1)}...D_{x_n}^{(\alpha_n)}, \qquad , D_{x_j}^{(l)} = \prod_{m=0}^{l}\left(\frac{1}{2\pi i}\frac{\partial}{\partial_{x_j}} - m\right), \quad l \in \mathbb{N}.$$

The operators $D_x^{(\alpha)}$ are useful in the analysis in torus and details can be found in [24]. The symbol classes are then defined as follows:

Definition 3.1. For $m \in \mathbb{R}$, we say that a function $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ belong to $S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ if $\sigma(\xi, .) \in C^\infty(\mathbb{T}^n)$ for all $\xi \in \mathbb{Z}^n$ and for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, there exists a positive constant $M_{\alpha,\beta}$ such that

$$\left|(D_x^{(\beta)} \Delta_\xi^\alpha \sigma)(\xi, x)\right| \leq M_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}, \quad (\xi, x) \in \mathbb{Z}^n \times \mathbb{T}^n.$$

Therefore, the corresponding discrete pseudo-differential operator with symbol σ is given by

$$(T_\sigma f)(\xi) = \int_{\mathbb{T}^n} e^{2\pi i \xi . x} \sigma(\xi, x) \hat{f}(x) dx, \qquad \xi \in \mathbb{Z}^n.$$

Example 3.2. For a complex-valued function on $\mathbb{Z}^n$ its discrete Lplacian is given by

$$\Delta(f)(\xi) = \sum_{j=1}^n \partial_j \partial_j^* f(\xi) = \sum_{j=1}^n \partial_j^* \partial_j f(\xi), \quad \xi \in \mathbb{Z}^n,$$

where $\partial_j f(\xi) = f(\xi + e_j) - f(\xi)$ and $\partial_j^* f(\xi) = f(\xi - e_j) - f(\xi)$. Then one can get

$$\Delta(f)(\xi) = \sum_{j=1}^n \Big(f(\xi + e_j) - 2f(\xi) + f(\xi - e_j \Big)).$$

The discrete Laplacian Δ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^n)$ and one gets

$$\mathcal{F}_{\mathbb{Z}^n}(\Delta(f))(\xi) = -\sum_{j=1}^n |e^{i2\pi\xi_j} - 1|^2 \mathcal{F}_{\mathbb{Z}^n}(f)(\xi) = -4\left(\sum_{j=1}^n \sin^2(\pi\xi_j)\right)\mathcal{F}_{\mathbb{Z}^n}(f)(\xi).$$

The symbol of the discrete laplacian is the function

$$\sigma_\Delta(\xi) = \sum_{k=1}^n (e^{i2\pi\xi_k} - 1)^2.$$

Example 3.3. The discrete Riesz transforms R_j, $j = 1, ..., n$, associated with Δ are defined on $\ell^2(\mathbb{Z}^n)$ as the multiplier operators

$$\mathcal{F}_{\mathbb{Z}^n}(R_j(f))(\xi) = \frac{ie^{-i\pi\xi_j} \sin(\pi\xi_j)}{\left(\sum_{k=1}^n \sin^2(\pi\xi_k)\right)^{\frac{1}{2}}} \mathcal{F}_{\mathbb{Z}^n}(f)(\xi),$$

it can be interpreted as $R_j = \partial_j \Delta^{-\frac{1}{2}}$.

The following theorem gives the product of two discrete pseudo-differential operators.

Theorem 3.4. [4, 13] *Let* $\sigma \in S^{m_1}(\mathbb{Z}^n \times \mathbb{T}^n)$ *and* $\tau \in S^{m_2}(\mathbb{Z}^n \times \mathbb{T}^n)$. *Then the product* $T_\sigma T_\tau$ *of the pseudo-differential operators* T_σ *and* T_τ *is a pseudo-differential operator with symbol in* $S^{m_1+m_2}(\mathbb{Z}^n \times \mathbb{T}^n)$.

Definition 3.5. A symbol $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ is called elliptic of order m if there exist positive constants C and M such that

$$|\sigma(\xi, x)| \geq C(1 + |\xi|)^m$$

for all $\xi \in \mathbb{Z}^n$ and all $x \in \mathbb{T}^n$ with $|\xi| \geq M$.

The corresponding discrete pseudo-differential operator T_σ is called elliptic. The following theorem gives the parametrix for an elliptic discrete pseudo-differential operator.

Theorem 3.6. *Let* $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$, $m \in \mathbb{R}$, *be elliptic. Then there exists a symbol* $\tau \in S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$ *such that*

$$T_\sigma T_\tau = I + R, \qquad T_\tau T_\sigma = I + S$$

where R *and* S *are infinitely smoothing in the sense that they are pseudo-differential operators with symbol in* $\bigcap_{\mu \in \mathbb{R}} S^\mu(\mathbb{Z}^n \times \mathbb{T}^n)$.

Proof. Let σ be the symbol of the operator T_σ. In special case, by using of Proposition (3.6) in [21] and similar to Theorem (3.7) in [21], we consider a function $\tau_0(x, \xi) = \frac{1}{\sigma(x,\xi)} \in S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$. Now, we can apply theorem of composition of two pseudo-differential operators [21] to obtain

$$T_{\tau_0} T_\sigma = I + R_0 \textit{ and } T_\sigma T_{\tau_0} = I + S_0,$$

with $r_0(x, \xi), s_0(x, \xi) \in S^{-1}$. Using the formal expansion

$$I - R_0 + R_0^2 - ... \sim I + R \in \mathbf{OPS}^0$$

and setting $T_\tau = (I + R)T_{\tau_0} \in \mathbf{OPS}^{-m}$ we have

$$T_\tau T_\sigma = I + R, \; r(x, \xi) \in S^{-\infty}.$$

Similarly, we obtain $\tilde{T}_\tau \in \mathbf{OPS}^{-m}$ satisfying

$$T_\sigma \tilde{T}_\tau = I + S, \; s(x, \xi) \in S^{-\infty}.$$

But evaluating $(T_\tau T_\sigma)\tilde{T}_\tau = T_\tau(T_\sigma \tilde{T}_\tau)$ yields $T_\tau = \tilde{T}_\tau$ *mod* $\mathbf{OPS}^{-\infty}$. Therefore, $T_\sigma T_\tau = I$ *mod* $\mathbf{OPS}^{-\infty}$ and $T_\tau T_\sigma = I$ *mod* $\mathbf{OPS}^{-\infty}$. In fact we have that $T_\sigma T_\tau = I + R$ and $T_\tau T_\sigma = I + S$ where R and S are infinitely smoothing in the sense that they are pseudo-differential operators with symbols in $\cap_{k \in \mathbb{R}} S^k$. □

Now, let us recall the definition of the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$, on the lattice $\mathbb{Z}^n$ the space of all functions $\varphi : \mathbb{Z}^n \to \mathbb{C}$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$,

$$\sup_{\xi\in\mathbb{Z}^n}\left|\xi^\alpha(\Delta_\xi^\alpha\varphi)(\xi)\right| < \infty.$$

A sequence $\{\varphi_j\}$ of functions in $\mathcal{S}(\mathbb{Z}^n)$ is said to be converge to zero in $\mathcal{S}(\mathbb{Z}^n)$ if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, $\sup_{\xi\in\mathbb{Z}^n}\left|\xi^\alpha(\Delta_\xi^\alpha\varphi_j)(\xi)\right| \to 0$ as $j \to \infty$. A linear functional T on $\mathcal{S}(\mathbb{Z}^n)$ is called a tempered distribution if for any sequence $\{\varphi_j\}$ of functions in $\mathcal{S}(\mathbb{Z}^n)$ converging to 0, one has $T(\varphi_j) \to 0$ as $j \to \infty$. For $s \in \mathbb{R}$, it is usual denoted by J_s the pseudo-differential operator of which the symbol σ_s is defined by

$$\sigma_s(\xi) = (1 + |\xi|^2)^{\frac{s}{2}}, \qquad \xi \in \mathbb{Z}^n.$$

It is noteworthy that the symbol of J_s is in $S^s(\mathbb{Z}^n \times \mathbb{T}^n)$. The pseudo-differential operator J_s is often called the discrete Bessel potential of order s. Hence, for $s \in \mathbb{R}$, one can define ℓ^2-Sobolev space, $\mathcal{H}^{s,2}(\mathbb{Z}^n)$, to be the set of all tempered distributions u for which $J_s u \in \ell^2(\mathbb{Z}^n)$. Then $\mathcal{H}^{s,2}(\mathbb{Z}^n)$ is a Banach space with respect to the norm $\|.\|_{s,2}$ given by

$$\|u\|_{s,2} = \|J_{-s}u\|_2, \qquad u \in \mathcal{H}^{s,2}(\mathbb{Z}^n).$$

The following result is well-known.

Proposition 3.7. *For $s \in \mathbb{R}$, $J_{-s} : \mathcal{H}^{s,2}(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ is a surjective isometry.*

Proof. Since $\|u\|_{s,2} = \|J_{-s}u\|_2$ for any $u \in \mathcal{H}^{s,2}(\mathbb{Z}^n)$ it follows that $J_{-s} : \mathcal{H}^{s,2} \to \ell^2(\mathbb{Z}^n)$ is an isometry. For every $v \in \ell^2(\mathbb{Z}^n)$ we take $u := (J_{-s})^{-1}v$ then $J_{-s}u = v \in \ell^2(\mathbb{Z}^n)$. Hence, $J_s : \mathcal{H}^{s,2}(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ is surjection. ☐

Theorem 3.8. *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$, $m \in \mathbb{R}$. Then $T_\sigma : \mathcal{H}^{s,2}(\mathbb{Z}^n) \to \mathcal{H}^{s-m,2}(\mathbb{Z}^n)$ is abounded linear operator.*

Proof. We factorize the pseudo-differential operator T_σ as in following diagram and get

$$T_\sigma = (J_{m-s})^{-1} T_\tau J_{-s},$$

where $T_\tau = J_{m-s} T_\sigma (J_{-s})^{-1}$.

$$\begin{array}{ccc} \mathcal{H}^{s,2}(\mathbb{Z}^n) & \xrightarrow{T_\sigma} & \mathcal{H}^{s-m,2}(\mathbb{Z}^n) \\ {\scriptstyle J_{-s}}\big\downarrow & & \big\downarrow{\scriptstyle J_{m-s}} \\ \mathcal{H}^{0,2} = \ell^2(\mathbb{Z}^n) & \xrightarrow{T_\tau} & \ell^2(\mathbb{Z}^n) \end{array}.$$

By Theorem 3.4 and proposition 3.7 we can see that T_τ is a pseudo-differential operator with symbol S^0. Hence, by theorem (3.3) in [21], $T_\tau : \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ ia a bounded linear operator. Therefor, $T_\sigma : \mathcal{H}^{s,2}(\mathbb{Z}^n) \to \mathcal{H}^{s-m,2}(\mathbb{Z}^n)$ is bounded linear operator. □

The following result on spectral invariance [25] is well-known. See also Theorem 4.2 in [13] in this connection.

Theorem 3.9. *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ be such that the pseudo-differential operator $T_\sigma : \mathcal{H}^{\frac{m}{2},2}(\mathbb{Z}^n) \to \mathcal{H}^{-\frac{m}{2},2}(\mathbb{Z}^n)$ is invertible. Then σ is elliptic and T_σ^{-1} is an elliptic pseudo-differential operator with symbol in $S^{-m}(\mathbb{Z}^n \times \mathbb{T}^n)$.*

The following estimate is useful in continue of the paper.

Theorem 3.10. *Let $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ be such that the pseudo-differential operator $T_\sigma : \mathcal{H}^{\frac{m}{2},2}(\mathbb{Z}^n) \to \mathcal{H}^{-\frac{m}{2},2}(\mathbb{Z}^n)$ is invertible. Then there exist positive constants M_1 and M_2 such that*

$$M_1 \|T_\sigma u\|_{-\frac{m}{2},2} \leq \|u\|_{\frac{m}{2},2} \leq M_2 \|T_\sigma u\|_{-\frac{m}{2},2}, \qquad u \in \mathcal{H}^{\frac{m}{2},2}(\mathbb{Z}^n).$$

Proof. The inequality on the left side is obtained by the boundedness of the discrete pseudo-differential operator T_σ from Theorem 3.8. Moreover, by using of Theorem 3.8 and 3.9, there exists a positive constant M such that

$$\|u\|_{\frac{m}{2},2} = \|T_\sigma^{-1} T_\sigma u\|_{\frac{m}{2},2} \leq M \|T_\sigma u\|_{-\frac{m}{2},2}, \qquad u \in \mathcal{H}^{\frac{m}{2},2}(\mathbb{Z}^n).$$

□

We let $\lambda = (j,k)$, where j is the level of resolution and k is the location. We let J be the index set given by

$$J = \{\lambda = (j,k) \ : \ j = 0,1,2,...,\ k \in \mathbb{Z}^n\},$$

and for $\lambda = (j,k)$ in J, we define $|\lambda| := j$. Thus, as similar result in [10, 19], one can have the following result on the lattice $\mathbb{Z}^n$.

Theorem 3.11. *Suppose that $\Psi = \{\psi_\lambda \ : \ \lambda \in J\}$ and $\tilde{\Psi} = \{\tilde{\psi}_\lambda \ : \ \lambda \in J\}$ are biorthogonal collections in $\ell^2(\mathbb{Z}^n)$ for which the sequence $\{Q_n\}_{n=0}^\infty$ of the projection operators given by*

$$Q_n v = \sum_{j=0}^{n} \sum_{\xi \mathbb{Z}^n} \langle v, \tilde{\psi}_{j,k} \rangle_2 \psi_{j,k}, \qquad v \in \ell^2(\mathbb{Z}^n),$$

is uniformly bounded in the sense that there exists a positive constant M such that

$$\|Q_n v\|_{s,2} \leq M \|v\|_{s,2}, \qquad n = 0,1,2,\dots .$$

Then for every $v \in \mathcal{H}^{s,2}(\mathbb{Z}^n)$*, one can get*

$$\|v\|_{s,2} \sim \left(\sum_{\lambda\in J} 2^{2|\lambda|^s} |\langle v, \tilde{\psi}_\lambda\rangle_2|^2\right)^{\frac{1}{2}}, \qquad s \in (-\gamma^{'}, \gamma),$$

where $\gamma = \sup\{s \in \mathbb{R} \ : \ \psi \in \mathcal{H}^{s,2}(\mathbb{Z}^n)\}$ *and* $\gamma^{'} = \sup\{s \in \mathbb{R} \ : \ \tilde{\psi} \in \mathcal{H}^{s,2}(\mathbb{Z}^n)\}$.

It is noteworthy that γ and $\gamma^{'}$ are, respectively, less than or equal to the vanishing moments of ψ and $\tilde{\psi}$. The goal of this article is to use adaptive wavelets to compute numerically the inverse of an invertible discrete pseudo-differential operator $T_\sigma : \mathcal{H}^{m,2}(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$, where $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$ and $m = min\{\gamma, \gamma^{'}\}$. This means to solving the discrete pseudo-differential equation

$$T_\sigma u = f \tag{3.1}$$

on $\mathbb{Z}^n$ for all functions $u \in \mathcal{H}^{m,2}(\mathbb{Z}^n)$ and $f \in \ell^2(\mathbb{Z}^n)$. In order to do this, one can transform the equation 3.1 to the equation

$$T_\sigma^* T_\sigma u = T_\sigma^* f \tag{3.2}$$

on $\mathbb{Z}^n$, where T_σ^* denotes the discrete formal adjoint of T_σ. Furthermore, $T_\sigma^* T_\sigma$ is a discrete pseudo-differential operator T_τ or order $2m$ and $T_\sigma^* f \in \mathcal{H}^{-m,2}(\mathbb{Z}^n)$. Moreover, T_τ is asymmetric and there exist positive constants M and $M^{'}$ for which

$$M\|u\|_{m,2}^2 \leq \langle T_\tau u, u\rangle_2 \leq M^{'}\|u\|_{m,2}^2, \qquad u \in \mathcal{H}^{m,2}(\mathbb{Z}^n). \tag{3.3}$$

The right hand side inequality follows from Theorem 3.8. In fact, there exists a positive constant $M^{'}$ for which

$$\langle T_\tau u, u\rangle_2 \leq \left|\langle T_\tau u, u\rangle_2\right| \leq \|T_\tau u\|_{-m,2}\|u\|_{m,2} \leq M^{'}\|u\|_{m,2}^2, \qquad u \in \mathcal{H}^{m,2}(\mathbb{Z}^n).$$

On the other hand, one gets from Theorem 3.8 and 3.9 a positive constant M such that

$$\|u\|_{m,2}^2 = \|T_\sigma^{-1} T_\sigma u\|_{m,2}^2 \leq M\|T_\sigma\|_{m,2}^2, \qquad u \in \mathcal{H}^{m,2}(\mathbb{Z}^n).$$

Therefore, by abusing the notations, the problem 3.1 is the same as solving for $u \in \mathcal{H}^{m,2}(\mathbb{Z}^n)$ to the equation

$$T_\sigma u = f$$

on $\mathbb{Z}^n$ for any $f \in \mathcal{H}^{-m,2}(\mathbb{Z}^n)$, where T_σ is asymmetric discrete pseudo-differential operator of order $2m$ such that there exist positive constants $M^{'}$ and $M^{"}$ for which

$$M^{'}\|u\|_{m,2} \leq \|u\|_{T_\sigma} \leq M^{"}\|u\|_{m,2}, \qquad u \in \mathcal{H}^{m,2}(\mathbb{Z}^n),$$

where $\|u\|^2_{T_\sigma} = \langle T_\sigma u, u\rangle_2$. The existence of a positive constant $M^{'}$ such that

$$\|u\|^2_{T_\sigma} \geq M^{'}\|u\|_{m,2}, \qquad u \in \mathcal{H}^{m,2}(\mathbb{Z}^n),$$

is a condition related to Gårding's inequality on the symbol $\sigma \in S^m(\mathbb{Z}^n \times \mathbb{T}^n)$. For example, see the paper [30] in this connection. Adaptive wavelet methods in finding solutions to differential and integral equations can be found in [7, 12].

4. Residual estimate and error bounds on $\mathbb{Z}^n$

The process of calculating the inverse of the discrete pseudo-differential operator $T_\sigma : \mathcal{H}^{m,2}(\mathbb{Z}^n) \to \mathcal{H}^{-m,2}(\mathbb{Z}^n)$ numerically is equivalent to the computation of subspaces V_Λ of the form $V_\Lambda = \overline{span\{\psi_\lambda \ : \ \lambda \in \Lambda\}}$ that are adapted to the unique solution $u \in \mathcal{H}^{m,2}(\mathbb{Z}^n)$ of the discrete pseudo-differential equation

$$T_\sigma u = f \tag{4.1}$$

on $\mathbb{Z}^n$ for any function $f \in \mathcal{H}^{-m,2}(\mathbb{Z}^n)$.

To illustrate the equation 4.1, one can consider the Bessel potential operator as discrete pseudo-differential operator as follows:

The function $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ is a pseudo-differential symbol of order 1. According to Example 3.2 and since $1+|\xi|^2$ is the symbol $I-\Delta$, the associated discrete pseudo-differential operator

$$\langle D_x\rangle u = \int_{\mathbb{T}^n} e^{2\pi i.x.\xi}\langle\xi\rangle\hat{u}d\xi$$

can be considered as the square root of $I-\Delta$. More generally, one can consider a symbol $\langle\xi\rangle^m \in \mathcal{S}^m$ for every $m \in \mathbb{R}$ and $\langle D_x\rangle^m = (I-\Delta)^{\frac{m}{2}}$.

In order to do this, one can use the weak formulation of 4.1 to the effect of finding a solution $u_\Lambda \in V_\Lambda$ for which

$$\langle T_\sigma u_\Lambda, v\rangle_2 = \langle f, v\rangle_2, \qquad v \in V_\Lambda. \tag{4.2}$$

In other words, for an arbitrary tolerance eps, one attempts to find a subset Λ of J for which the Galerkin approximation $u_\Lambda \in V_\Lambda$ given by 4.2 satisfies the estimate $\|u_\Lambda - u\|_{m,2} \leq eps$. This is to be achieved by successively upgrading Λ based on appropriate a posteriori estimates of a current Galerkin approximation u_Λ. Consider the residual term $r_\Lambda = T_\sigma u_\Lambda - f$ which is the same as $r_\Lambda = T_\sigma(u_\Lambda - u)$. Therefore,, by Theorem 3.10, one can find positive constants M_1 and M_2 such that

$$M_1\|r_\Lambda\|_{-m,2} \leq \|u_\Lambda - u\|_{m,2} \leq M_2\|r_\Lambda|\|_{-m,2},$$

for all subsets Λ of J. Hence, one can find positive constants M_3 and M_4 for which

$$M_3\Big(\sum_{\lambda\in J\backslash\Lambda} 2^{-2m|\lambda|}|\langle r_\Lambda,\psi_\lambda\rangle_2|^2\Big)^{\frac{1}{2}} \le \|r_\Lambda\|_{-m,2} \le M_4\Big(\sum_{\lambda\in J\backslash\Lambda} 2^{-2m|\lambda|}|\langle r_\Lambda,\psi_\lambda\rangle_2|^2\Big)^{\frac{1}{2}}.$$

For $\lambda\in J\backslash\Lambda$, define $\delta_\lambda := 2^{-m|\lambda|}|\langle r_\Lambda,\psi_\lambda\rangle_2|$. From $u_\Lambda\in V_\Lambda$ implies that $u_\Lambda=\sum_{\lambda'\in\Lambda} u_{\lambda'}\psi_{\lambda'}$, where $u_{\lambda'}=\langle u_\lambda,\tilde{\psi}_{\lambda'}\rangle_2$. Thus, for $\lambda\in J\backslash\Lambda$,

$$\delta_\lambda = 2^{-m}|\lambda|\Big|f_\lambda-\sum_{\lambda'\in\Lambda}\langle T_\sigma\psi_{\lambda'},\psi_\lambda\rangle_2 u_{\lambda'}\Big|.$$

Let μ be the Hölder exponent of $\partial^\gamma\varphi$. Therefore, for all positive numbers ϵ and δ with $\delta<\mu-\frac{1}{2}$, one can choose positive numbers ϵ_1 and ϵ_2 for which $\epsilon_1^{2(\tilde{r}+1)}+2^{-\frac{\delta}{\epsilon_2}}\le\epsilon$, where $\tilde{r}$ is the vanishing moment of $\tilde{\varphi}$. For any $\lambda\in J$ and for an arbitrary positive number ϵ, one can define the tolerance set

$$J_{\lambda,\epsilon} := \{\lambda'\in J\ :\ ||\lambda|-|\lambda'||\le\epsilon_2^{-1},\ 2^{\min\{|\lambda|,|\lambda'|\}}d(supp(\psi_\lambda),supp(\psi_{\lambda'}))\le\epsilon_1^{-1}\}.$$

Therefore, one can obtain the following lemma, which is Lemma 4.2 in [12].

Lemma 4.1. *For $\lambda\in J\backslash\Lambda$, let e_λ be defined by $e_\lambda=\sum_{\lambda'\in\Lambda\backslash J_{\lambda,\epsilon}} 2^{-m|\lambda|}\langle T_\sigma\psi_{\lambda'},\psi_\lambda\rangle_2 u_{\lambda'}$. Then there exists a positive constant M_5 such that*

$$\Big(\sum_{\lambda\in J\backslash\Lambda}|e_\lambda|^2\Big)^{\frac{1}{2}}\le M_5\|Q'_\Lambda f\|_{-m,2},$$

where $Q'_\Lambda f=\sum_{\lambda\in\Lambda}\langle f,\psi_\lambda\rangle_2\tilde{\psi}_\lambda$.

For $\lambda\in J\backslash\Lambda$, one can get the following estimates:

$$\begin{aligned}\delta_\lambda &= 2^{-m|\lambda|}\Big|f_\lambda-\Big(\sum_{\lambda'\in\Lambda\cap J_{\lambda,\epsilon}}+\sum_{\lambda'\in\Lambda\backslash J_{\lambda,\epsilon}}\Big)\langle T_\sigma\psi_{\lambda'},\psi_\lambda\rangle_2 u_{\lambda'}\Big|\\ &\le |d_\lambda|+|e_\lambda|,\end{aligned}$$

where $d_\lambda=2^{-m|\lambda|}\Big|f_\lambda-\sum_{\lambda'\in\Lambda\cap J_{\lambda,\epsilon}}\langle T_\sigma\psi_\lambda,\psi_\lambda\rangle_2 u_{\lambda'}\Big|$.

Suppose that $\mathcal{N}_{\Lambda,\epsilon}$ is the set of all indices in the complement of Λ with influence set intersecting Λ. In other words,

$$\mathcal{N}_{\Lambda,\epsilon} := \{\lambda\in J\backslash\Lambda\ :\ J_{\lambda,\epsilon}\cap\Lambda\ne\emptyset\}.$$

It can be shown that $\mathcal{N}_{\Lambda,\epsilon} = \cup_{\lambda' \in \Lambda} J_{\lambda',\epsilon}$ and $\mathcal{N}_{\Lambda,\epsilon}$ has at most a finite number of elements. Therefore,

$$\lambda' \in J \backslash (\Lambda \cup \mathcal{N}_{\Lambda,\epsilon}) \Rightarrow J_{\lambda',\epsilon} \cap \Lambda = \emptyset.$$

Since $f \in \mathcal{H}^{-m,2}(\mathbb{Z}^n)$ if and only if $\sum_{\lambda \in J} 2^{-2m|\lambda|}|f_\lambda|^2 < \infty$, it follows that

$$\sum_{\lambda \in J \backslash (\Lambda \cup \mathcal{N}_{\Lambda,\epsilon})} 2^{-2m|\lambda|}|f_\lambda|^2$$

can be made arbitrarily small by choosing Λ appropriately. Consequently,

$$\begin{aligned}\sum_{\lambda \in J \backslash (\Lambda \cup \mathcal{N}_{\Lambda,\epsilon})} 2^{-2m|\lambda|}|f_\lambda|^2 &= \sum_{\lambda \in J} 2^{2m|\lambda|}|f_\lambda|^2 - \sum_{\lambda \in (\Lambda \cup \mathcal{N}_{\Lambda,\epsilon})} 2^{-2m|\lambda|}|f_\lambda|^2 \\ &= \|f - Q'_{\Lambda \cup \mathcal{N}_{\Lambda,\epsilon}} f\|_{-m,2} \sim \inf_{v \in \tilde{V}_{\Lambda \cup \mathcal{N}_{\Lambda,\epsilon}}} \|f - v\|_{-m,2} \\ &\leq \inf_{v \in \tilde{V}_\Lambda} \|f - v\|2_{-m,2}.\end{aligned}$$

Now, one can apply the basic assumptions to the effect that there are positive constants M_6 and M_7 for which

$$M_6 \|Q'_\Lambda f\|_{-m,2} \leq M_7 \|f\|_{-m,2}$$

and

$$\left(\sum_{\lambda \in J \backslash \Lambda} 2^{-2m|\lambda|}|f_\lambda|^2 \right)^{\frac{1}{2}} \leq M_7 \inf_{v \in \tilde{V}_\Lambda} \|f - v\|_{-m,2}$$

for all subsets Λ of J. Suppose that $\lambda \in J \backslash \Lambda$, consider a_λ by

$$a_\lambda := 2^{-m|\lambda|} \left| \sum_{\lambda' \in \Lambda \cap J_{\Lambda,\epsilon}} \langle T_\sigma \psi_{\lambda'}, \psi_\lambda \rangle_2 u_{\lambda'} \right|.$$

Proposition 4.2. *Under the assumptions of Lemma 4.1, one gets*

$$\|u_\Lambda - u\|_{m,2} \leq M_2 M_4 \left\{ \left(\sum_{\lambda \in \mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2 \right)^{\frac{1}{2}} + M_6 \epsilon \|f\|_{-m,2} + M_7 \inf_{v \in \tilde{V}_\Lambda} \|f - v\|_{-m,2} \right\}$$

and

$$\left(\sum_{\lambda \in \mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2 \right)^{\frac{1}{2}} \leq \frac{1}{M_1 M_3} \|u_\Lambda - u\|_{m,2} + M_6 \epsilon \|f\|_{-m,2} + M_7 \inf_{v \in \tilde{V}_\Lambda} \|f - v\|_{-m,2}.$$

Theorem 4.3. *Assume that* $\Lambda \subset \tilde{\Lambda} \subset J$. *Then*

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap \mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2 \right)^{\frac{1}{2}} \leq \frac{1}{M_1 M_3} \|u_{\tilde{\Lambda}} - u_\lambda\|_{m,2} + M_6 \epsilon \|f\|_{-m,2} + M_7 \inf_{v \in \tilde{V}\Lambda} \|f - v\|_{-m,2}.$$

Proof. Suppose that $\lambda \in \tilde{\Lambda}$. Then

$$\langle T_\sigma u_\Lambda, \psi_\lambda \rangle_2 = \langle T_\sigma (u_\Lambda - u_{\tilde{\Lambda}}), \psi_\lambda \rangle_2 + f_\lambda.$$

Thus, $d_\lambda(\Lambda, \epsilon) \leq 2^{-m|\lambda|} |\langle T_\sigma (u_\Lambda - u_{\tilde{\Lambda}}), \psi_\lambda \rangle_2| |e_\lambda|$. Furthermore,

$$\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} 2^{-m|\lambda|} |\langle T_\sigma (u_\Lambda - u_{\tilde{\Lambda}}), \psi_\lambda \rangle_2|^2 \leq \frac{1}{M_3^2} \|T_\sigma (u_\Lambda - u_{\tilde{\Lambda}})\|_{-m,2}^2 \leq \frac{1}{M_1^2 M_3^2} \|u_\Lambda - u_{\tilde{\Lambda}}\|_{m,2}^2.$$

Therefore, from Lemma 4.1 one can get,

$$\left(\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} d_\lambda(\Lambda, \epsilon)^2 \right)^{\frac{1}{2}} \leq \frac{1}{M_1 M_3} \|u_\Lambda - u_{\tilde{\Lambda}}\|_{m,2}^2 + M_5 \epsilon \|Q_\Lambda' f\|_{-m,2}.$$

Hence, $|a_\lambda(\Lambda, \epsilon)| \leq |d_\lambda(\Lambda, \epsilon)| + 2^{-m|\lambda|} |f_\lambda|$, and the proof is complete. □

5. An adaptive algorithm

In this section, we show that for a set $\tilde{\Lambda}$ containing Λ, the solutions in $V_{\tilde{\Lambda}}$ approximate the actual solutions better than the one in V_Λ. In order to do this, we recall our assumptions at the end of Section 2 that the discrete pseudo-differential operator T_σ is symmetric and there exist positive constants M_8 and M_9 for which

$$M_8 \|u\|_{m,2} \leq \|u\|_{T_\sigma} \leq M_9 \|u\|_{m,2}, \qquad u \in \mathcal{H}^{m,2}(\mathbb{Z}^n),$$

where $\|u\|_{T_\sigma}^2 = \langle T_\sigma u, u \rangle_2$.

Theorem 5.1. *Let eps be a given tolerance. For $\theta^* \in (0,1)$, we define the number $M_e := \frac{1}{M_1 M_3} + \frac{1-\theta^*}{2 M_2 M_4}$. Suppose that μ^* is a positive number for which $\mu^* M_e \leq \frac{1-\theta^*}{2(2-\theta^*) M_2 M_4}$. Assume that ϵ is the positive number defined by $\epsilon := \frac{\mu^* eps}{2 M_6 \|f\|_{-m,2}}$. Suppose that Λ is a subset of J for which*

$$M_7 \inf_{v \in \tilde{V}_\lambda} \|f - v\|_{-m,2} \leq \frac{1}{2} \mu^* eps.$$

Then for all subsets $\tilde{\Lambda}$ of J such that $\Lambda \subset \tilde{\Lambda}$ and

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap \mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2 \right)^{\frac{1}{2}} \geq (1 - \theta^*) \left(\sum_{\lambda \in \mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2 \right)^{\frac{1}{2}},$$

there exists a number $\kappa \in (0,1)$ such that $\|u - u_{\tilde{\Lambda}}\|_{T_\sigma} \leq \kappa \|u - u_\Lambda\|_{T_\sigma}$.

Proof. Let us consider the assumption $\|u_\Lambda - u\|_{m,2} \geq \frac{eps}{C_e}$. By using of Proposition 4.2 and Theorem 4.3,

$$\begin{aligned}\|u_{\tilde{\Lambda}} - u_\Lambda\|_{m,2} \geq M_1M_3\Bigg\{\Bigg(\sum_{\lambda\in\tilde{\Lambda}\cap\mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2\Bigg)^{\frac{1}{2}} - M_6\epsilon\|f\|_{-m,2} - M_7\inf_{v\in\tilde{\Lambda}}\|f-v\|_{-m,2}\Bigg\} \\ \geq M_1M_3\Bigg\{(1-\theta^*)\Bigg(\frac{1}{M_2M_4}\|u-u_\Lambda\|_{m,2} - M_6\epsilon\|f\|_{-m,2} \\ - M_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}\Bigg) - M_6\epsilon\|f\|_{-m,2} - M_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}\Bigg\} \\ \geq M_1M_3\Bigg((1-\theta^*)\frac{1}{M_2M_4}\|u-u_\Lambda\|_{m,2} - (2-\theta^*)M_6\epsilon\|f\|_{-m,2} \\ - (2-\theta^*)M_7\inf_{v\in\tilde{V}_\Lambda}\|f-v\|_{-m,2}\Bigg).\end{aligned}$$

Thus,

$$\|u_{\tilde{\Lambda}} - u_\Lambda\|_{m,2} \geq M_1M_3\Bigg(\frac{1-\theta^*}{M_2M_4}\|u-u_\Lambda\|_{m,2} - (2\theta^*)\mu^*eps\Bigg).$$

Indeed,

$$\|u_{\tilde{\Lambda}} - u_\Lambda\|_{m,2} \geq M_1M_3\Bigg(\frac{1-\theta^*}{M_2M_4} - (2-\theta^*)\mu^*M_e\Bigg)\|u-u_\Lambda\|_{m,2} \geq \frac{M_1M_3(1-\theta^*)}{2M_2M_4}\|u-u_\Lambda\|_{m,2}.$$

On the other hand,

$$\begin{aligned}\|u_{\tilde{\Lambda}} - u_\Lambda\|_{m,2}^2 = \langle T_\sigma u_{\tilde{\Lambda}} - T_\sigma u_\Lambda, u_{\tilde{\lambda}} - u_\Lambda\rangle_2 = \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - \langle T_\sigma u_{\tilde{\Lambda}}, u_\Lambda\rangle_2 - \langle T_\sigma u_\Lambda, u_{\tilde{\Lambda}}\rangle_2 \\ = \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - \langle f, u_\Lambda\rangle_2 - \langle u_\Lambda, f\rangle_2 = \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 \\ - \langle T_\sigma u_\Lambda, u_\Lambda\rangle_2 - \langle u_\Lambda, T_\sigma u_\Lambda\rangle_2 = \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 - \|u_\Lambda\|_{T_\sigma}^2. \quad (5.1)\end{aligned}$$

Moreover,

$$\begin{aligned}\|u - u_{\tilde{\Lambda}}\|_{m,2}^2 = \langle T_\sigma u - T_\sigma u_{\tilde{\Lambda}}\rangle_2 = \|u\|_{T_\sigma}^2 + \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 - \langle T_\sigma u, u_{\tilde{\Lambda}}\rangle_2 - \langle T_\sigma u_{\tilde{\Lambda}}, u\rangle_2 \\ = \|u\|_{T_\sigma}^2 + \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 - \langle f, u_{\tilde{\Lambda}}\rangle_2 - \langle u_{\tilde{\Lambda}}, f\rangle_2 = \|u\|_{T_\sigma}^2 + \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2 \\ - \langle T_\sigma u_{\tilde{\Lambda}}, u_{\tilde{\Lambda}}\rangle_2 - \langle u_{\tilde{\Lambda}}, T_\sigma u_{\tilde{\Lambda}}\rangle_2 = \|u\|_{T_\sigma}^2 - \|u_{\tilde{\Lambda}}\|_{T_\sigma}^2. \quad (5.2)\end{aligned}$$

Furthermore,

$$\begin{aligned}\|u - u_\Lambda\|_{T_\sigma}^2 = \langle T_\sigma u - T_\sigma u_\Lambda, u - u_\Lambda\rangle_2 = \|u\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - \langle T_\sigma u, u_\Lambda\rangle_2 - \langle T_\sigma u_\Lambda, u\rangle_2 \\ = \|u\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 - \langle f, u_\Lambda\rangle_2 - \langle u_\Lambda, f\rangle_2 = \|u\|_{T_\sigma}^2 + \|u_\Lambda\|_{T_\sigma}^2 \\ - \langle T_\sigma u_\Lambda, u_\Lambda\rangle_2 - \langle u_\Lambda, T_\sigma u_\Lambda\rangle_2 = \|u\|_{T_\sigma}^2 - \|u\|_{T_\sigma}^2. \quad (5.3)\end{aligned}$$

Hence, by using of the relations 5.1, 5.2 and 5.3, $\|u_{\tilde{\Lambda}} - u_\Lambda\|^2_{T_\sigma} = \|u - u_\Lambda\|^2_{T_\sigma} = \|u - u_{\tilde{\Lambda}}\|^2_{T_\sigma}$, or equivalently $\|u - u_{\tilde{\Lambda}}\|^2_{T_\sigma} + \|u_{\tilde{\Lambda}} - u_\Lambda\|^2_{T_\sigma} = \|u - u_\Lambda\|^2_{T_\sigma}$. On the other side,

$$\begin{aligned}\|u_{\tilde{\Lambda}} - u_\Lambda\|_{T_\sigma} \geq M_8\|u_{\tilde{\Lambda}} - u_\Lambda\|_{m,2} &\geq \frac{M_1M_3M_8(1-\theta^*)}{2M_2M_4}\|u - u_\Lambda\|_{m,2} \\ &\geq \frac{M_1M_3M_8(1-\theta^*)}{2M_2M_4M_9}\|u - u_\Lambda\|_{T_\sigma}. \qquad (5.4)\end{aligned}$$

Therefore,

$$\begin{aligned}\|u - u_{\tilde{\Lambda}}\|^2_{T_\sigma} &= \|u - u_\Lambda\|^2_{T_\sigma} - \|u_{\tilde{\Lambda}} - u_\Lambda\|^2_{T_\sigma} \\ &\leq \|u - u_\Lambda\|^2_{T_\sigma} - \left(\frac{M_1M_3M_8(1-\theta^*)}{2M_2M_4M_9}\right)^2\|u - u_\Lambda\|^2_{T_\sigma} = \kappa\|u - u_\Lambda\|^2_{T_\sigma},\end{aligned}$$

where $\kappa = \sqrt{1 - \left(\frac{M_1M_3M_8(1-\theta^*)}{2M_2M_4M_9}\right)^2}$. □

We can give an adaptive algorithm by the following steps:

Suppose that $\theta^* \in (0,1)$ and the desired accuracy eps are given, we proceed as follows:

Step 1: Compute $\epsilon = \frac{\mu^* eps}{2M_6\|f\|_{-m,2}}$.

Step 2: Determine an index set $\Lambda \subset J$ such that

$$M_7 \inf_{v\in\tilde{V}_\lambda} \|f - v\|_{-m,2} < \frac{1}{2}\mu^* eps.$$

Step 3: Compute the Galerkin solution u_Λ with respect to V_Λ.

Step 4: Compute

$$\eta_{\Lambda,\epsilon} = \left(\sum_{\lambda\in\mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2\right)^{\frac{1}{2}}.$$

If $\eta_{\Lambda,\epsilon} < eps$, then we stop and accept u_Λ as a solution. Otherwise, go to the next step.

Step 5: Determine an index set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset J$ and

$$\left(\sum_{\lambda\in\tilde{\Lambda}\cap\mathcal{N}_{\Lambda,\epsilon}} a_\lambda^2\right)^{\frac{1}{2}} \geq (1-\theta^*)\eta_{\Lambda,\epsilon},$$

and go to Step 3 with Λ replaced by $\tilde{\Lambda}$.

6. Conflict of interest statement

There is no conflict of interest for the authors.

7. Acknowledgements:

The authors would like to acknowledge the financial support of Bozorgmehr University of Qaenat for this research under contract number 39197.

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