Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **12**(1)(2023), 11-16 (Research paper)

B-Spline Finite Element Method for Solving Linear System of Second-Order Boundary Value Problems

A. Yazdani [1] and S. Gharbavi [2]
Department of Mathematics, University of Mazandaran, Babolsar, Iran.
[1] yazdani@umz.ac.ir
[2] s.gharbavi@yahoo.com

Abstract. In this paper, we solve a linear system of second-order boundary value problems by using the quadratic B-spline finite element method (FEM). The performance of the method is tested on one model problem. Comparisons are made with both the analytical solution and some recent results.The obtained numerical results show that the method is efficient.

Keywords: Finite element method; Quadratic B-splines ; Boundary Value Problems.

2000 Mathematics subject classification: 34B05; Secondary 80M10.

1. Introduction

System of ordinary differential equations have been applied to many problems in physics, engineering, biology and so on. There are many publications dealing with the linear system of second-order boundary value problems. For instance, B-spline method has been proposed in [4].

Spline functions are a class of piecewise polynomials which satisfy continuity properties depending on the degree of the polynomials. They have highly desirable characteristics which have made them a powerful mathematical tool for numerical approximations. Spline functions are a

[1]Corresponding author: yazdani@umz.ac.ir
Received: 25 November 2013
Revised: 23 February 2014
Accepted: 24 February 2014

set of continuous combinations of B-splines that used as trial functions in the Galerkin methods [5, 7, 12, 13, 14]. The finite element method was introduced and analyzed for semilinear parabolic problems by Zlamal in [16]. Later Xiong and Chen [15] studied superconvergency of triangular quadratic finite element for semilinear elliptic problem and illustrated the effectiveness of the proposed method.

The quadratic B-splines incorporated with finite element methods have been proven to give very smooth solutions[1, 2, 3, 8, 9, 10, 11, 6], and the use of the quadratic B-splines as shape functions in the finite element method guarantees continuity of the first and second-order derivatives of trial solutions at the mesh points.

In this paper, we present and analyze the B-spline finite element method for solution of a linear system of second-order boundary value problems. The paper is organized as follows: in section 2, the properties of the quadratic B-spline finite element are discussed, the numerical experiments and data comparisons are provided to verify the accuracy and efficiency. The last section is conclusion.

2. Analysis of B-spline finite element method

We consider the following linear system of second-order boundary value problem:

$$\begin{cases} u''+a_1(x)u'+a_2(x)u+a_3(x)v''+a_4(x)v'+a_5(x)v=f_1(x),\\ v''+b_1(x)v'+b_2(x)v+b_3(x)u''+b_4(x)u'+b_5(x)u=f_2(x),\\ u(0)=u(1)=0, \qquad v(0)=v(1)=0, \end{cases} \tag{2.1}$$

where $a_i(x)$,$b_i(x)$,$f_1(x)$ and $f_2(x)$ are given functions, and $a_i(x)$, $b_i(x)$ are continuous, $i=1,2,3,4,5$.

Define $H_0^1(I)$, $I=(0,1)$ by

$$H_0^1(I)=\left\{w|\ w\in H^1(I),,\ w(0)=w(1)=0\right\}. \tag{2.2}$$

The variational problem accordance with (2.1) is: Find $u,v\in H_0^1(0,1)$ such that

$$\begin{cases} -\langle u',w'\rangle+\langle a_1u',w\rangle+\langle a_2u,w\rangle-\langle a_3v',w'\rangle+\langle a_4v',w\rangle+\langle a_5v,w\rangle\\ =\langle f_1,w\rangle,\\ -\langle v',w'\rangle+\langle b_1v',w\rangle+\langle b_2v,w\rangle-\langle b_3u',w'\rangle+\langle b_4u',w\rangle+\langle a_5u,w\rangle\\ =\langle f_2,w\rangle, \end{cases} \tag{2.3}$$

for all $w\in H_0^1(0,1)$, where $\langle u,w\rangle=\int_0^1 uwdx$.

The interval $\overline{I}=[0,1]$ is divided into N finite elements of equal length h by the knots x_i $(i=0,1,\cdots,N)$ such that $0=x_0<x_1<\cdots<x_N=1$ and $h=x_{i+1}-x_i=\frac{1}{N}$.The set of B-splines $\{\phi_{-1},\phi_0,\cdots,\phi_N\}$ form a

basis for the functions defined on $[0,1]$. Quadratic B-splines ϕ_m with required properties are defined by

$$\phi_m = \frac{1}{h^2}\begin{cases} (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2 + 3(x_m - x), & [x_{m-1}, x_m], \\ (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2, & [x_m, x_{m+1}], \\ (x_{m+2}-x)^2 & [x_{m+1}, x_{m+2}], \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

where $h = x_{m+1} - x_m$, $m = -1, 0, \cdots, N$. The quadratic B-spline $\phi_m(x)$ and its first derivative vanishes outside the interval $[x_{m-1}, x_{m+2}]$. The value of ϕ_m and its first derivative $\phi'(x)$ at the knots are given by:

x	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$\phi_m(x)$	0	1	1	0
$\phi'_m(x)$	0	$2/h$	$-2/h$	0

Let

$$u_N(x) = \sum_{j=-1}^{N} C_j \phi_j(x), \qquad v_N(x) = \sum_{j=-1}^{N} D_j \phi_j(x), \tag{2.5}$$

be an approximate solution of (2.1), where C_j and D_j are unknown real coefficients which must be determined. Each B-spline covers three intervals so that three B-splines $\phi_{m-1}, \phi_m, \phi_{m+1}$ cover each finite element $[x_m, x_{m+1}]$. All other B-splines are zero in this region.

Using Eq. (2.4), the nodal value u_m, v_m, u'_m and v'_m at the knot x_m can be expressed in the terms of the coefficients C_j and D_j as

$$\begin{aligned} u_m &:= u_N(x_m) = C_{m-1} + C_m, & v_m &:= v_N(x_m) = D_{m-1} + D_m, \\ u'_m &:= u'_N(x_m) = \frac{2}{h}(C_m - C_{m-1}), & v'_m &:= v'_N(x_m) = \frac{2}{h}(D_m - D_{m-1}). \end{aligned} \tag{2.6}$$

Since $u_N(x)$ and $v_N(x)$ must satisfy the boundary conditions $u_N(0) = u_N(1) = 0$ and $v_N(0) = v_N(1) = 0$, we get $C_{-1} = -C_0$, $C_N = -C_{N-1}$, $D_{-1} = D_0$ and $D_N = D_{N-1}$. Hence, we have

$$u_N(x) = \sum_{j=0}^{N-1} C_j \psi_j(x), \qquad v_N(x) = \sum_{j=0}^{N-1} D_j \psi_j(x), \tag{2.7}$$

where $\psi_0 = \phi_0(x) - \phi_{-1}(x)$, $\psi_m = \phi_m,\ m = 1, 2, \cdots, N-2$ and $\psi_{N-1} = \phi_{N-1} - \phi_N(x)$. Hence $2N$ unknowns C_m, D_m , $m = 0, 1, \cdots, N-1$ must be determined.

According to Galerkin method, the weight function $w(x)$ in Eqs.(2.3) is chosen as $w(x) = \psi_n(x),\ n = 0, 1, \cdots, N-1$.

Putting Eqs.(2.7) in Eqs.(2.3), we have a system of linear equations.

This system can be written in the matrix-vector form as follows:

$$RX = F, \tag{2.8}$$

where

$$X = \Big[C_0, C_1, \cdots, C_{N-1}, D_0, D_1, \cdots, D_{N-1}\Big]^T,$$

$$F = \Big[\int_0^1 f_1\psi_0 dx, \cdots, \int_0^1 f_1\psi_{N-1}dx, \int_0^1 f_2\psi_0 dx, \cdots, \int_0^1 f_2\psi_{N-1}dx\Big]^T,$$

$$R = \begin{bmatrix} M_1 & | & M_2 \\ -- & -- & -- \\ M_3 & | & M_4 \end{bmatrix}_{2N\times 2N},$$

and four tridiagonal submatrices M_1, M_2, M_3, M_4 as follows:

$$(M_1)_{ij} = -\int_0^1 \psi'_j\psi'_i dx + \int_0^1 a_1(x)\psi'_j\psi_i dx + \int_0^1 a_2(x)\psi_j\psi_i dx$$

$$(M_2)_{ij} = -\int_0^1 a_3(x)\psi'_j\psi'_i dx + \int_0^1 a_4(x)\psi'_j\psi_i dx + \int_0^1 a_5(x)\psi_j\psi_i dx$$

$$(M_3)_{ij} = -\int_0^1 \psi'_j\psi'_i dx + \int_0^1 b_1(x)\psi'_j\psi_i dx + \int_0^1 b_2(x)\psi_j\psi_i dx$$

$$(M_4)_{ij} = -\int_0^1 b_3(x)\psi'_j\psi'_i dx + \int_0^1 b_4(x)\psi'_j\psi_i dx + \int_0^1 b_5(x)\psi_j\psi_i dx$$

where $\quad i = 0, \cdots, N-1, \quad j = 0, \cdots, N-1.$

Example 2.1. Consider the following system of second-order boundary value problem

$$\begin{cases} u''(x) + xu(x) + xv(x) = f_1(x), \\ v''(x) + 2xv(x) + 2xu(x) = f_2(x), \end{cases}$$

subject to boundary conditions $u(0) = u(1) = 0$, $v(0) = v(1) = 0$ where $0 < x < 1$, $f_1(x) = 2$ and $f_2(x) = -2$. The exact solution $u(x)$, $v(x)$ are $x^2 - x$ and $x - x^2$, respectively. The initial interval $[0, 1]$ is divided into $N = 41$, finite elements of equal length $h = \frac{1}{N}$. The observed maximum absolute errors for various values x are given in Table 1.

Table 1. The maximum absolute error for Example 2.1 when $h = \frac{1}{41}$.

x	*Absolute error*
0.0	0.0
0.2	1.58673×10^{-4}
0.4	1.19036×10^{-4}
0.6	7.93776×10^{-5}
0.8	3.96886×10^{-6}
1.0	0.0

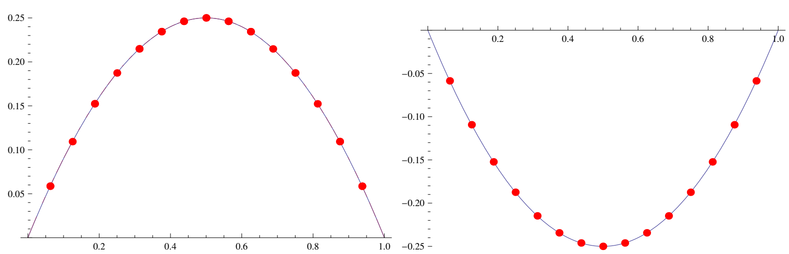

Figure 1. Result for Example 2.1 with $u(x) = x^2 - x$ and $v(x) = x - x^2$.

3. Conclusion

In this paper, B-spline finite element method using quadratic B-spline basis functions has been successfully used to develop the solution of linear system of second-order boundary value problems. We have seen that the numerical technique presented here is capable enough of producing numerical solution of high accuracy. The B-spline FEM is very beneficial for getting the numerical solutions of the differential equations when continuity is the basic requirement. Given technique is flexible enough and can be applied to other complex problems which are difficult to solve directly.

References

[1] E. N. Aksan, Quadratic B-spline finite element method for numerical solution of the Burgers equation, *Applied Mathematics and Computation,* **174** *(2006) 884-896.*

[2] A. R. Bahadir, Application of cubic B-spline finite element technique to the thermistor problem, *Applied Mathematics and Computation,* **149** *(2004) 379-387.*

[3] N. Caglar, H. Caglar, B-spline solution of singular boundary value problems,*Applied Mathematics and Computation,* **182** *(2006) 1509-1513.*

[4] N. Caglar, H. Cagler, B-spline method for solving linear system of second-order boundary value problems,*Computers and Mathematics with Applications,* **57** *(2009) 757-762.*

[5] S. Dhawan, S. Kapoor, S. Kumar, Numerical method for advection diffusion equation using FEM and B-splines, *Journal of Computational Science,* **3** *(2012) 429-437.*

[6] B. Dongmei, Z. Luming, Numerical studies on a novel split-step quadratic B-spline finite element method for the coupled Schrodinger–KdV equations, *Commun. Nonlinear Sci. Numer. Simulat.,* **16** *(2011) 1263-1273.*

[7] L. R. T. Gardner, G. A. Gardner, I. Dag, A B-spline finite element method for the regularized long wave equation, *Commum. Numer. Meth. Eng.,* **11** *(1995) 59-68.*

[8] D. Idris, M. Naci Ozer, Approximation of the RLW equation by the least square cubic B-spline finite element method, *Applied Mathematical Modelling,* **25** *(2001) 221-231.*

[9] S. Kutluay, A. Esen, A B-spline finite element method for the thermistor problem with the modified electrical conductivity, *Applied Mathematics and Computation,* **156** *(2004) 621-632.*

[10] S. Kutluay, A. Esen, I. Dag, Numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method,*Computational and Applied Mathematics,* **167** *(2004) 21-33.*

[11] Q. Lin, Y. Hong Wua, R. Loxton, S. Lai, Linear B-spline finite element method for the improved Boussinesq equation, *Journal of Computational and Applied Mathematics,* **224** *(2009) 658-667.*

[12] T. Ozis, A. Esen, S. Kutluay, Numerical solution of Burgers equation by quadratic B-spline finite elements,*Applied Mathematics and Computation,* **165** *(2005) 237-249.*

[13] L. Ronglin, N. Guangzheng, Y. Jihui, B-spline finite element method in polar coordinates,*Finite Elements in Analysis and Design,* **28** *(1998) 337-346.*

[14] D. Sharma, R. Jiwari, S. Kumar, Numerical Solution of Two Point Boundary Value Problems Using Galerkin-Finite Element Method,*International Journal of Nonlinear Science,* **13***(2012) 204-210.*

[15] Z. G. Xiong, C. M. Chen, Superconvergence of triangular quadratic finite element method with interpolated coefficients for nonlinear elliptic problem, *Acta Math. Sci.,* **26** *(2006) 174-182 .*

[16] M. Zlamal, A finite element solution of the nonlinear heat equation, *RAIRO Model. Anal. Numer.,* **14** *(1980) 203-216.*