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Interpolation Inequalities via Berezin Radius

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Abstract. In this article, we obtain new inequalities for Berezin radius. We have some improvements and interpolations of Berezin radius inequalities via operator convex function. These results offer several general forms and refinements of some known inequalities in the literature.

Keywords: Berezin number, Functional Hilbert space, Berezin norm, Convex function.

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1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle ., . \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathbb{B}(\mathcal{H})$. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be positive if $\langle Tx, x\rangle \geq 0$ for each $x \in \mathcal{H}$. We write $T > 0$ if T is positive and invertible. For $T \in \mathbb{B}(\mathcal{H})$, T^* denotes the adjoint of T. We write $|T| = (T^*T)^{1/2}$ and $|T^*| = (TT^*)^{1/2}$. Every $T \in \mathbb{B}(\mathcal{H})$ can be decomposition as $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$.

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This decomposition is called the Cartesian decomposition of T. The numerical radius of $T \in \mathbb{B}(\mathcal{H})$, denoted by $w(T)$, is defined as $w(T) = \sup\{|\langle Tu, u\rangle| : u \in \mathcal{H} \text{ and } \|u\| = 1\}$. Recall that the operator norm of $T \in \mathbb{B}(\mathcal{H})$ is defined by $\|T\| = \sup\{\|Tu\| : u \in \mathcal{H} \text{ and } \|u\| = 1\}$. It is easy to verify that $w(.)$ defines a norm on $\mathbb{B}(\mathcal{H})$. Furthermore, it is equivalent to the operator norm on $\mathbb{B}(\mathcal{H})$, satisfies

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

A recently released book [7] discusses several improvements of the inequalities in (1.1) and related conclusions. The reader can also view the papers [9, 8, 11, 29] and references.

Next we turn our attention to a functional Hilbert space (FHS). A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of all complex-valued functions on a non-empty set Ω, which has the property that point evaluations are continuous, i.e., for every $\tau \in \Omega$ the map $E_\tau : \mathcal{H} \to \mathbb{C}$ defined by $E_\tau(h) = h(\tau)$, is continuous. Riesz representation theorem ensures that for each $\tau \in \Omega$ there exists a unique $k_\tau \in \mathcal{H}$ such that $h(\tau) = \langle h, k_\tau\rangle$ for all $h \in \mathcal{H}$. The collection of k_τ for all $\tau \in \Omega$ is called the reproducing kernel of $\mathcal{H}$ and the collection of $\widehat{k}_\tau := \frac{k_\tau}{\|k_\tau\|_{\mathcal{H}}}$ for all $\tau \in \Omega$ is called the normalized reproducing kernel of $\mathcal{H}$. For any $T \in \mathbb{B}(\mathcal{H})$, the Berezin symbol of T is a function $\widetilde{T}$ on Ω defined as $\widetilde{T}(\tau) := \left\langle T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle$, for each $\tau \in \Omega$, which was introduced by Berezin [2]. The Berezin set (or range) of T is denoted by $\text{Ber}(T)$ and is defined as $\text{Ber}(T) := \left\{\widetilde{T}(\tau) : \tau \in \Omega\right\}$. The Berezin number (or radius) of T, denoted by $\text{ber}(T)$ and the Berezin norm of T, denoted by $\|T\|_{\text{Ber}}$, are respectively defined as

$$\text{ber}(T) := \sup_{\tau\in\Omega}\left|\widetilde{T}(\tau)\right| \text{ and } \|T\|_{\text{Ber}} := \sup_{\tau\in\Omega}\left\|T\widehat{k}_\tau\right\|$$

(see, [25, 26]). For $T, S \in \mathbb{B}(\mathcal{H})$ it is clear from the definition of the Berezin number and the Berezin norm that the following properties hold:

(B1) $\text{ber}(\alpha T) = |\alpha|\,\text{ber}(T)$ for all $\alpha \in \mathbb{C}$,
(B2) $\text{ber}(T + S) \leq \text{ber}(T) + \text{ber}(S)$,
(B3) $\text{ber}(T) \leq \|T\|_{\text{ber}}$,
(B4) $\|\alpha T\|_{\text{ber}} = |\alpha|\,\|T\|_{\text{ber}}$ for all $\alpha \in \mathbb{C}$,
(B5) $\|T + S\|_{\text{ber}} \leq \|T\|_{\text{ber}} + \|S\|_{\text{ber}}$,
(B6) $\|T\|_{\text{ber}} = \|T^*\|_{\text{ber}}$ and $\text{ber}(T) = \text{ber}(T^*)$.

In [10], it is proved that $\|T\|_{\text{Ber}} = \text{ber}(T)$, if $T \in \mathbb{B}(\mathcal{H})$ is positive. It is clear from the definition that $\text{Ber}(T) \subseteq W(T)$ and so

$$\text{ber}(T) \leq w(T) \leq \|T\|. \tag{1.2}$$

The Berezin number inequalities have been studied by many mathematicians over the years, for the latest and recent results we refer the readers to see [3, 5, 6, 12, 15, 16, 18, 19, 20, 21, 30, 31, 32, 34].

In 2021, the following inequalities has been shown Huban et al. ([22])

$$\frac{1}{4}\left\||T|^2+|T^*|^2\right\|_{\text{ber}} \le \text{ber}(T) \le \frac{1}{2}\left\||T|^2+|T^*|^2\right\|_{\text{ber}} \tag{1.3}$$

and

$$\text{ber}^r(S^*T) \le \frac{1}{2}\left\||T|^r+|S|^r\right\|_{\text{ber}}. \tag{1.4}$$

Also, the same authors (see, [23, 24]) have proved

$$\text{ber}(T) \le \frac{1}{2}\left\||T|+|T^*|\right\|_{\text{ber}} \le \frac{1}{2}\left(\|T\|_{\text{ber}}+\|T\|_{\text{ber}}^{1/2}\right) \tag{1.5}$$

and

$$\text{ber}^{2r}(T) \le \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\|_{\text{ber}}, \text{ where } r \ge 1. \tag{1.6}$$

In this article, we obtain new inequalities for Berezin radius. We have some improvements and interpolations of Berezin radius inequalities via operator convex function. These results offer several general forms and refinements of some known inequalities in the literature. The bounds obtained here improve on the earlier ones studied in [17, 22].

2. Auxiliary Theorems

To reach our goal in this present article we begin with the following sequence of lemmas. The first lemma is aritmetic-geometric mean inequality for usual norm.

Lemma 2.1 ([8])**.** *Let $T,S\in\mathbb{B}(\mathcal{H})$ be positive, then*

$$\|TS\| \le \frac{1}{4}\|T+S\|^2. \tag{2.1}$$

Lemma 2.2 ([27])**.** (*i*) *Let $T\in\mathbb{B}(\mathcal{H})$. If $f,g:[0,\infty]\to[0,\infty]$ are continuous functions satisfying $f(t)\,g(t)=t$ for all $t\ge 0$, then*

$$|\langle Tx,y\rangle| \le \|f(|T|)\,x\|\,\|g(|T^*|)\,y\|,\ \forall x,y\in\mathcal{H}. \tag{2.2}$$

(*ii*) *Let $x\in\mathcal{H}$ with $\|x\|=1$. Then*

$$|\langle Tx,x\rangle|^2 \le \langle |T|\,x,x\rangle\,\langle |T^*|\,x,x\rangle. \tag{2.3}$$

Lemma 2.3 ([28])**.** *If $T,S\in\mathbb{B}(\mathcal{H})$ are positive operators, then we have $\|T+S\|\le\|T\|+\|S\|$ iff $\|TS\|=\|T\|\,\|S\|$.*

Now, we remember the definition of operator convex function. It says that: A real-valued continuous function f on an interval J is denoted operator convex if

$$f((1-t)\,T+tS) \le (1-t)\,f(T)+tf(S)$$

in the operator order for all $t \in [0,1]$ and for every self-adjoint operator T and S on a Hilbert space $\mathcal{H}$ whose spectra are contained in J. If either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$, the function $f(t) = t^r$ is operator convex.

Lemma 2.4 ([14]). *Let $T \in \mathbb{B}(\mathcal{H})$ be a self-adjoint operator and let $x \in \mathcal{H}$ be an unit vector. If f is a convex function on an interval containing the spectrum of T, then*

$$f(\langle Tx, x\rangle) \leq \langle f(T)x, x\rangle. \tag{2.4}$$

If f is a concave, then inequality (2.4) holds in the reverse direction.

Lemma 2.5 ([13]). *Let $f : J \to \mathbb{R}$ be an operator convex function on an interval J. Let T and S be two self-adjoint operators with spectra in J. Then*

$$f\left(\frac{T+S}{2}\right) \leq \int_0^1 f((1-t)T + tS)\,dt \leq \frac{1}{2}(f(T) + f(S)). \tag{2.5}$$

If f is non-negative, then the operator inequality (2.5) can be reduced to the following norm inequality:

$$\left\|f\left(\frac{T+S}{2}\right)\right\| \leq \left\|\int_0^1 f((1-t)T + tS)\,dt\right\| \leq \frac{1}{2}\|f(T) + f(S)\|. \tag{2.6}$$

Lemma 2.6. *If $f : [0,d] \to [0,\infty]$, $(d > 0)$ is an increasing convex function with $f(0) = 0$ and $\alpha \in [0,1]$. Then we have*

$$f(\alpha x) = \alpha f(x). \tag{2.7}$$

Lemma 2.7 ([1]). *Let f be a nonnegative increasing convex function on $[0,\infty)$ and let $T, S \in \mathbb{B}(\mathcal{H})$ be a positive operators. Then*

$$\|f((1-v)T + vS)\| \leq \|(1-v)f(T) + vf(S)\| \tag{2.8}$$

for every $0 \leq v \leq 1$.

Lemma 2.8 ([22]). *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS and let $T, S \in \mathbb{B}(\mathcal{H})$ be a self adjoint. Then*

$$\|T + S\| \leq \sqrt{\mathrm{ber}^2(T + iS) + \|T\|\,\|S\| + \mathrm{ber}(ST)}. \tag{2.9}$$

Lemma 2.9 ([4]). *Let $T \in \mathbb{B}(\mathcal{H})$ be a positive operator. Then for all $\lambda \in \Omega$*

$$\left(\widetilde{T}(\lambda)\right)^\alpha \leq \frac{1}{\mu}\widetilde{T^\alpha}(\lambda),\ \alpha \geq 1, \tag{2.10}$$

where $\mu = 1 + 2(\alpha - 1)\left(1 - \frac{\widetilde{T^{1/2}}(\lambda)}{\left(\widetilde{T}(\lambda)\right)^{1/2}}\right)$.

Lemma 2.10 ([33])**.** *If $T \in \mathbb{B}(\mathcal{H})$ be a hyponormal, i.e. $T^*T - TT^* \geq 0$, $v = \min\{\lambda, 1-\lambda\}$, where $0 \leq \lambda \leq 1$, then*

$$\operatorname{ber}(T) \leq \frac{1}{\zeta} \frac{\||T| + |T^*|\|_{\text{ber}}}{2}, \tag{2.11}$$

where $\zeta \geq 1$, $\zeta = \inf_{\xi \in \Omega}\left\{K\left(\frac{\widetilde{|T|}(\xi)}{\widetilde{|T^*|}(\xi)}, 2\right)^v\right\}$.

3. Main Results

In this section, we mainly establish several refinement of Berezin radius inequalities (1.3). Furthermore, the main goal of this section is to present new interpolation inequalities of some known inequalities for the numerical radius by using the properties of operator convex functions.

3.1. **Some refinement of Berezin radius inequalities.** We now prove the following norm inequalities.

Theorem 3.1. *If $\mathcal{H} = \mathcal{H}(\Omega)$ is a FHS and $T, S \in \mathbb{B}(\mathcal{H})$, then the following inequalities hold:*

$$\|T+S\|_{\text{ber}}^2 \leq \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + \frac{1}{2}\|T^*T + S^*S\|_{\text{ber}} + \operatorname{ber}(T^*S)$$

and

$$\|T+S\|_{\text{ber}}^2 \leq \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + \frac{1}{2}\|TT^* + SS^*\|_{\text{ber}} + \operatorname{ber}(TS^*).$$

Proof. Let τ, υ be an arbitrary. Then we have

$$\begin{aligned}
&\left|\left\langle (T+S)\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 \\
&\leq \left(\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right| + \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|\right)^2 \\
&= \left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + 2\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right| \\
&= \left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + 2\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\left\langle \widehat{k}_\tau, S\widehat{k}_\upsilon\right\rangle\right| \\
&\leq \left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \left\|T\widehat{k}_\tau\right\|\left\|S\widehat{k}_\tau\right\| + \left|\left\langle T\widehat{k}_\tau, S\widehat{k}_\upsilon\right\rangle\right| \\
&\leq \left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \frac{1}{2}\left(\left\|T\widehat{k}_\tau\right\| + \left\|S\widehat{k}_\tau\right\|\right) + \left|\left\langle T\widehat{k}_\tau, S\widehat{k}_\upsilon\right\rangle\right| \\
&\leq \left|\left\langle T\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^2 + \frac{1}{2}\left\langle (T^*T + S^*S)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left|\left\langle T^*S\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|,
\end{aligned}$$

where the fourth inequality follows from Bozano's inequality ([11]), i.e., if $x, y, e \in \mathcal{H}$ and $\|e\| = 1$, then we have

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|). \tag{3.1}$$

Now, taking the supremum over all $\tau, \upsilon \in \Omega$ with $\tau = \upsilon$, we get

$$\begin{aligned}\sup_{\tau\in\Omega}\left|\left\langle (T+S)\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^2 &\le \sup_{\tau\in\Omega}\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^2 + \sup_{\tau\in\Omega}\left|\left\langle S\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^2 \\ &+ \sup_{\tau\in\Omega}\frac{1}{2}\left\langle (T^*T+S^*S)\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left|\left\langle T^*S\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\end{aligned}$$

which to equivalent

$$\|T+S\|_{\text{ber}}^2 \le \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + \frac{1}{2}\|T^*T+S^*S\|_{\text{ber}} + \text{ber}\,(T^*S)\,. \quad (3.2)$$

By replacing T by T^* and S by S^* in (3.2), we get

$$\|T+S\|_{\text{ber}}^2 \le \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + \frac{1}{2}\|TT^*+SS^*\|_{\text{ber}} + \text{ber}\,(TS^*)\,.$$

This completes the proof. □

In [22, Theorem 3.1], Huban et al. obtained another refinement of the second inequality in (1.2), the authors proved that

$$\frac{1}{4}\|T^*T+TT^*\|_{\text{ber}} \le \text{ber}^2\,(T) \le \frac{1}{2}\|T^*T+TT^*\|_{\text{ber}}\,. \quad (3.3)$$

Based on the above norm inequalities we obtain the following refinement of Huban et al.'s inequality (3.3).

Theorem 3.2. *If $\mathcal{H} = \mathcal{H}(\Omega)$ is a FHS and $T \in \mathbb{B}(\mathcal{H})$, then we have*

$$\begin{aligned}&\frac{1}{4}\|T^*T+TT^*\|_{\text{ber}} \\ &\le \frac{1}{8}\left[\left(\|T+T^*\|_{\text{ber}}^2 + \|T-T^*\|_{\text{ber}}^2\right) + \frac{1}{2}\left(\|T+T^*\|_{\text{ber}}^2 - \|T-T^*\|_{\text{ber}}^2\right)^2\right]^{\frac{1}{2}} \\ &\le \text{ber}^2\,(T)\,.\end{aligned}$$

Proof. Let $T = S + iR$ be the Cartesian decomposition of T. Then S and R are self-adjoint, and $T^*T + TT^* = 2\left(S^2 + R^2\right)$. It is clear that

$$\frac{1}{4}\|T^*T+TT^*\|_{\text{ber}} = \frac{1}{2}\left\|S^2+R^2\right\|_{\text{ber}}\,. \quad (3.4)$$

From the identity (3.4) and Theorem 3.1, we get

$$\begin{aligned}
&\frac{1}{4}\left\|T^*T+TT^*\right\|_{\mathrm{ber}}\\
&=\frac{1}{2}\left\|S^2+R^2\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}\left[\left\|S\right\|_{\mathrm{ber}}^4+\left\|R\right\|_{\mathrm{ber}}^4+\frac{1}{2}\left\|S^4+R^4\right\|_{\mathrm{ber}}+\mathrm{ber}\left(S^2R^2\right)\right]^{1/2}\\
&\leq\frac{1}{2}\left[\left\|S\right\|_{\mathrm{ber}}^4+\left\|R\right\|_{\mathrm{ber}}^4+\frac{1}{2}\left(\left\|S\right\|_{\mathrm{ber}}^4+\left\|R\right\|_{\mathrm{ber}}^4\right)+\left\|S\right\|_{\mathrm{ber}}^2\left\|R\right\|_{\mathrm{ber}}^2\right]^{1/2}.
\end{aligned}$$

The rest of the proof is easily illustrated with a basic calculation. □

The following theorem shows that inequality (3.5) is a refinement of inequality given in Theorem 3.2.

Theorem 3.3. *If $\mathcal{H}=\mathcal{H}(\Omega)$ is a FHS and $T\in\mathbb{B}(\mathcal{H})$, then we have*

$$\begin{aligned}
&\frac{1}{4}\left\|T^*T+TT^*\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{8}\left[\left(\left\|T+T^*\right\|_{\mathrm{ber}}^2+\left\|T-T^*\right\|_{\mathrm{ber}}^2\right)+\frac{3}{4}\left(\left\|T+T^*\right\|_{\mathrm{ber}}^2+\left\|T-T^*\right\|_{\mathrm{ber}}^2\right)^2\right]^{\frac{1}{2}}\\
&\leq\mathrm{ber}^2\left(T\right). \qquad (3.5)
\end{aligned}$$

Proof. Now let us prove the first inequality in (3.5). In fact, according to the identity (3.4), the AM-GM inequality for usual norm (see, [8]), the inequality $\mathrm{ber}^2\left(S^2+iR^2\right)\leq\left\|S^4+R^4\right\|_{\mathrm{ber}}$ and the inequality (2.1), we have that

$$\begin{aligned}
&\frac{1}{4}\left\|T^*T+TT^*\right\|_{\mathrm{ber}}\\
&=\frac{1}{2}\left\|S^2+R^2\right\|_{\mathrm{ber}}\\
&\leq\frac{1}{2}\left[\mathrm{ber}^2\left(S^2+iR^2\right)+\left\|S^2\right\|_{\mathrm{ber}}\left\|R^2\right\|_{\mathrm{ber}}+\mathrm{ber}\left(S^2R^2\right)\right]^{\frac{1}{2}}\\
&\leq\frac{1}{2}\left[\left\|S^4+R^4\right\|_{\mathrm{ber}}+\left\|S^2\right\|_{\mathrm{ber}}\left\|R^2\right\|_{\mathrm{ber}}+\left\|S^2R^2\right\|_{\mathrm{ber}}\right]^{\frac{1}{2}}\\
&\leq\frac{1}{2}\left[\left\|S\right\|_{\mathrm{ber}}^4+\left\|R\right\|_{\mathrm{ber}}^4+\left\|S\right\|_{\mathrm{ber}}^2\left\|R\right\|_{\mathrm{ber}}^2+\frac{1}{4}\left\|S^2+R^2\right\|_{\mathrm{ber}}^2\right]^{\frac{1}{2}}\\
&\leq\frac{1}{2}\left[\left\|S\right\|_{\mathrm{ber}}^4+\left\|R\right\|_{\mathrm{ber}}^4+\frac{1}{2}\left(\left\|S\right\|_{\mathrm{ber}}^4+\left\|R\right\|_{\mathrm{ber}}^4\right)+\frac{1}{4}\left(\left\|S\right\|_{\mathrm{ber}}^2+\left\|R\right\|_{\mathrm{ber}}^2\right)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

The desired first inequality in (3.5) is obtained. We will now prove the second inequality in (3.5). We have $\left\|S\right\|_{\mathrm{ber}}\leq\mathrm{ber}\left(T\right)$ and $\left\|R\right\|_{\mathrm{ber}}\leq$

$\mathrm{ber}(T)$. So

$$\frac{1}{4}\left\|T^*T+TT^*\right\|_{\mathrm{ber}}$$
$$\le \frac{1}{2}\left[\|S\|_{\mathrm{ber}}^4+\|R\|_{\mathrm{ber}}^4+\frac{1}{2}\left(\|S\|_{\mathrm{ber}}^4+\|R\|_{\mathrm{ber}}^4\right)+\frac{1}{4}\left(\|S\|_{\mathrm{ber}}^2+\|R\|_{\mathrm{ber}}^2\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{2}\left[4\mathrm{ber}^4(T)\right]^{\frac{1}{2}}$$
$$\le \mathrm{ber}^2(T)$$

which gives the desired the second inequality in (3.5). This completes the proof. □

Theorem 3.4. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS and $T,S\in\mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}\|T+S\|_{\mathrm{ber}} &\le \sqrt{\|T+iS\|_{\mathrm{ber}}^2+\|T\|_{\mathrm{Ber}}\|S\|_{\mathrm{Ber}}+\mathrm{ber}(S^*T)} \\ &\le \|T\|_{\mathrm{ber}}+\|S\|_{\mathrm{ber}}\end{aligned} \tag{3.6}$$

Proof. Let τ,υ be an arbitrary. Then we have

$$\begin{aligned}\left|\left\langle (T+S)\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2 &\le \left(\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|+\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|\right)^2 \\ &= \left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+2\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right| \\ &= \left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+2\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\left\langle \widehat{k}_\tau,S\widehat{k}_\upsilon\right\rangle\right| \\ &\le \left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left\|T\widehat{k}_\tau\right\|\left\|S\widehat{k}_\tau\right\|+\left|\left\langle T\widehat{k}_\tau,S\widehat{k}_\upsilon\right\rangle\right| \\ &= \left|\left\langle T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle+i\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left\|T\widehat{k}_\tau\right\|\left\|S\widehat{k}_\tau\right\|+\left|\left\langle S^*T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right| \\ &= \left|\left\langle (T+iS)\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left\|T\widehat{k}_\tau\right\|\left\|S\widehat{k}_\tau\right\|+\left|\left\langle S^*T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|,\end{aligned}$$

where the second inequality follows from the inequality $(|a|+|b|)^2=|a|^2+|b|^2+2|ab|$ and the fifth inequality follows from the inequality $|a+ib|^2=|a|^2+|b|^2$. Hence we have

$$\begin{aligned}&\left|\left\langle (T+S)\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2 \\ &\le \left|\left\langle (T+iS)\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^2+\left\|T\widehat{k}_\tau\right\|\left\|S\widehat{k}_\tau\right\|+\left|\left\langle S^*T\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|.\end{aligned}$$

Now, taking the supremum over $\tau,\upsilon\in\Omega$ with $\tau=\upsilon$ in the above inequality, we have

$$\|T+S\|_{\mathrm{ber}}^2 \le \|T+iS\|_{\mathrm{ber}}^2+\|T\|_{\mathrm{Ber}}\|S\|_{\mathrm{Ber}}+\mathrm{ber}(S^*T).$$

Thus

$$\|T+S\|_{\text{ber}} \leq \sqrt{\|T+iS\|_{\text{ber}}^2 + \|T\|_{\text{Ber}}\|S\|_{\text{Ber}} + \text{ber}(S^*T)}.$$

We will now prove the second inequality of (3.6). We have

$$\begin{aligned}(\|T\|_{\text{ber}} + \|S\|_{\text{ber}})^2 &\leq \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + 2\|T\|_{\text{ber}}\|S\|_{\text{ber}} \\ &= \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + \|T\|_{\text{ber}}\|S\|_{\text{ber}} + \|T\|_{\text{ber}}\|S\|_{\text{ber}} \\ &\geq \|T\|_{\text{ber}}^2 + \|S\|_{\text{ber}}^2 + \|T\|_{\text{Ber}}\|S\|_{\text{Ber}} + \|T\|_{\text{ber}}\|S\|_{\text{ber}} \\ &\geq \|T+iS\|_{\text{ber}}^2 + \|T\|_{\text{Ber}}\|S\|_{\text{Ber}} + \text{ber}(S^*T),\end{aligned}$$

and, so

$$\sqrt{\|T+iS\|_{\text{ber}}^2 + \|T\|_{\text{Ber}}\|S\|_{\text{Ber}} + \text{ber}(S^*T)} \leq \|T\|_{\text{ber}} + \|S\|_{\text{ber}},$$

which gives the desired the second inequality in (3.6). □

Theorem 3.5. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS. If $T \in \mathbb{B}(\mathcal{H})$ and f, g are non-negative continuous function on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then we get*

$$\begin{aligned}\text{ber}^{2r}(T) &\leq \frac{1}{4}\left\|f^{4r}(|T|) + g^{4r}(|T^*|)\right\|_{\text{ber}} + \frac{1}{4}\left\|f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right\|_{\text{ber}} \\ &\leq \frac{1}{4}\left\|f^{4r}(|T|) + g^{4r}(|T^*|)\right\|_{\text{ber}} + \frac{1}{2}\text{ber}\left(f^{2r}(|T|)g^{2r}(|T^*|)\right)\end{aligned} \tag{3.7}$$

for all $r \geq 1$.

Proof. Let τ be an arbitrary. Then, from the inequalities (2.2), Hölder-McCarthy inequality and AM-GM inequality, we get

$$\begin{aligned}&\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2r} \\ &\leq \left\langle f^2(|T|)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^r \left\langle g^2(|T^*|)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^r \\ &\leq \left\langle f^{2r}(|T|)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle g^{2r}(|T^*|)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\ &\leq \left(\frac{\left\langle f^{2r}(|T|)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle g^{2r}(|T^*|)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle}{2}\right)^2 \\ &\leq \frac{1}{4}\left\langle \left(f^{2r}(|T|) + g^{2r}(|T^*|)\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2 \\ &\leq \frac{1}{4}\left\langle \left(f^{2r}(|T|) + g^{2r}(|T^*|)\right)^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\ &= \frac{1}{4}\left\langle \left(f^{4r}(|T|) + g^{4r}(|T^*|) + f^{2r}(|T|)g^{2r}(|T^*|) + g^{2r}(|T^*|)f^{2r}(|T|)\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle,\end{aligned}$$

and so

$$\left|\left\langle T\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|^{2r}$$
$$\leq\frac{1}{4}\left\langle\left(f^{4r}\left(|T|\right)+g^{4r}\left(|T^{*}|\right)+f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)+g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right)\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle.$$

Taking the supremum over $\tau\in\Omega$ in the above inequality, we have

$$\begin{aligned}
&\operatorname{ber}^{2r}(T)\\
&\leq\frac{1}{4}\left\|f^{4r}\left(|T|\right)+g^{4r}\left(|T^{*}|\right)+f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)+g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right\|_{\text{ber}}\\
&\leq\frac{1}{4}\left(\left\|f^{4r}\left(|T|\right)+g^{4r}\left(|T^{*}|\right)\right\|_{\text{ber}}+\left\|f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)+g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right\|_{\text{ber}}\right)\\
&=\frac{1}{4}\left\|f^{4r}\left(|T|\right)+g^{4r}\left(|T^{*}|\right)\right\|_{\text{ber}}+\frac{1}{4}\left\|f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)+g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right\|_{\text{ber}}.
\end{aligned}$$

From the (B6) and (B2) feature, we also observe that

$$\begin{aligned}
\operatorname{ber}\left(f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)\right)&=\frac{1}{2}\operatorname{ber}\left(f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)\right)+\frac{1}{2}\operatorname{ber}\left(g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right)\\
&\geq\frac{1}{2}\operatorname{ber}\left(f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)+g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right)\\
&\geq\frac{1}{2}\left\|f^{2r}\left(|T|\right)g^{2r}\left(|T^{*}|\right)+g^{2r}\left(|T^{*}|\right)f^{2r}\left(|T|\right)\right\|_{\text{ber}}
\end{aligned}$$

as required to prove. The theorem is proven. □

Corollary 3.6. *If we put $r=1$ in inequality (3.7), then we have*

$$\begin{aligned}
\operatorname{ber}^{2}(T)&\leq\frac{1}{4}\left\|f^{4}\left(|T|\right)+g^{4}\left(|T^{*}|\right)\right\|_{\text{ber}}+\frac{1}{4}\left\|f^{2}\left(|T|\right)g^{2}\left(|T^{*}|\right)+g^{2}\left(|T^{*}|\right)f^{2}\left(|T|\right)\right\|_{\text{ber}}\\
&\leq\frac{1}{4}\left\|f^{4}\left(|T|\right)+g^{4}\left(|T^{*}|\right)\right\|_{\text{ber}}+\frac{1}{2}\operatorname{ber}\left(f^{2}\left(|T|\right)g^{2}\left(|T^{*}|\right)\right),
\end{aligned}$$

which is the result of [17].

Taking $f(t)=t^{1-v}$ and $g(t)=t^{v}$ with $0\leq v\leq 1$ in inequality (3.7), we also get the following inequality:

Corollary 3.7. *If $T\in\mathbb{B}(\mathcal{H})$, then we have*

$$\begin{aligned}
\operatorname{ber}^{2r}(T)&\leq\frac{1}{4}\left\||T|^{4r(1-v)}+|T^{*}|^{4rv}\right\|_{\text{ber}}+\frac{1}{4}\left\||T|^{2r(1-v)}|T^{*}|^{2rv}+|T^{*}|^{2rv}|T|^{2r(1-v)}\right\|_{\text{ber}}\\
&\leq\frac{1}{4}\left\||T|^{4r(1-v)}+|T^{*}|^{4rv}\right\|_{\text{ber}}+\frac{1}{2}\operatorname{ber}\left(|T|^{2r(1-v)}|T^{*}|^{2rv}\right).
\end{aligned}$$

Corollary 3.8. *If we put $r=1$ in Corollary 3.7, then we have*

$$\begin{aligned}
\operatorname{ber}^{2}(T)&\leq\frac{1}{4}\left\||T|^{4(1-v)}+|T^{*}|^{4v}\right\|_{\text{ber}}+\frac{1}{4}\left\||T|^{2(1-v)}|T^{*}|^{2v}+|T^{*}|^{2v}|T|^{2(1-v)}\right\|_{\text{ber}}\\
&\leq\frac{1}{4}\left\||T|^{4(1-v)}+|T^{*}|^{4v}\right\|_{\text{ber}}+\frac{1}{2}\operatorname{ber}\left(|T|^{2(1-v)}|T^{*}|^{2v}\right).
\end{aligned}$$

In particular

$$\begin{aligned}\operatorname{ber}^2(T) &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\|_{\text{ber}}+\frac{1}{4}\left\||T|\,|T^*|+|T^*|\,|T|\right\|_{\text{ber}}\\ &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\|_{\text{ber}}+\frac{1}{2}\operatorname{ber}(|T|\,|T^*|)\,. \end{aligned}\tag{3.8}$$

Remark 3.9. Gürdal and Başaran (see, [17, Corollary 2]) have proven the first inequality of Corollary 3.7 and inequality (3.8), and in [22, Corolary 3.3], Huban et al. have proved the second inequality of inequality (3.8). From Corollary 3.8, we have

$$\frac{1}{2}\left\||T|\,|T^*|+|T|\,|T^*|\right\|_{\text{ber}}\leq \operatorname{ber}(|T|\,|T^*|)\,.\tag{3.9}$$

Theorem 3.10. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS. If $T,S\in\mathbb{B}(\mathcal{H})$ and $r\geq 1$, then*

$$\begin{aligned}\operatorname{ber}^{2r}(S^*T) &\leq \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}+\frac{1}{4}\left\||T|^{2r}|S|^{2r}+|S|^{2r}|T|^{2r}\right\|_{\text{ber}}\\ &\leq \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}+\frac{1}{2}\operatorname{ber}\left(|T|^{2r}|S|^{2r}\right)\\ &\leq \frac{1}{2}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}\end{aligned}\tag{3.10}$$

Proof. Following the same procedure as in Theorem 3.5 and Cauchy-Schwarz inequality

$$\left|\left\langle S^*T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^{2r}\leq\left\langle |T|^{2r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle |S|^{2r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle,$$

we obtain the first inequality and second inequality in (3.10). Then, from the inequality in [9, Corollary 3.16], we get

$$\begin{aligned}&\frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}+\frac{1}{2}\operatorname{ber}\left(|T|^{2r}|S|^{2r}\right)\\ &\leq \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}+\frac{1}{2}\left\||T|^{2r}|S|^{2r}\right\|_{\text{ber}}\\ &\leq \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}+\frac{1}{2}\left\|\left(\frac{|T|^{4r}+|S|^{4r}}{2}\right)\right\|_{\text{ber}}\\ &\leq \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}+\frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}}\\ &=\frac{1}{2}\left\||T|^{4r}+|S|^{4r}\right\|_{\text{ber}},\end{aligned}$$

which gives the desired the last inequality in (3.10). This completes the proof. ∎

Corollary 3.11. *Let $T \in \mathbb{B}(\mathcal{H})$ be an operator and $r \geq 1$. If we take $v = \frac{1}{2}$ in Remark 3.9, then we have*

$$\begin{aligned} \mathrm{ber}^{2r}(T) &\leq \frac{1}{4}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{\mathrm{ber}} + \frac{1}{4}\left\| |T|^{r}|T^*|^{r} + |T|^{r}|T^*|^{r} \right\|_{\mathrm{ber}} \\ &\leq \frac{1}{4}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}\left(|T|^{r}|T^*|^{r}\right) \ \textit{(by (3.9))} \\ &\leq \frac{1}{4}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{\mathrm{ber}}. \end{aligned}$$

The above inequality is an improvement of inequality (1.3).

In the future theorem, we have the improvement of the second inequality in Corollary 3.11.

Theorem 3.12. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS, $T \in \mathbb{B}(\mathcal{H})$ and $r \geq 1$. Then we have*

$$\mathrm{ber}^{2r}(T) \leq \frac{1}{4\varsigma\gamma}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{\mathrm{ber}} + \frac{1}{2\varsigma\gamma}\mathrm{ber}\left(|T|^{r}|T^*|^{r}\right),$$

where

$$\varsigma = \inf\left\{ 1 + 2(r-1)\left(1 - \frac{\widetilde{|T|^{\frac{1}{2}}}(\tau)}{\left(\widetilde{|T|}(\tau)\right)^{\frac{1}{2}}} \right) \right\},$$

$$\gamma = \inf\left\{ 1 + 2(r-1)\left(1 - \frac{\widetilde{|T^*|^{\frac{1}{2}}}(\tau)}{\left(\widetilde{|T^*|}(\tau)\right)^{\frac{1}{2}}} \right) \right\}.$$

Proof. Let $\widehat{k}_\tau$ be a normalized reproducing kernel. It follows from (2.3), (2.10) and (3.1) that

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2r} \\
&\leq \left\langle |T| \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^r \left\langle |T^*| \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^r \\
&\leq \frac{1}{\varsigma\gamma}\left\langle |T^*|^r \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle \widehat{k}_\tau, |T^r| \widehat{k}_\tau\right\rangle \\
&\leq \frac{1}{2\varsigma\gamma}\left(\left\||T^r| \widehat{k}_\tau\right\| \left\||T^*|^r \widehat{k}_\tau\right\| + \left|\left\langle |T^r| \widehat{k}_\tau, |T^*|^r \widehat{k}_\tau\right\rangle\right|\right) \\
&\leq \frac{1}{4\varsigma\gamma}\left(\left\||T^r| \widehat{k}_\tau\right\|^2 + \left\||T^*|^r \widehat{k}_\tau\right\|^2\right) + \frac{1}{2\varsigma\gamma}\left|\left\langle |T^r|\,|T^*|^r \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right| \\
&= \frac{1}{4\varsigma\gamma}\left(\left\langle |T|^{2r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |T^*|^{2r} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right) + \frac{1}{2\varsigma\gamma}\left|\left\langle |T^r|\,|T^*|^r \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right| \\
&= \frac{1}{4\varsigma\gamma}\left\langle \left(|T|^{2r} + |T^*|^{2r}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \frac{1}{2\varsigma\gamma}\left|\left\langle |T^r|\,|T^*|^r \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|.
\end{aligned}$$

Taking the supremum over $\tau \in \Omega$ in the above inequality, we have

$$\begin{aligned}
\sup_{\tau\in\Omega}\left|\left\langle T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2r} &\leq \sup_{\tau\in\Omega}\frac{1}{4\varsigma\gamma}\left\langle \left(|T|^{2r} + |T^*|^{2r}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + \sup_{\tau\in\Omega}\frac{1}{2\varsigma\gamma}\left|\left\langle |T^r|\,|T^*|^r \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|
\end{aligned}$$

which is equivalent to

$$\operatorname{ber}^{2r}(T) \leq \frac{1}{4\varsigma\gamma}\left\||T|^{2r} + |T^*|^{2r}\right\|_{\text{ber}} + \frac{1}{2\varsigma\gamma}\operatorname{ber}\left(|T|^r\,|T^*|^r\right),$$

as required. This completes the proof. □

3.2. **Operator convex function in Berezin radius inequalities.** The main idea of this section is to present new interpolation inequalities of same known for the Berezin radius by using the properties operator convex functions.

Theorem 3.13. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a FHS, $T \in \mathbb{B}(\mathcal{H})$ be hyponormal and f be nonnegative increasing operator convex function on $[0, \infty)$. Then for $\zeta \geq 1$ we have*

$$f(\operatorname{ber}(T)) \leq \frac{1}{2\zeta}\left\|f(|T|) + f(|T^*|)\right\|_{\text{ber}}. \tag{3.11}$$

Proof. Let $R \in \mathbb{B}(\mathcal{H})$ and R be a self-adjoint operator, f be non-negative increasing operator convex function on $[0,\infty)$. Then

$$\begin{aligned} f\left(\|R\|_{\text{ber}}\right) &= f\left(\sup_{\tau\in\Omega}\left\langle R\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right) = \sup_{\tau\in\Omega} f\left(\left\langle R\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right) \\ &\leq \sup_{\tau\in\Omega}\left\langle f(R)\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle \text{ (by (2.4))} \\ &= \|f(R)\|_{\text{ber}}\,. \end{aligned}$$

It follows from Lemma 2.10 that

$$\operatorname{ber}(T) \leq \left\|\frac{1}{2\zeta}\left(|T| + |T^*|\right)\right\|_{\text{ber}}, \tag{3.12}$$

and, so

$$f(\operatorname{ber}(T)) \leq f\left(\left\|\frac{1}{2\zeta}\left(|T| + |T^*|\right)\right\|_{\text{ber}}\right).$$

Since $\frac{1}{2\zeta}\left(|T| + |T^*|\right)$ is a self-adjoint operator, then we have

$$\begin{aligned} & f\left(\left\|\frac{1}{2\zeta}\left(|T| + |T^*|\right)\right\|_{\text{ber}}\right) \\ &\leq \left\|f\left(\frac{1}{2\zeta}\left(|T| + |T^*|\right)\right)\right\|_{\text{ber}} \\ &\leq \left\|\frac{1}{\zeta}f\left(\frac{1}{2}\left(|T| + |T^*|\right)\right)\right\|_{\text{ber}} \text{ (by (2.7))} \\ &\leq \frac{1}{\zeta}\left\|\frac{1}{2}\left(f(|T|) + f(|T^*|)\right)\right\|_{\text{ber}} \text{ (by (2.8))} \\ &\leq \frac{1}{2\zeta}\left\|\left(f(|T|) + f(|T^*|)\right)\right\|_{\text{ber}}. \end{aligned}$$

This completes the proof. □

Corollary 3.14. *Let $T \in \mathbb{B}(\mathcal{H})$ be hyponormal. Then*

$$f(\operatorname{ber}(T)) \leq \left\|f\left(\frac{1}{2\zeta}\left(|T| + |T^*|\right)\right)\right\|_{\text{ber}} \leq \frac{1}{2\zeta}\left\|\left(f(|T|) + f(|T^*|)\right)\right\|_{\text{ber}}. \tag{3.13}$$

Taking $f(t) = t^r$ with $r \geq 1$ in inequality (3.13), we also get the following inequality.

Corollary 3.15. *Let $T \in \mathbb{B}(\mathcal{H})$ be hyponormal. Then*
(*i*) $\operatorname{ber}^r(T) \leq \frac{1}{2\zeta}\left\||T|^r + |T^*|^r\right\|_{\text{ber}}$.
(*ii*) $\operatorname{ber}^r(T) \leq \left\|\left(\frac{1}{2\zeta}\left(|T| + |T^*|\right)\right)^r\right\|_{\text{ber}}$.

Remark 3.16. If we pay attention to the case for $r=1$, we have

$$\operatorname{ber}(T) \leq \left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)\right\|_{\operatorname{ber}} \leq \frac{1}{2\zeta}\left\|\left(|T|+|T^*|\right)\right\|_{\operatorname{ber}}.$$

Theorem 3.17. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS, $T\in\mathbb{B}(\mathcal{H})$ be hyponormal and f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned} f(\operatorname{ber}(T)) &\leq \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2\zeta}\left(|T|+|T^*|\right)\right)+t\operatorname{ber}(T)\,I\right)dt\right\|_{\operatorname{ber}} \\ &\leq \frac{1}{2\zeta}\left\|\left(f(|T|)+f(|T^*|)\right)\right\|_{\operatorname{ber}}. \end{aligned}$$

Proof. Since

$$\left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)\operatorname{ber}(T)\,I\right\|_{\operatorname{ber}} = \left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)\right\|_{\operatorname{ber}}\left\|\operatorname{ber}(T)\,I\right\|_{\operatorname{ber}}.$$

From Lemma 2.1, we have

$$\left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)\operatorname{ber}(T)\,I\right\|_{\operatorname{ber}} = \left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)\right\|_{\operatorname{ber}} + \operatorname{ber}(T). \quad (3.14)$$

By using inequality (3.12) and equality (3.14) we can find

$$2\operatorname{ber}(T) \leq \left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)+\operatorname{ber}(T)\right\|_{\operatorname{ber}}.$$

Then we get

$$\begin{aligned} f(\operatorname{ber}(T)) &\leq f\left(\frac{1}{2}\left\| \frac{1}{2\zeta}\left(|T|+|T^*|\right)+\operatorname{ber}(T)\right\|_{\operatorname{ber}}\right) \\ &\leq \left\| f\left(\frac{1}{2}\left(\frac{1}{2\zeta}\left(|T|+|T^*|\right)+\operatorname{ber}(T)\right)\right)\right\|_{\operatorname{ber}} \quad \text{(by (2.4))} \\ &\leq \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2\zeta}\left(|T|+|T^*|\right)\right)+t\operatorname{ber}(T)\,I\right)dt\right\|_{\operatorname{ber}} \quad \text{(by (2.6))} \\ &\leq \frac{1}{2}\left\| f\left(\frac{1}{2\zeta}\left(|T|+|T^*|\right)\right)\right\|_{\operatorname{ber}} + \frac{1}{2}f(\operatorname{ber}(T)) \quad \text{(by (2.6))} \\ &\leq \left\| f\left(\frac{1}{2\zeta}\left(|T|+|T^*|\right)\right)\right\|_{\operatorname{ber}} \quad \text{(by (2.10))} \\ &\leq \frac{1}{2\zeta}\left\|\left(f(|T|)+f(|T^*|)\right)\right\|_{\operatorname{ber}} \quad \text{(by (3.13))}. \end{aligned}$$

This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.17.

Corollary 3.18. *Let $T \in \mathbb{B}(\mathcal{H})$ be hyponormal. Then*

$$f(\operatorname{ber}(T)) \leq \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)+t\operatorname{ber}(T)I\right)dt \right\|_{\operatorname{ber}} \leq \left\| f\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)\right\|_{\operatorname{ber}}.$$

Corollary 3.19. *Let $T \in \mathbb{B}(\mathcal{H})$ be hyponormal. Then*

(*i*) $\operatorname{ber}(T) \leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)+t\operatorname{ber}(T)I\right)^2 dt \right\|_{\operatorname{ber}}^{1/2} \leq \frac{1}{2\zeta}\|(|T|+|T^*|)\|_{\operatorname{ber}}$.

(*ii*) $\operatorname{ber}(T) \leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)+t\operatorname{ber}(T)I\right)^2 dt \right\|_{\operatorname{ber}}^{1/2} \leq \left\|\frac{1}{2\zeta}(|T|+|T^*|)\right\|_{\operatorname{ber}}$.

Corollary 3.20. *Let $T \in \mathbb{B}(\mathcal{H})$ be hyponormal. Then*

(*i*) $\operatorname{ber}^r(T) \leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)+t\operatorname{ber}(T)I\right)^r dt \right\|_{\operatorname{ber}} \leq \frac{1}{2\zeta}\|(|T|^r+|T^*|^r)\|_{\operatorname{ber}}$.

(*ii*) $\operatorname{ber}^r(T) \leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)+t\operatorname{ber}(T)I\right)^r dt \right\|_{\operatorname{ber}} \leq \left\|\left(\frac{1}{2\zeta}(|T|+|T^*|)\right)^r\right\|_{\operatorname{ber}}$.

Theorem 3.21. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS, $T \in \mathbb{B}(\mathcal{H})$ and f be nonnegative increasing operator convex function on $[0,\infty)$. Then*

$$f(\operatorname{ber}^r(T)) \leq \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right)+t\operatorname{ber}^r(T)I\right)dt \right\|_{\operatorname{ber}} \leq \left\| f\left(\frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right)\right\|_{\operatorname{ber}} \tag{3.15}$$

for all $\alpha \in (0,1)$, $r \geq 1$.

Proof. Indeed, by the proof of [23, Teorem 3.2] we get

$$\operatorname{ber}^r(T) \leq \frac{1}{2}\left\| |T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right\|_{\operatorname{ber}}, \; \alpha \in (0,1), r \geq 1.$$

Therefore, by using the some arguments in Theorem 3.17, (3.15) follows. The theorem is proved. □

Next inequality follows from Lemma 2.7 and the inequality (3.15).

Corollary 3.22. *Let $T \in \mathbb{B}(\mathcal{H})$ be hyponormal. Then*

$$\begin{aligned} f\left(\operatorname{ber}^r(T)\right) &\leq \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right)+t\operatorname{ber}^r(T)\, I\right) dt \right\|_{\operatorname{ber}} \\ &\leq \frac{1}{2}\left\| f\left(|T|^{2\alpha r}\right)+f\left(|T^*|^{2(1-\alpha)r}\right)\right\|_{\operatorname{ber}} \end{aligned}$$

for all $\alpha \in (0,1)$, $r \geq 1$.

Considering $f(t)=t^2$ in Theorem 3.21 and Corollary 3.22, we have the following corollaries.

Corollary 3.23. *If $T \in \mathbb{B}(\mathcal{H})$, then we have*

$$\begin{aligned} \operatorname{ber}^r(T) &\leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right)+t\operatorname{ber}^r(T)\, I\right)^2 dt \right\|_{\operatorname{ber}}^{1/2} \\ &\leq \left\| \frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right\|_{\operatorname{ber}} \end{aligned}$$

and

$$\begin{aligned} \operatorname{ber}^r(T) &\leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right)+t\operatorname{ber}^r(T)\, I\right)^2 dt \right\|_{\operatorname{ber}}^{1/2} \\ &\leq \left(\frac{1}{2}\left\|\left(|T|^{4\alpha r}+|T^*|^{4(1-\alpha)r}\right)\right\|_{\operatorname{ber}}\right)^{1/2} \end{aligned}$$

for all $\alpha \in (0,1)$, $r \geq 1$.

Considering $f(t)=t$ in Theorem 3.21, then we get the following inequality.

Corollary 3.24. *Let $T \in \mathbb{B}(\mathcal{H})$. Then we have*

$$\begin{aligned} \operatorname{ber}^r(T) &\leq \left\| \int_0^1 \left((1-t)\left(\frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right)+t\operatorname{ber}^r(T)\, I\right) dt \right\|_{\operatorname{ber}} \\ &\leq \left\| \frac{1}{2}\left(|T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right)\right\|_{\operatorname{ber}} \end{aligned}$$

for all $\alpha \in (0,1)$, $r \geq 1$.

Theorem 3.25. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS. For any $T, S \in \mathbb{B}(\mathcal{H})$, $\alpha \in (0,1)$ and $r \geq 1$, we have the inequality*

$$\operatorname{ber}^{2r}(S^*T) \leq \left\| \alpha\, |T|^{\frac{2r}{\alpha}}+(1-\alpha)\, |S|^{\frac{2r}{(1-\alpha)}} \right\|_{\operatorname{ber}}. \qquad (3.16)$$

Proof. Let $\widehat{k}_\tau$ be a normalized reproducing kernel. From Cauchy-Schwarz inequality, Hölder-McCarthy inequalities,Weighted AM-GM inequality and convexity of $f(t) = t^r$, we have

$$\begin{aligned}
\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right| &\le \left\langle T^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle S^*S\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&= \left\langle \left[(T^*T)^{\frac{1}{\alpha}}\right]^\alpha \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle \left[(S^*S)^{\frac{1}{(1-\alpha)}}\right]^{(1-\alpha)} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\le \left\langle (T^*T)^{\frac{1}{\alpha}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^\alpha \left\langle (S^*S)^{\frac{1}{(1-\alpha)}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\le \alpha \left\langle (T^*T)^{\frac{1}{\alpha}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + (1-\alpha) \left\langle (S^*S)^{\frac{1}{(1-\alpha)}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\le \left[\alpha \left\langle (T^*T)^{\frac{1}{\alpha}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{2r} + (1-\alpha) \left\langle (S^*S)^{\frac{1}{(1-\alpha)}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{2r}\right]^{\frac{1}{2r}} \\
&\le \left[\alpha \left\langle (T^*T)^{\frac{2r}{\alpha}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle + (1-\alpha) \left\langle (S^*S)^{\frac{2r}{(1-\alpha)}} \widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right]^{\frac{1}{2r}}.
\end{aligned}$$

Hence

$$\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2r} \le \left\langle \left(\alpha (T^*T)^{\frac{2r}{\alpha}} + (1-\alpha) (S^*S)^{\frac{2r}{(1-\alpha)}}\right) \widehat{k}_\tau, \widehat{k}_\tau\right\rangle.$$

Taking the supremum over $\tau \in \Omega$ in the above inequality, we have

$$\operatorname{ber}^{2r}(S^*T) \le \left\|\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha) |S|^{\frac{2r}{(1-\alpha)}}\right\|_{\text{ber}}.$$

This completes the proof. □

Corollary 3.26. *For $\alpha = \frac{1}{2}$, we obtain the following inequality:*

$$\operatorname{ber}^{2r}(S^*T) \le \frac{1}{2} \left\||T|^{4r} + |S|^{4r}\right\|_{\text{ber}}$$

for all $r \ge 1$.

If we take $S = I$ and $S = T^*$, respectively, we have the following corollary.

Corollary 3.27. *For any $T \in \mathbb{B}(\mathcal{H})$ and any $\alpha \in (0,1)$, $r \ge 1$, we get the inequalities*

$$\operatorname{ber}^{2r}(T) \le \left\|\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha) I\right\|_{\text{ber}}$$

and

$$\operatorname{ber}^{2r}\left(T^2\right) \le \left\|\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha) |T^*|^{\frac{2r}{(1-\alpha)}}\right\|_{\text{ber}},$$

respectively. Moreover, we get

$$\|T\|_{\text{ber}}^{4r} \le \left\|\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha) |T^*|^{\frac{2r}{(1-\alpha)}}\right\|_{\text{ber}}.$$

Theorem 3.28. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS. Let $T,S\in\mathbb{B}(\mathcal{H})$ and f be nonnegative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned} f\left(\mathrm{ber}^{2r}(S^*T)\right) &\leq \left\|\int_0^1 f\left((1-t)\left(\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|S|^{\frac{2r}{(1-\alpha)}}\right)+t\,\mathrm{ber}^{2r}(T)I\right)dt\right\|_{\mathrm{ber}} \\ &\leq \left\|f\left(\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|T^*|^{\frac{2r}{(1-\alpha)}}\right)\right\|_{\mathrm{ber}} \end{aligned}$$

for all $\alpha\in(0,1)$, $r\geq 1$.

Proof. Using the inequality (3.16) and proceeding similarly as Theorem 3.17 we can reach the required inequalities. □

In particular, if $f(t)=t^2$ and $f(t)=t$, respectively, then we obtain the following interpolation inequalities of (3.16).

Corollary 3.29. *Let $T,S\in\mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned} \mathrm{ber}^{4r}(S^*T) &\leq \left\|\int_0^1\left((1-t)\left(\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|S|^{\frac{2r}{(1-\alpha)}}\right)+t\,\mathrm{ber}^{2r}(T)I\right)^2dt\right\|_{\mathrm{ber}}^{\frac{1}{2}} \\ &\leq \left\|\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|S|^{\frac{2r}{(1-\alpha)}}\right\|_{\mathrm{ber}} \end{aligned}$$

and

$$\begin{aligned} \mathrm{ber}^{2r}(S^*T) &\leq \left\|\int_0^1\left((1-t)\left(\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|S|^{\frac{2r}{(1-\alpha)}}\right)+t\,\mathrm{ber}^{2r}(T)I\right)dt\right\|_{\mathrm{ber}} \\ &\leq \left\|\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|S|^{\frac{2r}{(1-\alpha)}}\right\|_{\mathrm{ber}} \end{aligned}$$

for all $\alpha\in(0,1)$, $r\geq 1$.

Theorem 3.30. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a FHS. Let $T,S\in\mathbb{B}(\mathcal{H})$ and f be nonnegative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned} f\left(\mathrm{ber}^{2r}(S^*T)\right) &\leq \left\|\int_0^1 f\left((1-t)\left(\alpha|T|^{\frac{2r}{\alpha}}+(1-\alpha)|S|^{\frac{2r}{(1-\alpha)}}\right)+t\,\mathrm{ber}^{2r}(T)I\right)dt\right\|_{\mathrm{ber}} \\ &\leq \frac{1}{2}\left\|\alpha f\left(|T|^{\frac{2r}{\alpha}}\right)+(1-\alpha)f\left(|S|^{\frac{2r}{(1-\alpha)}}\right)\right\|_{\mathrm{ber}}. \end{aligned}$$

for all $\alpha\in(0,1)$, $r\geq 1$.

Proof. If we apply Lemma 2.7 to Theorem 3.30, then we get required inequalities. □

Corollary 3.31. *If we take $\alpha=\frac{1}{2}$ and $f(t)=t$, then we have*

$$\begin{aligned} \mathrm{ber}^{2r}(S^*T) &\leq \left\|\int_0^1(1-t)\left(\frac{1}{2}\left(|T|^{4r}+|S|^{4r}\right)\right)+t\,\mathrm{ber}^{2r}(T)I\,dt\right\|_{\mathrm{ber}} \\ &\leq \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|_{\mathrm{ber}}. \end{aligned}$$

Corollary 3.32. *For any $T \in \mathbb{B}(\mathcal{H})$ and $\alpha \in (0,1)$, $r \geq 1$, we get the following inequalities:*

$$\begin{aligned}\operatorname{ber}^{4r}(S^*T) &\leq \left\| \int_0^1 \left((1-t)\left(\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha)|S|^{\frac{2r}{(1-\alpha)}} \right) + t\operatorname{ber}^{2r}(T) I \right)^2 dt \right\|_{\operatorname{ber}} \\ &\leq \frac{1}{2} \left\| \alpha |T|^{\frac{4r}{\alpha}} + (1-\alpha)|S|^{\frac{4r}{(1-\alpha)}} \right\|_{\operatorname{ber}},\end{aligned}$$

$$\begin{aligned}\operatorname{ber}^{2r}(T) &\leq \left\| \int_0^1 \left((1-t)\left(\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha) I \right) + t\operatorname{ber}^{2r}(T) I \right)^2 dt \right\|_{\operatorname{ber}} \\ &\leq \frac{1}{2} \left\| \alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha) I \right\|_{\operatorname{ber}}\end{aligned}$$

and

$$\begin{aligned}\operatorname{ber}^{2r}(T^2) &\leq \left\| \int_0^1 \left((1-t)\left(\alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha)|T^*|^{\frac{2r}{(1-\alpha)}} \right) + t\operatorname{ber}^{2r}(T) I \right) dt \right\|_{\operatorname{ber}} \\ &\leq \left\| \alpha |T|^{\frac{2r}{\alpha}} + (1-\alpha)|T^*|^{\frac{2r}{(1-\alpha)}} \right\|_{\operatorname{ber}}.\end{aligned}$$

References

[1] J. Aujla and F. Silva, Weak majorization inequalities and convex functions, *Linear Algebra Appl.* **369***(2003), 217-233.*

[2] F. A. Berezin, Covariant and contravariant symbols for operators, *Math. USSR-Izvestiya* **6***(1972), 1117-1151.*

[3] H. Başaran, M. Gürdal, and A. N. Güncan, Some operator inequalities associated with Kantorovich and Hölder-McCarthy inequalities and their applications, *Turkish J. Math.* **43***(2019), 523-532.*

[4] H. Başaran and M. Gürdal, Berezin number inequalities via inequality, *Honam Math. J.* **43***(2021), 523-537.*

[5] H. Başaran and V. Gürdal, Berezin radius and Cauchy-Schwarz inequality, *Montes Taurus J. Pure Appl. Math.* **5***(2023), 16-22.*

[6] H. Başaran, M. B. Huban, and M. Gürdal, Inequalities related to Berezin norm and Berezin number of operators, *Bull. Math. Anal. App.* **14***(2022), 1-11.*

[7] P. Bhunia, S. S. Dragomir, M. S. Moslehian, and K. Paul, Lectures on numerical radius inequalities, Infosys Science Foundation Series, Infosys Science Foundation Series in Mathematical Sciences, Springer Cham., (2022).

[8] R. Bhatia and F. Kittaneh, Notes on aritmetic-geometric mean inequalities, *Linear Algebra Appl.* **308***(2000), 203-211.*

[9] P. Bhunia and K. Paul, Refinements of norms and numerical radius, *Rocky Mountain J. Math.* **51***(2021), 1953-1965.*

[10] P. Bhunia, K. Paul, and A. Sen, Inequalities involving Berezin norm and Berezin number, *Complex Anal. Oper. Theory* **17***(2023). https://doi.org/10.1007/s11785-022-01305-9*

[11] M. L. Buzano, Generalizzatione della disuguaglianza di Cauchy-Schwarz, *Rend. Semin. Mat. Univ. Politech. Torino* **31***(1974), 405-409.*

[12] I. Chalendar, E. Fricain, M. Gürdal, and M. Karaev, Compactness and Berezin symbols, *Acta Sci. Math. (Szeged)* **78***(2012), 315-329.*

[13] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions, *Appl. Math. Comput.* **218***(2011), 766-772.*

[14] T. Furuta, H. Mićić, J. Pečarić, and Y. Seo, Mond-Peμcarić, Method in Operator Inequalities, Zagreb, Element, 2005.

[15] M. T. Garayev and M. W. Alomari, Inequalities for the Berezin number of operators and related questions, *Complex Anal. Oper. Theory* **15***(2021), 1-30.*

[16] M. Gürdal and H. Başaran, A-Berezin number of operators, *Proc. Inst. Math. Mech.* **48***(2022), 77-87.*

[17] M. Gürdal and H. Başaran, Advanced refinements of Berezin number inequalities, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. in press, 2023.*

[18] M. Gürdal and N. Duman, Berezin number inequalities involving superquadratic functions, *Casp. J. Math. Sci.* **10***(2021), 18-27.*

[19] M. T. Garayev, M. Gürdal and A. Okudan, Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators, *Math. Inequal. Appl.* **19***(2016), 883-891.*

[20] M. T. Garayev, M. Gürdal, and S. Saltan, Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems, *Positivity* **21***(2017), 1615-1623.*

[21] M. T. Garayev, H. Guedri, M. Gürdal, and G. M. Alsahli, On some problems for operators on the reproducing kernel Hilbert space, *Linear Multilinear Algebra* **69***(2021), 2059-2077.*

[22] M. B. Huban, H. Başaran, and M. Gürdal, New upper bounds related to the Berezin number inequalities, *J. Inequal. Spec. Funct.* **12***(2021), 1-12.*

[23] M. B. Huban, H. Başaran, and M. Gürdal, Some new inequalities via Berezin numbers, *Turk. J. Math. Comput. Sci.* **14***(2022), 129-137.*

[24] M. B. Huban, H. Başaran, and M. Gürdal, Berezin number inequalities via convex functions, *Filomat* **36***(2022), 2333-2344.*

[25] M. T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, *J. Funct. Anal.* **238***(2006), 181-192.*

[26] M. T. Karaev, Reproducing kernels and Berezin symbols techniques in various questions of operator theory, *Complex Anal. Oper. Theory* **7***(2013), 983-1018.*

[27] F. Kittaneh, Notes on some inequalities for Hilbert Space operators, *Publ. Res. Inst. Math. Sci.* **24***(1988), 283-293.*

[28] F. Kittaneh, Norm inequalities for sums of positive operators, *J. Operator Theory* **48***(2002), 95–103.*

[29] H. Lu, Interpolation inequalities of numerical radius, *J. Math. Ineq.* **16***(2022), 1633-1644.*

[30] R. Tapdigoglu, New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space, *Oper. Matrices* **15***(2021), 1031–1043.*

[31] U. Yamancı and M. Gürdal, On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space, *New York J. Math.* **23** *1531-1537 (2017).*

[32] U. Yamancı, M. Gürdal, and M. T. Garayev, Berezin number inequality for convex function in reproducing kernel Hilbert space, *Filomat* **31** *5711-5717 (2017).*

[33] U. Yamancı and İ. M. Karlı, Further refinements of the Berezin number inequalities on operators, *Linear Multilinear Algebra* **70***(2022), 5237-5246.*

[34] U. Yamancı, R. Tunç, and M. Gürdal, Berezin numbers, Grüss type inequalities and their applications, *Bull. Malays. Math. Sci. Soc.* **43***(2020), 2287-2296.*