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On rough $\mathcal{I}$-statistical ϕ-convergence

Ömer Kişi [1] and Mehmet Gürdal [2]
[1] Department of Mathematics, Bartin University, 74100, Bartın, Turkey
[2] Department of Mathematics, Suleyman Demirel University, 32260, Isparta, Turkey

Abstract. In this paper, we introduce the concept of rough $\mathcal{I}$-statistical ϕ-convergence of real numbers as a generalization of rough statistical convergence as well as $\mathcal{I}$-statistical ϕ-convergence. We study some of its fundamental properties. We obtain some results for rough $\mathcal{I}$-statistical ϕ-convergence by introducing the rough $\mathcal{I}$-statistical-ϕ limit set. So our main objective is to find out the different behaviour of the new convergence concept based on rough $\mathcal{I}$-statistical-ϕ limit set.

Keywords: Statistical convergence, rough convergence, ϕ-convergence, statistically ϕ-convergence.

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1. Introduction

The notion of statistical convergence was first presented by Fast [16]. The main idea behind statistical convergence was the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is denoted and defined by

$$d(A) = \lim_{n} \frac{1}{n} \left| \{k \in A : k \leq n\} \right|,$$

where the vertical bars indicate the cardinality of the enclosed set. Clearly, $d(\mathbb{N} \setminus A) + d(A) = 1$ and $A \subseteq B$ implies $d(A) \leq d(B)$. It

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is obvious that when A is a finite set then $d(A) = 0$. A real-valued sequence $w = (w_k)$ is said to be statistically convergent to the number w_0 if for each $\varepsilon > 0$,

$$d(\{k \in \mathbb{N} : |w_k - w_0| \geq \varepsilon\}) = 0.$$

Following that, researchers such as Braha et al. [2], Gürdal [13], Mohiuddine et al. [17, 22, 23], Mursaleen and Başar [24], Nuray and Ruckle [26], Temizsu and Et [37], along with numerous others, delved into a more detailed examination of statistical convergence from the perspective of sequence spaces (refer, for instance, to [4, 5, 6, 7, 25, 35]).

In 2001, the notion of $\mathcal{I}$-convergence was developed by Kostyrko et. al. [20] mainly as a common generalization of usual and statistical convergence. A family of subsets $\mathcal{I} \subset 2^Y$ is known as an ideal on a non-empty set Y, provided that for each $R, S \in \mathcal{I}$ implies $R \cup S \in \mathcal{I}$ and for each $R \in \mathcal{I}$ with $S \subset R$ means $S \in \mathcal{I}$. An ideal $\mathcal{I}$ is named non-trivial when $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$. Additionally, a non-trivial ideal $\mathcal{I} \subset 2^Y$ is called an admissible ideal in Y if $\mathcal{I} \supset \{\{u\} : u \in Y\}$. When $\mathcal{I}$ is an ideal in a set Y, then the class $\mathcal{F}(\mathcal{I}) = \{Y - K : K \in \mathcal{I}\}$ is called known as the filter associated with the ideal $\mathcal{I}$. When we take $Y = \mathbb{N}$, a real-valued sequence $w = (w_k)$ is said to be $\mathcal{I}$-convergent to w_0, provided that for each $\eta > 0$,

$$\{k \in \mathbb{N} :| w_k - w_0 |\geq \varepsilon\} \in \mathcal{I} \text{ (or } \{k \in \mathbb{N} :| w_k - w_0 |< \varepsilon\} \in \mathcal{F}(\mathcal{I})).$$

It is obvious that for $\mathcal{I} = \mathcal{I}_f = \{A \subset \mathbb{N} : |A| < \infty\}$, the above definition reduces to the definition of usual convergence and for $\mathcal{I} = \mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, the above definition turns to the definition of statistical convergence. In a way, $\mathcal{I}$-convergence serves as a comprehensive framework for studying the characteristics of various convergence types within a unified context. For further details on $\mathcal{I}$-convergence, additional information can be found in [14, 15, 19, 33, 34].

So in some sense, $\mathcal{I}$-convergence provides a general framework to investigate the properties of several types of convergence in a single frame. For more information on $\mathcal{I}$-convergence, one may refer to [14, 15, 19, 33, 34].

Savaş and Das [31] introduced the innovative concept of $\mathcal{I}$-statistical convergence, merging statistical convergence with $\mathcal{I}$-convergence. They investigated its fundamental properties and derived significant implications. Following their work, additional research on this subject has been undertaken by scholars including Debnath and Choudhury [8], Debnath and Rakshit [9], and others.

Exploring a different path, the notion of rough convergence made its debut through independent introductions by Burgin [3] and H. X. Phu [28]. While their concepts bore significant resemblance, Burgin delved into the fuzzy setting, whereas Phu formulated it specifically for finite-dimensional normed spaces.

Let r be a non-negative real number. A sequence $w = (w_k)$ in a normed linear space $(Y, \|\cdot\|)$ is said to be rough convergent to $u_0 \in Y$ with roughness degree r, if for each $\varepsilon > 0$, there is an $N = (N_\varepsilon)$ so that for all $k \geq N$, $\|w_k - u_0\| < r + \varepsilon$. Symbolically, it is represented as $w_k \overset{r-\|.\|}{\longrightarrow} w_0$.

Phu [28] initially illustrated that the set $LIM^r w$ exhibits bounded, closed, and convex characteristics, extensively delving into the fundamental aspects of this captivating notion. It is worth mentioning that the idea of rough convergence has practical applications in numerical analysis. In a follow-up study [29], Phu further expanded the investigation of rough convergence into the realm of infinite-dimensional normed spaces.

The combination of rough convergence and statistical convergence led to the development of rough statistical convergence by Aytar in 2008 [1]. This concept was further independently extended to rough ideal convergence by Dündar and Çakan [10], as well as Pal et al. [27]. Additionally, rough $\mathcal{I}$-statistical convergence was investigated by Malik et al. [21] and Savaş et. al [32]. For an in-depth exploration in this area, one can refer to [11, 12], which provides an extensive collection of references.

An Orlicz function [30] is a function $\phi : \mathbb{R} \to \mathbb{R}$ such that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\phi(x) = 0 \iff x = 0 \text{ and } \phi(x) \to \infty \text{ as } x \to \infty,$$

where $\mathbb{R}$, $\mathbb{R}^+$, and ϕ stands for the set of all real numbers, set of all positive real numbers, and Orlicz function respectively.

Rao and Ren [30] emphasize the significant roles and applications of Orlicz functions in various domains, including economics and stochastic problems.

An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the Δ_2 condition, if there exists a $M > 0$ such that $\phi(2x) \leq M\phi(x)$, for every $x \in \mathbb{R}^+$.

Example 1.1. (i) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = |x|$ is an Orlicz function.
(ii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^3$ is not an Orlicz function.
(iii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2$ is an Orlicz function satisfying the Δ_2 condition.
(iv) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the Δ_2 condition.

In 2019, Khusnussaadah and Supama [18] introduced the concept of ϕ-convergence, and later, Supama extended it to statistical ϕ-convergence by employing the Orlicz function ϕ. A more recent generalization to $\mathcal{I}$-statistically ϕ-convergence was presented by Debnath and Choudhury [8], who established several novel results in this context.

In this paper, we introduce the concept of rough $\mathcal{I}$-statistically ϕ-convergence on $\mathbb{R}$, incorporating rough statistical convergence and ϕ-convergence. This serves as a broader framework encompassing rough $\mathcal{I}$-statistical convergence and $\mathcal{I}$-statistical ϕ-convergence. We initially define the rough $\mathcal{I}$-statistical ϕ-limit set and subsequently establish its convexity under the condition that ϕ is convex. Additionally, we present various findings related to the rough $\mathcal{I}$-statistical ϕ-limit set, distinguishing it from the previously explored rough $\mathcal{I}$-statistical limit set by Savaş et al. [32].

2. Definitions and Preliminaries

We recall the concepts of rough convergence, rough statistical convergence, $\phi-$convergence are as follows:

Definition 2.1. [28] Let $w = (w_k)$ be a sequence of real numbers and r be a non-negative real number. A sequence (w_k) is said to be rough convergent to $w_0 \in \mathbb{R}$, denoted by $w_k \xrightarrow{r} w_0$, if

$$\forall \varepsilon > 0,\ \exists k_\varepsilon \in \mathbb{N} : k \geq k_\varepsilon \Longrightarrow |w_k - w_0| < r + \varepsilon$$

Definition 2.2. [1] Let $w = (w_k)$ be a sequence of real numbers and r be a non-negative real number. A sequence (w_k) is said to be rough statistically convergent to $x_* \in \mathbb{R}$, denoted by $w_k \xrightarrow{r-st} w_0$ if

$$\forall \varepsilon > 0,\ \delta\left(\{k \in \mathbb{N} : |w_k - w_0| \geq r + \varepsilon\}\right) = 0$$

Definition 2.3. [1] Let $w = (w_k)$ be a sequence of real numbers and r be a non-negative real number. Then the set

$$st - LIM^r w = \left\{w_0 \in \mathbb{R} : w_k \xrightarrow{r} w_0\right\}$$

is known as the rough statistical limit set of $w = (w_k)$.

Definition 2.4. [18] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $w = (w_k)$ is said to be ϕ-convergent to w_0 if $\lim_k \phi(w_k - w_0) = 0$. In this case, w_0 is called the ϕ-limit of (w_k) and denoted by $\phi - \lim w = w_0$. If a sequence (w_k) is ϕ-convergent to w_0, we denote it by $w_k \xrightarrow{\phi} w_0$.

Definition 2.5. [36] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $w = (w_k)$ is said to be statistically ϕ-convergent to $w_0 \in \mathbb{R}$ denoted by $w_k \xrightarrow{st-\phi} w_0$, if

$$\forall \varepsilon > 0,\ d\left(\{k \in \mathbb{N} : \phi(w_k - w_0) \geq \varepsilon\}\right) = 0.$$

Definition 2.6. [8] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $w = (w_k)$ is said to be $\mathcal{I}$-statistically ϕ-convergent to $w_0 \in \mathbb{R}$ for every ε, $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k - w_0\right) \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

w_0 is called $\mathcal{I}$-statistical ϕ-limit of the sequence (w_k) and we demonstrate, $\mathcal{I}_S$-$\phi \lim w_k = w_0$.

3. Main results

Definition 3.1. Assume $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $r \geq 0$ be a real number. A sequence $w = (w_k)$ is called to be rough $\mathcal{I}$-statistically ϕ-convergent to $w_0 \in \mathbb{R}$ if for every ε, $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k - w_0\right) \geq r + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

w_0 is called rough $\mathcal{I}$-statistical ϕ-limit of the sequence (w_k) and we demonstrate, $w_k \overset{r-\mathcal{I}_S-\phi}{\to} w_0$ or r-$\mathcal{I}_S$-$\phi \lim w_k = w_0$.

Condition 1: If we take $r = 0$, then we get $\mathcal{I}$-statistical ϕ-convergence.

Condition 2: If we take $\phi(w) = |w|$, we obtain rough $\mathcal{I}$-statistical convergence [32].

Condition 3: If we take $\phi(w) = |w|$ and $r = 0$ both, we get the $\mathcal{I}$-statistical convergence [31].

Therefore, our main interest is to deal with the case $r > 0$.

If $w_k \overset{r-\mathcal{I}_S-\phi}{\to} w_0$, w_0 is an r-$\mathcal{I}_S$-ϕ limit point of $w = (w_k)$, which is usually no more unique (for $r > 0$). So, we consider the rough $\mathcal{I}$-statistical ϕ-limit set of $w = (w_k)$ defined by

$$\mathcal{I} - st - \mathrm{LIM}^{r-\phi} w = \left\{ w_0 \in \mathbb{R} : w_k \overset{r-\mathcal{I}_S-\phi}{\to} w_0 \right\}.$$

A sequence $w = (w_k)$ is called to be rough $\mathcal{I}$-statistically ϕ-convergent if $\mathcal{I} - st - \mathrm{LIM}^{r-\phi} w \neq \emptyset$. In this case, r is called a rough $\mathcal{I}$-statistical-ϕ convergence degree of (w_k).

Let us illustrate by an example:

Example 3.2. Let $p \geq 1$ and $u \geq 2$ be any two given natural numbers. Suppose $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(w) = |w|^p$, be the Orlicz function. Consider the sequence $w = (w_k)$ defined by

$$w_k := \begin{cases} k, & \text{if } k = n^u \text{ for some } n \in \mathbb{N} \\ (-1)^k, & \text{if not.} \end{cases}$$

Then,

$$\mathcal{I}-st-\mathrm{LIM}^{r-\phi}w=\begin{cases}\left[1-\sqrt[p]{r},\sqrt[p]{r}-1\right], & \text{for } r\geq 1\\ \emptyset, & \text{for } r<1.\end{cases}$$

As a result, $w=(w_k)$ is rough $\mathcal{I}$-statistically ϕ-convergent for $r\geq 1$ but not rough $\mathcal{I}$-statistically ϕ-convergent for $r<1$. Also, the examined sequence is neither $\mathcal{I}$-statistically convergent nor $\mathcal{I}$-statistically ϕ-convergent.

Hence, we can observe that rough $\mathcal{I}$-statistical ϕ-convergence is a generalization of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-statistically ϕ-convergent both.

But, we state that rough $\mathcal{I}$-statistical ϕ-convergence and rough $\mathcal{I}$-statistical convergence are two independent notions. The following two examples will illustrate the fact.

Example 3.3. Let $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(w)=\sqrt{|w|}$, be an Orlicz function. Consider the sequence $w=(w_k)$ defined as

$$w_k=\begin{cases}0, & \text{if } k \text{ is odd}\\ 1, & \text{if } k \text{ is even.}\end{cases}$$

Then,

$$\mathcal{I}-st-\mathrm{LIM}^{r}w=\begin{cases}\left[1-r,r\right], & \text{for } r\geq \frac{1}{2}\\ \emptyset, & \text{for } r<\frac{1}{2}.\end{cases}$$

But

$$\mathcal{I}-st-\mathrm{LIM}^{r-\phi}w=\begin{cases}\left[1-r^2,r^2\right], & \text{for } r\geq \frac{1}{\sqrt{2}}\\ \emptyset, & \text{for } r<\frac{1}{\sqrt{2}}.\end{cases}$$

That is to say, for $\frac{1}{2}\leq r<\frac{1}{\sqrt{2}}$, the sequence is rough $\mathcal{I}$-statistically convergent but not rough $\mathcal{I}$-statistically ϕ-convergent and if $r\geq\frac{1}{\sqrt{2}}$, the sequence is convergent in both the sense.

Example 3.4. Let $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(w)=w^2$, be an Orlicz function. Consider the sequence $w=(w_k)$ defined in Example 3.3. Then, according to the definition,

$$\mathcal{I}-st-\mathrm{LIM}^{r-\phi}w=\begin{cases}\left[1-\sqrt{r},\sqrt{r}\right], & \text{for } r\geq \frac{1}{4}\\ \emptyset, & \text{for } r<\frac{1}{4}.\end{cases}$$

But,

$$\mathcal{I}-st-\mathrm{LIM}^{r}w=\begin{cases}\left[1-r,r\right], & \text{for } r\geq \frac{1}{2}\\ \emptyset, & \text{for } r<\frac{1}{2}.\end{cases}$$

So, for $\frac{1}{4}\leq r<\frac{1}{2}$, the sequence is rough $\mathcal{I}$-statistically ϕ-convergent but not rough $\mathcal{I}$-statistically convergent and if $r\geq\frac{1}{2}$, the sequence has both type of convergence.

Definition 3.5. A sequence $w = (w_k)$ is $\mathcal{I}$-statistically ϕ-bounded if there exists a real number $B > 0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k\right) \geq B \right\} \right| \geq \delta \right\} \in \mathcal{I} \tag{3.1}$$

for every $\delta > 0$.

Theorem 3.6. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function which satisfies Δ_2 condition. A sequence $w = (w_k)$ is $\mathcal{I}$-statistically ϕ-bounded if and only if $\exists r \in \mathbb{R}$ with $r \geq 0$ such that $\mathcal{I} - st{-}LIM^{r-\phi} w \neq \emptyset$.*

Proof. Let's assume that $w = (w_k)$ is $\mathcal{I}$-statistically ϕ-bounded. From the Definition 3.5, (3.1) is provided. By taking $A = \{k \leq n : \phi(w_k) \geq B\}$, we get

$$\mathcal{I}\text{-} \lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k\right) \geq B \right\} \right| = 0.$$

Let $r' = \sup\{\phi(w_k) : k \in A^c\}$. Then the set $\mathcal{I} - st{-}\mathrm{LIM}^{r'-\phi} w$ contains the origin. So, we obtain $\mathcal{I} - st{-}\mathrm{LIM}^{r-\phi} w \neq \emptyset$ for $r \neq r'$.

For the converse part, assume $\mathcal{I} - st{-}\mathrm{LIM}^{r-\phi} w \neq \emptyset$ for some $r \geq 0$. Let $w_0 \in \mathcal{I} - st{-}\mathrm{LIM}^{r-\phi} w$. Then we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k - w_0\right) \geq r + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I} \tag{3.2}$$

for all $\varepsilon, \delta > 0$. Since the function ϕ satisfies Δ_2 condition, so there exists $Q \in \mathbb{R}^+$ such that $\phi(2w) \leq Q\phi(w)$ for every $w \in \mathbb{R}$. Now we assert that

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \tfrac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k\right) \geq \tfrac{Q}{2}\left(r + \phi\left(w_0\right) + \varepsilon\right) \right\} \right| \geq \delta \right\} \\ &\quad \subseteq \left\{ n \in \mathbb{N} : \tfrac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k - w_0\right) \geq r + \varepsilon \right\} \right| \geq \delta \right\} \end{aligned} \tag{3.3}$$

is true when $\varepsilon, \delta > 0$. So, let's assume

$$s \in \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k\right) \geq \frac{Q}{2}\left(r + \phi\left(w_0\right) + \varepsilon\right) \right\} \right| \geq \delta \right\}.$$

Then, we obtain

$$\begin{aligned} \frac{Q}{2}\left(r + \phi\left(w_0\right) + \varepsilon\right) &\leq \phi\left(w_s\right) = \phi\left(w_s - w_0 + w_0\right) \\ &= \phi\left(\frac{1}{2}\left(2\left(w_s - w_0\right)\right) + \frac{1}{2}\left(2w_0\right) \right) \\ &\leq \frac{1}{2}\left(\phi\left(2\left(w_s - w_0\right)\right)\right) + \frac{1}{2}\left(\phi\left(2w_0\right)\right) \\ &\leq \frac{Q}{2}\left(\phi\left(w_s - w_0\right)\right) + \frac{Q}{2}\left(\phi\left(w_0\right)\right), \end{aligned}$$

where the third inequality follows from the convexity and the fourth inequality follows from the Δ_2 condition. So, we get

$$r + \phi\left(w_0\right) + \varepsilon \leq \phi\left(w_s - w_0\right) + \phi\left(w_0\right)$$

namely, $\phi(w_s - w_0) \geq r + \varepsilon$. To put it another way,

$$s \in \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi(w_k - w_0) \geq r + \varepsilon \right\} \right| \geq \delta \right\},$$

indicates that (3.3) is accurate. We may infer from (3.2) and (3.3) that for any $\varepsilon, \delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi(w_k) \geq \frac{Q}{2} \left(r + \phi(w_0) + \varepsilon \right) \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

$w = (w_k)$ is therefore rough $\mathcal{I}$-statistically ϕ-bounded. This completes the proof. □

Theorem 3.7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function. If $y_0 \in \mathcal{I} - st{-}LIM^{r_0 - \phi} w$ and $y_1 \in \mathcal{I} - st{-}LIM^{r_1 - \phi} w$, then we have*

$$y_\lambda = (1 - \lambda) y_0 + \lambda y_1 \in \mathcal{I} - st - LIM^{(1-\lambda) r_0 + \lambda r_1 - \phi} w$$

for $\lambda \in [0, 1]$.

Proof. Let $\varepsilon, \delta > 0$. Assume $y_0 \in \mathcal{I} - st{-}\mathrm{LIM}^{r_0 - \phi} w$ and $y_1 \in \mathcal{I} - st{-}\mathrm{LIM}^{r_1 - \phi} w$. So

$$K_1 = \left\{ n \in \mathbb{N} : \tfrac{1}{n} \left| \left\{ k \leq n : \phi(w_k - y_0) \geq r_0 + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I},$$
$$K_2 = \left\{ n \in \mathbb{N} : \tfrac{1}{n} \left| \left\{ k \leq n : \phi(w_k - y_1) \geq r_1 + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

$M = \mathbb{N} \setminus (K_1 \cup K_2) \in \mathcal{F}(\mathcal{I})$ and so M have to be a infinite set. Let $s \in M$ then $d(T_1) = 0$, where

$$T_1 = \{ k \leq s : \phi(w_k - y_0) \geq r_0 + \varepsilon \}$$

and $d(T_2) = 0$, where

$$T_2 = \{ k \leq s : \phi(w_k - y_1) \geq r_1 + \varepsilon \}.$$

Now, for every $k \in T_1^c \cap T_2^c$ and $\lambda \in [0, 1]$,

$$\begin{aligned} \phi(w_k - y_\lambda) &= \phi(w_k - (1 - \lambda) y_0 - \lambda y_1) \\ &= \phi((1 - \lambda) w_k + \lambda w_k - (1 - \lambda) y_0 - \lambda y_1) \\ &= \phi((1 - \lambda)(w_k - y_0) + \lambda (w_k - y_1)) \\ &\leq (1 - \lambda) \phi(w_k - y_0) + \lambda \phi(w_k - y_1) \\ &< (1 - \lambda)(r_0 + \varepsilon) + \lambda (r_1 + \varepsilon) \\ &= r + \varepsilon, \; y_\lambda = (1 - \lambda) r_0 + \lambda r_1. \end{aligned}$$

Since $d(T_1^c \cap T_2^c) = 1$, we obtain

$$\frac{1}{s} \left| \left\{ k \leq s : \phi(w_k - (1 - \lambda) y_0 - \lambda y_1) \geq r + \varepsilon \right\} \right| < \delta.$$

Hence

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \phi(w_k - (1 - \lambda) y_0 - \lambda y_1) \geq r + \varepsilon \right\} \right| < \delta \right\} \supseteq M \in \mathcal{F}(\mathcal{I}),$$

which gives that

$$y_{\lambda} = (1-\lambda)\, y_0 + \lambda y_1 \in \mathcal{I} - st - LIM^{(1-\lambda) r_0 + \lambda r_1 - \phi} w$$

for $\lambda \in [0,1]$. As a result, $\mathcal{I} - st-\mathrm{LIM}^{(1-\lambda) r_0 + \lambda r_1 - \phi} w$ is convex. Then the desired result has been obtained. □

Remark 3.8. If we take $r_0 = r_1 = r$, then we have

$$y_{\lambda} = (1-\lambda)\, y_0 + \lambda y_1 \in \mathcal{I} - st - \mathrm{LIM}^{r-\phi} w \text{ for } \lambda \in [0,1].$$

namely, the set $\mathcal{I} - st-\mathrm{LIM}^{r-\phi} x$ is convex.

Savaş et al. [32, Theorem 3.1] stated that $\operatorname{diam}(\mathcal{I} - st - LIM^{r} w) \leq 2r$ for the sequence $w = (w_k)$. However, this might not be the case in the case of rough $\mathcal{I}$-statistical ϕ-convergence.

Example 3.9. Let $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(w) = \sqrt{|w|}$, be an Orlicz function and $w = (w_k)$ be defined as

$$w_k = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even.} \end{cases}$$

Hence, we get

$$\begin{aligned} \operatorname{diam}\left(\mathcal{I} - st - LIM^{r-\phi} w\right) &= \sup\left\{|p-q| : p, q \in \mathcal{I} - st - LIM^{r-\phi} w\right\} \\ &= 2r^2 - 1 > 2r, \end{aligned}$$

for any $r > \frac{1+\sqrt{3}}{2}$.

We can thus conclude from the example above that the diameter of $\mathcal{I} - st - LIM^{r-\phi} w$ relies on the ϕ function that we have taken into account. The ϕ-image of the diameter of $\mathcal{I} - st - LIM^{r-\phi} w$ does not exceed Qr for some $Q > 0$, but, for a convex ϕ function that satisfies the Δ_2 requirement, the following theorem follows.

Theorem 3.10. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function which satisfies Δ_2 condition. Then there exists a real number $Q > 0$ such that $\phi\left(\operatorname{diam}\left(\mathcal{I} - st - LIM^{r-\phi} w\right)\right) \leq Qr$.*

Proof. Since ϕ is even and non decreasing on $\mathbb{R}^{+}$, we get

$$\begin{aligned} &\phi\left(\operatorname{diam}\left(\mathcal{I} - st - LIM^{r-\phi} w\right)\right) \\ &= \phi\left(\sup\left\{|x_* - y_*| : x_*, y_* \in \mathcal{I} - st - LIM^{r-\phi} w\right\}\right) \\ &= \sup\left\{\phi(x_* - y_*) : x_*, y_* \in \mathcal{I} - st - LIM^{r-\phi} w\right\}. \end{aligned} \tag{3.4}$$

From Δ_2 condition, there exists a $Q > 0$ in $\mathbb{R}$ such that $\forall w \in \mathbb{R}, \phi(2w) \leq Q\phi(w)$. Now, as $x_*, y_* \in \mathcal{I} - st-\mathrm{LIM}^{r-\phi} w$, so for each $\varepsilon, \delta > 0$, $K_1 \in \mathcal{I}$

and $K_2 \in \mathcal{I}$ where

$$K_1 = \left\{ n \in \mathbb{N} : \tfrac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k - x_*\right) \geq r + \tfrac{\varepsilon}{Q} \right\} \right| \geq \delta \right\},$$
$$K_2 = \left\{ n \in \mathbb{N} : \tfrac{1}{n} \left| \left\{ k \leq n : \phi\left(w_k - y_*\right) \geq r + \tfrac{\varepsilon}{Q} \right\} \right| \geq \delta \right\}.$$

Clearly, $M = \mathbb{N} \setminus \left(K_1 \cup K_2\right) \in \mathcal{F}\left(\mathcal{I}\right)$ and so M have to be a infinite set. Let $s \in M$ then $\phi\left(w_s - x_*\right) < r + \frac{\varepsilon}{Q}$ and $\phi\left(w_s - y_*\right) < r + \frac{\varepsilon}{Q}$. Hence

$$\begin{aligned} \phi\left(x_* - y_*\right) &= \phi\left(x_* - w_s + w_s - y_*\right) \\ &\leq \frac{Q}{2}\phi\left(w_s - x_*\right) + \frac{Q}{2}\phi\left(w_s - y_*\right) < Qr + \varepsilon. \end{aligned}$$

Since ε arbitrary, so we get $\phi\left(x_* - y_*\right) \leq Qr$ and clearly from equation (3.4), we obtain that $\phi\left(\operatorname{diam}\left(\mathcal{I} - st - LIM^{r-\phi}x\right)\right) \leq Qr$. This completes the proof. □

Malik et al. ([21, Theorem 3.8]) also proved that for any $\mathcal{I}$-statistically bounded sequence $w = (w_k)$, $\mathcal{I} - S\Gamma_w \subseteq \mathcal{I} - st{-}\mathrm{LIM}^{diam(\Gamma_w)}w$. However, this conclusion also appears to be false in the situation of rough $\mathcal{I}$-statistical ϕ-convergence. The example that follows supports our assertion. However, this conclusion also appears to be false in the situation of rough $\mathcal{I}$-statistical ϕ-convergence. The example that follows supports our assertion.

Example 3.11. Let $\phi : \mathbb{R} \to \mathbb{R}$, $\phi\left(w\right) = w^2$, be an Orlicz function. Consider the sequence $w = (w_k)$ defined as

$$w_k = \begin{cases} (-1)^k, & \text{if } k \text{ is not a perfect square} \\ k, & \text{if } k \text{ is a perfect square.} \end{cases}$$

Then, $\mathcal{I} - S\Gamma_w = \{-1, 1\}$. So, we get $\operatorname{diam}\left(\mathcal{I} - S\Gamma_w\right) = 2$. But $\mathcal{I} - st - LIM^{2-\phi}w = \left[1 - \sqrt{2}, -1 + \sqrt{2}\right] \not\supseteq \mathcal{I} - S\Gamma_w$.

Further, Savaş et al. ([32, Theorem 3.5]) demonstrated that for any arbitrary $c \in \mathcal{I} - S\Gamma_w$ of a sequence $w = (w_k)$, $|x_* - c| \leq r$ for all $x_* \in \mathcal{I} - st - LIM^r w$. For rough $\mathcal{I}$-statistical ϕ-convergence, however, it is not true.

Example 3.12. Consider Example 3.3. It is clear that $\mathcal{I} - S\Gamma_w = \{0, 1\}$ and $\mathcal{I} - st - LIM^{5-\phi}w = [-24, 25]$. If we take $c = 0$ and $x_* = 20$, then $|x_* - c| \leq 5$ does not holds.

Definition 3.13. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A real number γ is said to be $\mathcal{I}$-statistically ϕ-cluster point of a sequence $w = (w_k)$, if for every $\varepsilon > 0$,

$$d_{\mathcal{I}}\left(\left\{k \in \mathbb{N} : \phi\left(w_k - \gamma\right) < \varepsilon\right\}\right) \neq 0,$$

where

$$d_{\mathcal{I}}(A)=\mathcal{I}-\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:k\in A\}\right|,$$

if exists. The set of all $\mathcal{I}$-statistically ϕ-cluster points of a sequence $w=(w_k)$ is demonstrated by $\phi(\mathcal{I}-S\Gamma_w)$.

Theorem 3.14. *Let $\phi:\mathbb{R}\to\mathbb{R}$ be a convex Orlicz function which satisfies Δ_2 condition. Then, for any arbitrary $c\in\phi(\mathcal{I}-S\Gamma_w)$ and $x_*\in\mathcal{I}{-}st-LIM^{r-\phi}w$, there exists $Q>0$ in $\mathbb{R}$ such that $\phi(x_*-c)\leq Qr$.*

Proof. Since $c\in\phi(\mathcal{I}-S\Gamma_w)$, so for every $\varepsilon>0$,

$$d_{\mathcal{I}}(\{k\in\mathbb{N}:\phi(w_k-c)<\varepsilon\})\neq 0. \tag{3.5}$$

Now as ϕ supplies Δ_2 condition, so there exists $M\in\mathbb{R}^+$ such that $\phi(2w)\leq M\phi(w)$ for every $w\in\mathbb{R}$. We will prove that $Q=\frac{M}{2}$ is the positive real number that works here. If possible suppose

$$\phi(x_*-c)>\frac{Mr}{2} \tag{3.6}$$

Take $\varepsilon=\frac{2\left(\phi(x_*-c)-\frac{Mr}{2}\right)}{3M}$. We claim that the following inclusion holds

$$\{k\in\mathbb{N}:\phi(w_k-c)<\varepsilon\}\subseteq\{k\in\mathbb{N}:\phi(w_k-x_*)\geq r+\varepsilon\} \tag{3.7}$$

Suppose $s\in\{k\in\mathbb{N}:\phi(w_k-c)<\varepsilon\}$. Then, we have

$$\phi(w_s-c)<\frac{2\left(\phi(x_*-c)-\frac{Mr}{2}\right)}{3M} \tag{3.8}$$

Then, it can be proved by using the convexity, Δ_2 condition and the inequalities (3.8) that

$$2\phi(x_*-c)<\frac{3M}{2}\phi(w_s-x_*)-\frac{Mr}{2}.$$

Also, using the the inequalities (3.6), we get

$$\frac{Mr}{2}+\phi(x_*-c)<\frac{3M}{2}\phi(w_s-x_*)-\frac{Mr}{2},$$

$$\frac{3M}{2}\phi(w_s-x_*)>Mr+\frac{3M\varepsilon}{2}+\frac{Mr}{2}$$

and

$$\phi(w_s-x_*)>r+\varepsilon.$$

Thus, we have

$$s\in\{k\in\mathbb{N}:\phi(w_k-x_*)\geq r+\varepsilon\}$$

proving that the inclusion (3.7) is true. Combining (3.5) and (3.7) we conclude that

$$d_{\mathcal{I}}(\{k\in\mathbb{N}:\phi(w_k-x_*)\geq r+\varepsilon\})\neq 0,$$

which is a contradiction to the fact that $x_* \in st - LIM^{r-\phi}w$. This completes the proof. □

4. Conclusion

As crucial as identifying convergent sequences are those that do not meet the convergence requirement. Although not convergent, the presence of these form of sequences, which under certain circumstances exhibit traits like the notion of convergent sequence, has resulted in the creation of many types of convergence. One of them is the idea of rough convergence in finite dimensions normed spaces as described by Phu ([28]). This theory states that one may acquire a rough convergence of a sequence by increasing the range of convergence by an integer $r > 0$. Rough convergence has several fascinating applications in numerical analysis, it should be highlighted. Rough statistical convergence and rough ideal convergence are two topics that some mathematicians have recently started to investigate. The more broad idea in this theory, however, has not yet been investigated while taking into account the Orlicz functions ϕ and $\mathcal{I}$-statistical convergence concepts. This study makes three contributions to the field of rough theory and summability: (i) a kind of rough ideal statistically convergent via Orlicz function ϕ; (ii) the concept of $\mathcal{I}$-statistically ϕ-cluster point. These results can be utilized to study the convergence problems of double sequences having chaotic pattern in rough theory.

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