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On Ideal Completeness for Double Sequences on $\mathscr{L}-$ Fuzzy Normed Spaces

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Abstract. In this article, we have defined the ideal Cauchy and ideal bounded double sequences on $\mathscr{L}-$ fuzzy normed spaces, which are generalizations of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces, with the help of ideal convergence of double sequences, which can be accepted as a generalization of known convergence types, and examined their relationships.

Keywords: $L-$fuzzy normed space, ideal convergence, ideal Cauchy

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1. Introduction

Many valuable studies have been made and continue to be done on the concept of convergence of sequences[8, 9, 10, 13, 23, 27, 28, 29, 30]. Especially Savas has made very important contributions to the mathematics community on this subject[6, 17, 18, 19, 20, 21].

$\mathscr{L}-$ fuzzy normed spaces entered the literature as a generalization of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces [31, 1, 14, 16, 7, 12, 11, 3]based on some logical algebraic structures, which also enrich the notion of a $\mathscr{L}-$fuzzy metric space [4, 5].

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There is a vast literature of studies on this structure. In particular, some properties of a variant of the ideal convergence of sequences on $\mathscr{L}-$fuzzy normed spaces are given [3, 22]. However, generalizations of some well-known results are absent and in particular there is no literature on statistical boundedness conditions on sequences.

We come across double and triple sequences [2, 24, 26, 25], i.e. matrices, in many branches of science and engineering, and there are definitely situations where either the concept of ordinary convergence does not operate or the underlying space does not serve our intent.

In this study we give some results regarding ideal convergence for double sequences on $\mathscr{L}-$ fuzzy normed spaces and investigate the relationship between ideal convergent, ideal Cauchy[15] and ideal bounded double sequences, which will be newly introduced on $\mathscr{L}-$ fuzzy normed spaces.

2. Preliminaries

In this section we give some preliminaries on $L-$ normed spaces.

Definition 2.1. [22] Let $T : [0,1]\times[0,1] \to [0,1]$ be a function satisfying the conditions

(1) $T(x,y) = T(y,x)$
(2) $T(T(x,y),z) = T(x,T(y,z))$
(3) $T(x,1) = T(1,x) = x$
(4) $x \leq y, z \leq t$ then $T(x,z) \leq T(y,t)$

is called a triangular norm (or shortly $t-$norm).

Example 2.2. [22] The functions T_1, T_2 and T_3 given with,
$T_1(x,y) = \min\{x,y\}$,
$T_2(x,y) = xy$,
$T_3(x,y) = \max\{x+y-1, 0\}$
are some well-known examples of $t-$norms.

Definition 2.3. [22] Given a complete lattice $\mathscr{L} = (L, \preceq)$ and a set X which will be called the universe. A function

$$A : X \to L$$

is called an $L-$fuzzy set, or an $L-$set for short, on X. The family of all $L-$sets on a set X is denoted by L^X.

Intersection of two $L-$ sets on X is given by

$$(A \cap B)(x) = A(x) \wedge B(x)$$

for all $x \in X$. Similarly union of two $L-$sets and intersection and union of a family $\{A_i : i \in I\}$ of $L-$sets is given by

$$(A \cup B)(x) = A(x) \vee B(x)$$

$$\Big(\bigcap_{i\in I} A_i\Big)(x) = \bigwedge_{i\in I} A_i(x)$$

$$\Big(\bigcup_{i\in I} A_i\Big)(x) = \bigvee_{i\in I} A_i(x)$$

respectively.

We denote the smallest and the greatest elements of the complete lattice L by 0_L and 1_L, respectively. We also use the symbols $\succeq, \prec$ and $\succ$ on a given lattice $(L, \preceq)$, in the obvious meanings.

Definition 2.4. [22] A triangular norm ($t-$norm) on a complete lattice $\mathscr{L} = (L, \preceq)$ is a function $\mathscr{T} : L \times L \to L$ satisfying the following conditions for all $x, y, z, t \in L$:

(1) $\mathscr{T}(x, y) = \mathscr{T}(y, x)$
(2) $\mathscr{T}(\mathscr{T}(x, y), z) = \mathscr{T}(x, \mathscr{T}(y, z))$
(3) $\mathscr{T}(x, 1_L) = \mathscr{T}(1_L, x) = x$
(4) $x \preceq y$ and $z \preceq t$, then $\mathscr{T}(x, z) \preceq \mathscr{T}(y, t)$.

Definition 2.5. [22] A $t-$norm $\mathscr{T}$ on a complete lattice $\mathscr{L} = (L, \preceq)$ is called continuous, if for every pair of sequences (x_n) and (y_n) on L such that $(x_n) \to x \in L$ and $(y_n) \to y \in L$, one have the property that $\mathscr{T}(x_n, y_n) \to \mathscr{T}(x, y)$ with respect to the order topology on L.

Definition 2.6. [22] A mapping $\mathscr{N} : L \to L$ is called a negator on $\mathscr{L} = (L, \preceq)$ if,
N_1) $\mathscr{N}(0_L) = 1_L$
N_2) $\mathscr{N}(1_L) = 0_L$
N_3) $x \preceq y$ implies $\mathscr{N}(y) \preceq \mathscr{N}(x)$ for all $x, y \in L$.
If in addition,
N_4) $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in L$,
then the negator $\mathscr{N}$ is said to be involutive.

On the lattice $([0, 1], \leq)$ the function $\mathscr{N}_s : [0, 1] \to [0, 1]$ defined as $\mathscr{N}_s(x) = 1 - x$ is an example of a involutive negator, called standart negator on $[0, 1]$, which is used in the theory of fuzzy sets. Similarly, given the lattice $([0, 1]^2, \preceq)$ with the order

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \iff \mu_1 \leq \mu_2 \quad and \quad \nu_1 \geq \nu_2$$

for all $(\mu_i, \nu_i) \in [0, 1]^2$, $i = 1, 2$. Then the mapping $\mathscr{N}_1 : [0, 1]^2 \to [0, 1]^2$,

$$\mathscr{N}_1(\mu, \nu) = (\nu, \mu)$$

is an involutive negator used in the theory of intuitionistic fuzzy sets in the sense of Atanassov[1].

Definition 2.7. [22] Let V be a real vector space. $\mathscr{L} = (L, \preceq)$ be a complete lattice, $\mathscr{T}$ be a continuous $t-$norm on $\mathscr{L}$ and ρ be an $L-$set on $V \times (0, \infty)$ satisfying the following

(a) $\rho(x,t) \succ 0_L$ for all $x \in V, t > 0$
(b) $\rho(x,t) = 1_L$ for all $t > 0$ if and only if $x = \theta$
(c) $\rho(\alpha x, t) = \rho(x, \frac{t}{|\alpha|})$ for all $x \in V, t > 0 \quad and \quad \alpha \in \mathbb{R} - \{0\}$
(d) $\mathscr{T}(\rho(x,t), \rho(y,s)) \preceq \rho(x+y, t+s)$, for all $x, y \in V \quad and \quad t, s > 0$
(e) $\lim_{t\to\infty} \rho(x,t) = 1_L$ and $\lim_{t\to 0} \rho(x,t) = 0_L$ for all $x \in V - \{\theta\}$
(f) The mappings $f_x : (0,\infty) \to L$ given by $f(t) = \rho(x,t)$ are continuous.

In this case, the triple $(V, \rho\, \mathscr{T})$ is called an $\mathscr{L}-$fuzzy normed space or $\mathscr{L}-$normed space.

Definition 2.8. [22] A sequence (x_n) in an $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{T})$ is said to conergent to ℓ if, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$

$$\rho(x_n - \ell, t) \succ \mathscr{N}(\epsilon)$$

where $\mathscr{N}$ is a negator on $\mathscr{L}$, for each $\epsilon \in L - \{0_L\}$ and $t > 0$.

Definition 2.9. [22] A sequence (x_n) in an $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{T})$ is said to be Cauchy sequence if, there exists $n_0 \in \mathbb{N}$ such that, for all $m, n > n_0$

$$\rho(x_n - x_m, t) \succ \mathscr{N}(\epsilon)$$

where $\mathscr{N}$ is a negator on $\mathscr{L}$, for each $\epsilon \in L - \{0_L\}$ and $t > 0$. In this case, we show this convergence as $\mathscr{L} - \lim_n x_n = \ell$.

Definition 2.10. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Then a sequence $x = (x_k)$ is said to be bounded with respect to fuzzy norm ρ , provided that, for each $r \in L - \{0_L, 1_L\}$ and $t > 0$,

$$\rho(x_k, t) \succ \mathscr{N}(r)$$

for all $k \in \mathbb{N}$.

We will look at statistical convergence on $\mathscr{L}-$fuzzy normed spaces. Before we go any further, we should review some statistical convergence terminology [3]. If K is a subset of $\mathbb{N}$, the set of natural numbers, then its asymptotic density, denoted by $\delta\{K\}$, is

$$\delta\{K\} := \lim_n \frac{1}{n} \big|\{k \leq n : k \in K\}\big|$$

whenever the limit exists, with $|A|$ denoting the cardinality of the set A.

If the set $K(\epsilon) = \{k \leq n : |x_k - l| > \epsilon\}$ has the asymptotic density zero, i.e.

$$\lim_n \frac{1}{n}\{k \leq n : |x_k - l| > \epsilon\} = 0$$

then a number sequence $x = (x_k)$ is said to be statistically convergent to the l. In this scenario, we will write $st - \lim x = l$.

Although every convergent sequence is statistically convergent to the same limit, the converse is not always true.

Definition 2.11. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Then a sequence $x = (x_k)$ is statistically convergent to $l \in V$ with respect to fuzzy norm ρ, provided that, for each $\epsilon \in L - \{0_L\}$ and $t > 0$,

$$\delta\{k \in \mathbb{N} : \rho(x_k - l, t) \not\succ \mathscr{N}(\epsilon)\} = 0$$

or equivalently

$$\lim_m \frac{1}{m}\{j \leq m : \rho(x_k - l, t) \not\succ \mathscr{N}(\epsilon)\} = 0.$$

In this scenario, we will write $st_{\mathscr{L}} - \lim x = l$.

Definition 2.12. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Then a sequence $x = (x_k)$ is said to be statistically Cauchy with respect to the fuzzy norm ρ, provided that

$$\delta\{k \in \mathbb{N} : \rho(x_k - x_m, t) \not\succ \mathscr{N}(\epsilon)\} = 0$$

for each $\epsilon \in L - \{0_L\}$, $m \in \mathbb{N}$ and $t > 0$.

Definition 2.13. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Then a sequence $x = (x_k)$ is said to be statistically bounded with respect to fuzzy norm ρ, provided that there exists $r \in L - \{0_L, 1_L\}$ and $t > 0$ such that

$$\delta\{k \in \mathbb{N} : \rho(x_k, t) \not\succ \mathscr{N}(r)\} = 0$$

for each positive integer k.

Definition 2.14. [8] If X is a non-empty set then a family I of subsets of X is called an ideal in X if and only if

(a) $\emptyset \in I$,
(b) $A, B \in I$ implies $A \cup B \in I$,
(c) For each $A \in I$ and $B \subset A$ we have $B \in I$,

where $P(X)$ is the power set of X. I is called nontrivial ideal if $I \neq 0$ and $X \notin I$.

Definition 2.15. [8] Let X be a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if

(a) $\emptyset \notin I$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) For each $A \in F$ and $A \subset B$ we have $B \in F$.

Definition 2.16. [8] A nontrivial ideal I in X is called an admissible ideal if it contains all singletions, i.e., $\{x\} \in I$ for each $x \in X$.

Let $I \subset P(X)$ be a nontrivial ideal. Then a class $F(I) = \{M \subset X : M = X \setminus A$, for some $A \in I\}$ is a filter on X and is called the filter associated with the ideal I.

Definition 2.17. [8] An admissible ideal I is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from I there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcap_{n\in B_n} \in I$.

Definition 2.18. [8] Let $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then, a sequence $x = (x_k)$ is said to be $I-$ convergent to ℓ if, for every $\epsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - \ell| \geq \epsilon\} \in I.$$

In this case we write $I - \lim x = \ell$.

Definition 2.19. [8] Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$. Then, a sequence $x = (x_k)$ is said to be $I - Cauchy$ if, for every $\epsilon > 0$, there exist a number $N = N(\epsilon)$ such that

$$\{k \in \mathbb{N} : |x_k - x_N| \geq \epsilon\} \in I.$$

We will look at the concept of ideal convergence on $\mathscr{L}-$fuzzy normed spaces in this section. Throughout the paper we take I_1 as a nontrivial ideal in $\mathbb{N}$.

Definition 2.20. Let $(V, \rho, \mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space and I_1 be a nontrivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is I_1 convergent to $l \in V$ with respect to ρ fuzzy norm, provided that, for each $\epsilon \in L - \{0_L\}$ and $t > 0$,

$$\{k \in \mathbb{N} : \rho(x_k - l, t) \not\succ \mathscr{N}(\epsilon)\} \in I_1.$$

In this scenario, we will write $I_1^{\mathscr{L}} - \lim x = l$.

Theorem 2.21. *Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space and let I_1 be an admissible ideal. If $\lim x = l$, then $I_1^{\mathscr{L}} - \lim x = l$.*

The converse of the theorem is not true in general.

Theorem 2.22. *Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. If a sequence $x = (x_k)$ is I_1 convergent with respect to the $\mathscr{L}-$fuzzy norm ρ, then $I_1^{\mathscr{L}}-$limit is unique.*

Theorem 2.23. *Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space and I_1 be an admissible ideal. Then,*

(a) If $I_1^{\mathscr{L}} - \lim x_k = l_1$ and $I_1^{\mathscr{L}} - \lim y_k = l_2$ then $I_1^{\mathscr{L}} - \lim(x_k + y_k) = (l_1 + l_2)$

(b) If $I_1^{\mathscr{L}} - \lim x_k = l$ then $I_1^{\mathscr{L}} - \lim \alpha x_k = \alpha l$.

3. Ideal Cauchyness and Boundedness for Double Sequences

We will look at the concept of ideal convergence for double sequences on $\mathscr{L}-$fuzzy normed spaces in this section. Throughout the paper we take I_2 as a nontrivial ideal in $\mathbb{N} \times \mathbb{N}$.

Ideal Cauchy for double sequences with respect to $\mathscr{L}-$ fuzzy normed space will be given in this section, and also a new concept of ideal completeness will be defined.

First, let's recall the concept of double sequence and the definition of ideal double convergence in $\mathscr{L}$-fuzzy normed spaces.

For any given $\varepsilon > 0$, if there exists an integer N such that $\mid x_{jk} - l \mid < \varepsilon$ whenever $j, k > N$, a double sequence $x = (x_{jk})$ is said to be Pringsheim's convergent or shortly $P-$ convergent. This will be written as

$$\lim_{j,k \to \infty} x_{jk} = l$$

with j and k tending to infinity independently of one another.

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers, and let $K(m,n)$ be the numbers of (j,k) in K such that $j \leq m$ and $k \leq n$. Then we can define the two-dimensional analogue of natural density as follows: The lower asymptotic density of the set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\delta_2(K) = \liminf_{m,n} \frac{K(m,n)}{mn}$$

and if the sequence $\left(\frac{K(m,n)}{mn}\right)$ has a limit in the sense of Pringsheim, we say it has a double natural density, and it is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

Definition 3.1. Let $(V, \rho, \mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space and I_2 be a nontrivial ideal in $\mathbb{N}$. Then a sequence $x = (x_{jk})$ is I_2 convergent to $l \in V$ with respect to ρ fuzzy norm, provided that, for each $\epsilon \in L - \{0_L\}$ and $t > 0$,

$$\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \rho(x_{jk} - l, t) \not\succ \mathscr{N}(\epsilon)\} \in I_2.$$

In this scenario, we will write $I_2^{\mathscr{L}} - \lim x = l$.

Lemma 3.2. *Let $(V, \rho, \mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\epsilon \in L - \{0_L\}$ and $t > 0$:*

(a) $I_2^{\mathscr{L}} - \lim x = l$.
(b) $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \rho(x_{jk} - l, t) \not\succ \mathscr{N}(\epsilon)\} \in I_2$.
(c) $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \rho(x_{jk} - l, t) \succ \mathscr{N}(\epsilon)\} \in F(I_2)$.
(d) $I_2^{\mathscr{L}} - \lim \rho(x_{jk} - l, t) = 1_L$.

Definition 3.3. Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. Then, a sequence $a=(a_{mn})$ is said to be ideal double Cauchy with respect to $\mathscr{L}-$ fuzzy norm μ, if for every $\varepsilon \in L-\{0_L\}$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that for all $m,k \geq N$ *and* $n,l \geq M$ provided that

$$\delta_2\{(m,n)\in \mathbb{N}\times\mathbb{N} : \mu(a_{mn}-a_{kl},t) \not\succ \mathscr{N}(\varepsilon)\} \in I_2.$$

Theorem 3.4. *Every ideal convergent double sequence is ideal double Cauchy on $\mathscr{L}-$ fuzzy normed space.*

Proof. Let $a=(a_{mn})$ be a double sequence such that ideal convergent to ℓ with respect to $\mathscr{L}-$ fuzzy norm μ. For a given $\varepsilon>0$, choose $r>0$ such that,

$$\mathscr{K}(\mathscr{N}(r),\mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

For $t>0$ we can write,

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mu(a_{mn}-\ell,\frac{t}{2}) \succ \mathscr{N}(r)\}.$$

Take $(p,q)\in A$. Obviously, $\mu(a_{pq}-\ell,\frac{t}{2}) \succ \mathscr{N}(r)$. Also since,

$$\mu(\ell-a_{pq},\frac{t}{2}) = \mu(a_{pq}-\ell,\frac{\frac{t}{2}}{|-1|}) = \mu(a_{pq}-\ell,\frac{t}{2}) \succ \mathscr{N}(\varepsilon)$$

we have

$$\begin{aligned}\mu(a_{mn}-x_{pq},t) &= \mu\big((a_{mn}-\ell)+(\ell-a_{pq}),\frac{t}{2}+\frac{t}{2}\big)\\ &\succ \mathscr{K}\left(\mu(a_{mn}-\ell,\frac{t}{2}),(\nu(\ell-a_{pq},\frac{t}{2})\right)\\ &\succ \mathscr{K}\left(\mathscr{N}(r),\mathscr{N}(r)\right)\\ &\succ \mathscr{N}(\varepsilon).\end{aligned}$$

If we define a set $B=\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mu(a_{mn}-a_{pq},t)\succ\mathscr{N}(\varepsilon)\}$, then $A\subseteq B$. Since $\delta_2(A)\notin I_2$, $\delta_2(B)\notin I_2$. Thus, the double theta density of complement of B in I_2, i.e. $\delta_2(B^c)\in I_2$, which means $a=(a_{mn})$ is ideal double Cauchy. ☐

Definition 3.5. Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space and $a=(a_{mn})$ be a double sequence. Then, $a=(a_{mn})$ is said to be ideal double bounded with respect to $\mathscr{L}-$ fuzzy norm μ, provided that there exists $r\in L-\{0_L,1_L\}$ and $t>0$ such that

$$\delta_2\{(m,n)\in\mathbb{N}\times\mathbb{N} : \mu(a_{mn},t)\not\succ\mathscr{N}(r)\}\in I_2$$

for each positive integer m,n.

Theorem 3.6. *Every double bounded sequence on a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$, is ideal double bounded.*

Proof. Let (a_{mn}) be a double bounded sequence on $(V, \mu, \mathscr{K})$. Then, there exist $t > 0$ and $r \in L - \{0_L, 1_L\}$ such that $\mu(a_{mn}, t) \succ \mathscr{N}(r)$. In that case we have,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn}, t) \not\succ \mathscr{N}(r)\} = \emptyset$$

which yields

$$\delta_2\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn}, t) \not\succ \mathscr{N}(r)\} \in I_2.$$

Thus, (a_{mn}) is ideal double bounded. □

However the converse of this theorem does not hold in general as seen in the example below.

Example 3.7. Let $V = \mathbb{R}$ and $\mathscr{L} = (L, \leq)$ where L is the set of non-negative extended real numbers, that is $L = [0, \infty]$. Then, $0_L = 0, 1_L = \infty$. Define a $\mathscr{L}-$fuzzy norm ν on V by $\mu(x,t) = \frac{t}{|x|}$ for $x \neq 0$ and $\nu(0,t) = \infty$ for each $t \in (0, \infty)$. Consider the $t-$ norm $\mathscr{K}(a,b) = \min\{a,b\}$ on $\mathscr{L}$. Given the sequence,

$$x_{mn} = \begin{cases} m+n, & \text{if m+n is a prime number,} \\ \frac{1}{\tau(m+n)-2}, & \text{otherwise} \end{cases}$$

where, $\tau(m+n)$ denotes the number of positive divisors of $m+n$. Note that (x_{mn}) is not bounded since for each $t > 0$ and $r \in L - \{0, \infty\}$, for any prime number $m+n$ such that $rt \leq m+n$ we have

$$\mu(x_{mn}, t) = \mu(m+n, t) = \frac{t}{|\, m+n \,|} = \frac{t}{m+n} \not\succ \frac{1}{r} = \mathscr{N}(r).$$

However for $t = 1$ and any non-prime integer $m+n$, $r = 2$ satisfies

$$\mu(x_{mn}, 1) = \mu(\frac{1}{\tau(m+n)-2}, 1) = \frac{1}{|\frac{1}{\tau(m+n)-2}|} = |\tau(m+n)-2| > \frac{1}{2} = \mathscr{N}(r)$$

since $\tau(m+n) \neq 2$ for any non-prime $m+n$, and since the density of prime numbers converges zero by Prime Number Theorem we have,

$$\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk}, 1) \not> \mathscr{N}(2)\} \in I_2$$

suggesting that (x_{mn}) is ideal double bounded.

Theorem 3.8. *Every ideal double Cauchy sequence on a $\mathscr{L}-$fuzzy normed space $(V, \mu, \mathscr{K})$ is ideal double bounded.*

Proof. Let (a_{mn}) be a ideal double Cauchy on $(V, \mu, \mathscr{K})$. Then, for every $\epsilon \in L - \{0_L\}$ and $t > 0$, there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $m, k \geq N \;\; and \;\; n, l \geq M$ provided that

$$\delta_2\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - a_{kl}, t) \not\succ \mathscr{N}(\varepsilon)\} \in I_2.$$

Then,

$$\delta_2\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - a_{kl}, t) \succ \mathscr{N}(\varepsilon)\} \notin I_2.$$

Consider a number $(m,n) \in \mathbb{N} \times \mathbb{N}$ such that $\mu(a_{mn} - a_{kl}, 1) \succ \mathscr{N}(\varepsilon)$. Then, for $t = 2$

$$\mu(a_{mn}, 2) = \mu(a_{mn} - a_{kl} + a_{kl}, 2) \succ \mathscr{K}(\mu(a_{mn} - a_{kl}, 1), \mu(a_{kl}, 1)) \succ \mathscr{K}(\mathscr{N}(\varepsilon), \nu(x_{kl}, 1)).$$

Say $r := \mathscr{N}(\mathscr{K}(\mathscr{N}(\varepsilon), \mu(a_{kl}, 1)))$. Then,

$$\mu(a_{mn}, 2) \succ \mathscr{K}(\mathscr{N}(\varepsilon), \mu(a_{kl}, 1)) = \mathscr{N}(r),$$

which implies

$$\delta_2\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn}, 2) \succ \mathscr{N}(r)\} \notin I_2$$

or equivalently

$$\delta_2\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn}, 2) \not\succ \mathscr{N}(r)\} \in I_2$$

giving ideal double boundedness of (a_{mn}). □

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