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2D viscoelastic equation from the perspective of Lie groups

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Abstract. We investigate 2-dimensional viscoelastic equations with a view of Lie groups. In this sense, we answer question of the symmetry classification. We provide the algebra of symmetry and build the optimal system of Lie subalgebras. Reductions of similarities related to subalgebras are classified. In the end by using Bluman-Anco homotopy formula, we find local conservation laws of the viscoelastic equation.

Keywords: Lie algebras, Viscoelastic equation, Conservation laws, Reduction equations.

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1. Introduction

Viscoelastic equations are important mathematical models that have many applications in various sciences. Recently, the calculation of viscoelastic equations has been considered by different methods. We check out the following model

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \varepsilon \frac{\partial \Delta u(x,y,t)}{\partial t} - \gamma \Delta u(x,y,t) = f, \tag{1.1}$$

where f is a function. The Equation (1.1) has several applications, for example, it is applied to describe the heat transfer with memory

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materials, viscous elastic mechanics, loose medium pressure [5], nuclear reaction kinetics [14], Li et al. [10], used a proper orthogonal decomposition (POD) technique to reduce the finite volume element (FVE) method for two-dimensional (2D) viscoelastic equations. Error estimates of the reduced-order fully discrete FVE solution and its implementation are also provided in Ref. [10] for solving the reduced-order fully discrete FVE algorithm. Performing the Lie symmetry group procedure, the problem of symmetry classification for different equations is widely considered in various spaces [1, 2, 7, 8, 9]. On the other hand, the symmetry group approach or Lie's approach itself, which is a computational method algorithmic for finding group-invariant solutions, is significantly used in the resolution of differential equations. Using this procedure, one can find appropriate solutions through known ones, study the invariant solutions, and even decrease the order of ODEs [11, 3, 6, 4, 12]. Our aim in this paper is to investigate two-dimensional viscoelastic equations from Lie's point of view. Because Lie's theory is one of the useful and effective methods for solving nonlinear equations. Then we apply this method and obtain specified the symmetry algebra infinitesimal generators of Eq(1.1). According the optimal system of symmetry algebra can detect invariant solutions,which is relevant one-dimensional Lie algebra. In Lie's method Using symmetric algebra, the optimal 1-parameter device for viscoelastic equations can be found. In the following, more details are given in different sections of the article. This paper is divided into four sections. The second section are specified the symmetry algebra infinitesimal generators of Eq(1.1). In the next Section by using the symmetry group We obtain the one-parameter optimal system of Eq(1.1) . We find in section 4 similarity reduction corresponding to the infinitesimal symmetries of Eq(1.1) by using one-dimensional subalgebras. In the last section, we obtain the associated conservation laws for the equation using the direct method and provide conclusion remarks.

2. The symmetry algebra of Eq.(1.1)

Generally,

$$\Delta_\alpha(X, U^{(p)}) = 0, \qquad \alpha = 1, ..., t, \tag{2.1}$$

is a system of PDE of order pth, where $X = (x^1, ..., x^m)$ and $U = (u^1, ..., u^n)$ are m independent and n dependent variables respectively, and $U^{(i)}$ is the $i-$ order derivative of U with respect to x, $0 \leq i \leq p$. Infinitesimal transformations Lie group acts on both X and U, is:

$$\tilde{x}^i = x^i + \varepsilon \xi^i(X, U) + o(\varepsilon^2), \qquad i = 1, ..., m, \tag{2.2}$$

$$\tilde{u}^j = u^j + \varepsilon \phi_j(X, U) + o(\varepsilon^2), \qquad j = 1, ..., n, \tag{2.3}$$

where ξ^i and ϕ^j represent the infinitesimal transformations for $\{x^1, ..., x^p\}$ and $\{u^1, ..., u^q\}$, respectively. An arbitrary infinitesimal generator corresponding to the group of transformations (2.2) is

$$V = \sum_{i=1}^{p} \xi^i(X, U)\partial_{x^i} + \sum_{j=1}^{q} \phi_j(X, U)\partial_{u^j}. \tag{2.4}$$

Now in order to apply the Lie group procedure for Eq.(1.1), an infinitesimal transformation's one parameter Lie group is considered: (we use x, y and t instead of x^1, x^2 and x^3 respectively in order not to use index. So, $x^1 = x, x^2 = y, x^3 = t, u^1 = u, u^2 = f$),

$$\begin{aligned}
\tilde{x} &= x + \varepsilon\xi^1(x, y, t, u, f) + o(\varepsilon^2), \\
\tilde{y} &= y + \varepsilon\xi^2(x, y, t, u, f) + o(\varepsilon^2), \\
\tilde{t} &= t + \varepsilon\xi^3(x, y, t, u, f) + o(\varepsilon^2), && (2.5) \\
\tilde{u} &= u + \varepsilon\phi_1(x, y, t, u, f) + o(\varepsilon^2) && (2.6) \\
\tilde{f} &= f + \varepsilon\phi_1(x, y, t, u, f) + o(\varepsilon^2).
\end{aligned}$$

The corresponding symmetry generator is as follows:

$$\begin{gathered}
V = \xi^1(x, y, t, u, f)\partial_x + \xi^2(x, y, t, u, f)\partial_y + \xi^3(x, y, t, u, f)\partial_t + \\
\phi_1(x, y, t, u, f)\partial_u + \phi_2(x, y, t, u, f)\partial_f.
\end{gathered} \tag{2.7}$$

The proviso of being invariance corresponds to the equations:

$$Pr^{(3)}V[\frac{\partial^2 u(x, y, t)}{\partial t^2} - \varepsilon\frac{\partial \Delta u(x, y, t)}{\partial t} - \gamma\Delta u(x, y, t) - f] = 0, \quad \text{whenever}$$
$$\frac{\partial^2 u(x, y, t)}{\partial t^2} - \varepsilon\frac{\partial \Delta u(x, y, t)}{\partial t} - \gamma\Delta u(x, y, t) - f = 0.$$

Since ξ^1, ξ^2, ξ^3,ϕ_1 and ϕ_2 are only dependent on x,y, t ,$u$and f, setting the individual coefficients equal to zero, we have the following system of equations:

$$\begin{cases}
-a\xi^1_t = 0, & -a\xi^1_t = 0, \\
a\xi^1_f = 0, & -a\xi^1_t = 0, \\
a\xi^1_t = 0, & a\xi^2_{uf} = 0, \\
a\xi^1_f = 0, & a\xi^3_{uuf} = 0, \\
a\xi^1_{uu} = 0, & -2a\xi^1_f = 0, \\
-3a\phi^1_{ff} = 0, & -a\phi^1_{fff} = 0, \\
\vdots & \vdots
\end{cases}$$

The total number of these equations is 227. By solving these PDE equations, we earn the following result:

TABLE 1. Lie algebra for Eq.(1.1).

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	$-X_2$	0
X_2	*	0	0	X_1	0
X_3	*	*	0	0	0
X_4	X_2	$-X_1$	*	0	0
X_5	*	*	*	*	0

Theorem 2.1. *The point symmetries Lie group of equation* (1.1) *possesses a Lie algebra generated by* (2.7)*, whose coefficients are the following infinitesimals:*

$$\begin{aligned}
\xi^1 =& c_1 y + c_2 y,\\
\xi^2 =& -c_1 y + c_3,\\
\xi^3 =& c_4,\\
\phi_1 =& c_5 u + F_2(x,y,t),\\
\phi_2 =& -a(\frac{\partial^3}{\partial x^2 \partial t} F_2(x,y,t)) + c_5 f - a(\frac{\partial^3}{\partial t^3} F_2(x,y,t))\\
&- b(\frac{\partial^2}{\partial x^2} F_2(x,y,t)) - b(\frac{\partial^2}{\partial y^2} F_2(x,y,t))\\
&+ \frac{\partial^2}{\partial t^2} F_2(x,y,t) - a(\frac{\partial^3}{\partial y^2 \partial t} F_2(x,y,t)) - (\frac{\partial^2}{\partial t^2} F_2(x,y,t)),
\end{aligned} \tag{2.8}$$

where $c_i \in R$, $i = 1, ..., 5$ *and* $\alpha(u)$ *is a function satisfying Eq.*(1.1)*.*

Corollary 2.2. *Every point symmetry's one-parameter Lie group of Eq.*(1.1) *has the infinitesimal generators as follows:*

$$\begin{aligned}
X_1 &= \partial_x,\\
X_2 &= \partial_y,\\
X_3 &= \partial_t,\\
X_4 &= y\partial_x - x\partial_y,\\
X_5 &= u\partial_u + f\partial_f,\\
X_\alpha &= \alpha\partial_u.
\end{aligned} \tag{2.9}$$

We provide Lie algebra for Eq.(1.1) by Table (1). The expression $[X_i, X_j] = X_i X_j - X_j X_i$ determines the entry in row i^{th} and column j^{th}, $i, j = 1, ..., 5$.

For example, the flow of vector field X_4 in Corollary 2.2 is shown by

$$\Phi_\epsilon = (ysin(\epsilon) + xcos(\epsilon), ycos(\epsilon) - xsin(\epsilon), t).$$

The flow Φ_ϵ is plotted in Figures 1 and 2.

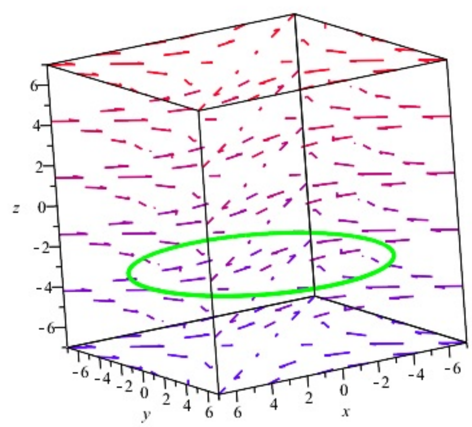

FIGURE 1. The plot of flow Φ_ϵ.

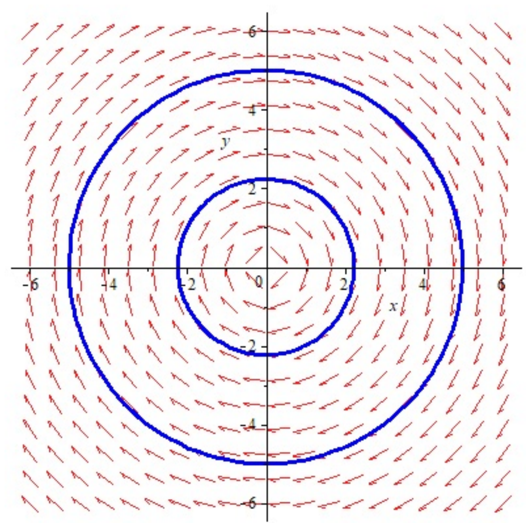

FIGURE 2. The projection of flow Φ_ϵ into the $(x, y, 0)$-plane.

3. Classification of one-dimensional subalgebras

Using the symmetry group, we can determine the one-parameter optimal system of Eq (1.1). It is important to obtain those subgroups which present different kinds of solutions. Thus, we need to search for invariant solutions that are not linked by a transformation in the full symmetry group. This subject leads to the notion of an optimal set of subalgebras. The problem of classifying one-dimensional subalgebras would be the

TABLE 2. Adjoint representation of the Lie algebra

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	$X_2 - sX_4$	X_3	X_4	X_5
X_2	$X_1 + sX_4$	X_2	X_3	X_4	X_5
X_3	X_1	X_2	X_3	X_4	X_5
X_4	$cos(s)X_1 - sin(s)X_2$	$sin(s)X_1 + cos(s)X_2$	X_3	X_4	X_5
X_5	X_1	X_2	X_3	X_4	X_5

same as the question of classifying the adjoint representation orbits. An optimal set of subalgebras problem is solved by considering one representative from every group of corresponding subalgebras [13] and [11]. The definition of the adjoint representation of each X_t, $t = 1, ..., 5$ would be:

$$\text{Ad}(exp(s.X_t).X_r) = X_r - s.[X_t, X_r] + \frac{s^2}{2}.[X_t, [X_t, X_r]] - \cdots, \quad (3.1)$$

where s is a parameter and $[X_t, X_r]$ is defined in Table (1) for $t, r = 1, \cdots, 5$ ([11],page 199). Let $\mathfrak{g}$, be the Lie algebra that produced by (2.9). We obtain the adjoint action for $\mathfrak{g}$ in Table (2).

Theorem 3.1. *One-dimensional subalgebras of Eq.*(1.1) *are as follows:*

1) $X_1 + c_1X_3 + c_2X_5$,
2) $X_3 + c_1X_3 + c_2X_5$,
3) $X_4 + c_1X_3 + c_2X_5$,
4) $X_3 + c_1X_5$,

where $c_i \in \mathbb{R}$ *are arbitrary numbers for* $i = 1, \cdots, 5$.

Proof. From Table (1), it is clear that the center of Lie algebra is $\langle X_3, X_5 \rangle$. Hence, it would be sufficient to determine the sub-algebras of

$$\langle X_1, X_2, X_4 \rangle.$$

For $t = 1, \cdots, 5$, the map:

$$\begin{cases} F_t^s : \mathfrak{g} \to \mathfrak{g} \\ X \mapsto \text{Ad}(\exp(sX_t).X) \end{cases}$$

TABLE 3. Lie invariants and similarity solution.

i	H_i	ξ_i	η_i	w_i	u_i	f_i
1	X_1	y	t	u	$h(\xi,\eta)$	$g(\xi,\eta)$
2	X_2	x	t	u	$h(\xi,\eta)$	$g(\xi,\eta)$
3	X_3	x	y	u	$h(\xi,\eta)$	$g(\xi,\eta)$
4	X_1+X_3	$x-t$	y	u	$h(\xi,\eta)$	$g(\xi,\eta)$
5	X_2+X_3	x	$y-t$	u	$h(\xi,\eta)$	$g(\xi,\eta)$

is a linear function. Considering basis $\{X_1,\cdots,X_5\}$, the matrixes M_t^s of F_t^s, $t=1,\cdots 5$ are given by:

$$M_1^s=\begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -s_1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2^s=\begin{bmatrix} 1 & 0 & 0 & s_2 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_3^s=\begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_4^s=\begin{bmatrix} cos(s_4) & -sin(s_4) & 0 & 0 & 0\\ sin(s_4) & cos(s_4) & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_5^s=\begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By applying these matrixes on a vector field $X=\sum_{i=1}^5 a_iX_i$ alternatively, we can simplify X as follows:

For $a_4\neq 0$, the coefficients of X_1 and X_2 can be disappeared by setting $s_2=-(a_4/a_1)$ and $s_1=(a_4/a_2)$ respectively. If needed, by scaling X, we suppose $a_4=1$. Thus, X turns into (3).

For $a_4=0$ and $a_2\neq 0$, the coefficients of X_1 can be disappeared by setting $s_3=-tan^{-1}(a_1/a_2)$. If needed, by scaling X, we suppose $a_2=1$. Thus, X turns into (2).

For $a_2=a_4=0$ and $a_1\neq 0$, if needed, by scaling X, we suppose $a_1=1$. Thus, X turns into (1).

For $a_1=a_2=0$ and $a_4=0$, X turns into (4). □

Table 4. Reduced equations regarding infinitesimal symmetries.

i	Reduction of equations
1	$h_{\eta\eta} - ah_{\xi\xi\eta} - ah_{\eta\eta\eta} - bh_{\xi\xi} - bh_{\eta\eta} - g = 0,$
2	$h_{\eta\eta} - ah_{\xi\xi\eta} - ah_{\eta\eta\eta} - bh_{\xi\xi} - bh_{\eta\eta} - g = 0,$
3	$h_{\eta\eta} - ah_{\xi\xi\eta} - ah_{\eta\eta\eta} - bh_{\xi\xi} - bh_{\eta\eta} - g = 0,$
4	$h_{\xi\xi} + ah_{\xi\xi\eta} + ah_{\eta\eta\xi} + ah_{\xi\xi\xi} - bh_{\xi\xi} - bh_{\eta\eta} - bh_{\xi\xi} - g = 0,$
5	$h_{\eta\eta} + ah_{\xi\xi\eta} + ah_{\eta\eta\eta} + ah_{\eta\eta\eta} - bh_{\xi\xi} - bh_{\xi\xi} - bh_{\eta\eta} - bh_{\eta\eta} - g = 0.$

4. Similarity reduction of Equation (1.1)

Here, we want to classify symmetry reduction of Eq.(1.1) concerning subalgebras of Theorem 3.1. We need to search for a new form of Equation (1.1) in specific coordinates so that it would reduce. Such a coordinate will be constructed by finding independent invariant ξ, η, h regarding the infinitesimal generator. So, expressing the equation in new coordinates applying the chain rule reduces the system. For 1-dimensional subalgebras in the Theorem 3.1 the similarity variables ξ_i, η_i, and h_i are listed in Table 3. Each similarity variable is applied to find the reduced PDE of Eq.(1.1) which, they are listed in Table 4.

For instance, we compute the invariants associated with subalgebra $H_5 := X_1 + X_3$ by integrating the following characteristic equation.

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{1} = \frac{du}{0}.$$

Hence, the similarity variables would be:

$$\xi = x, \qquad \eta = y - t, \qquad h = u,$$

Substituting the similarity variables in Eq.(1.1) and applying the chain rule it results that, the solution of Eq.(1.1) is:

$$u = h(\xi, \eta)$$

where $h(\xi, \eta)$ satisfies a reduced PDE with two variables as follows:

$$h_{\eta\eta} + ah_{\xi\xi\eta} + ah_{\eta\eta\eta} + ah_{\eta\eta\eta} - bh_{\xi\xi} - bh_{\xi\xi} - bh_{\eta\eta} - bh_{\eta\eta} - g = 0. \quad (4.1)$$

Subalgebra $X_1 + X_3$ and the reduced equation (4.1) are shown in Tables 3 and 4, by the case (5).

5. Conservation laws

One of the important classes of partial differential equations is the law of conservation, which is one of the important laws of nature. Due to its importance, many methods have been proposed to study conservation laws, and here we use a direct method to study conservation laws.

Let $P\{x;u\}$ be differential equation of order k with n independent variables $x=(x^1,\cdots x^n)$ and one dependent variable u, Which is given as follows

$$P[u]=P(x,u,\partial u,...,\partial^k u)=0. \tag{5.1}$$

Multiplying $\Lambda(x,u,\partial u,...,\partial^l u)$ in can give the conservation law $\Lambda[u]P[u]=D_i\varphi^i[u]=0$ for the differential equation $P\{x;u\}$ if and only if

$$E_U\left(\Lambda(x,U,\partial U,\cdots,\partial^I U)P(x,U,\partial U,\cdots,\partial^k U)\right)\equiv 0, \tag{5.2}$$

that $U(x)$ is an arbitrary function and E_U is the Euler operator with respect to U as follows

$$E_U=\partial U-D_i\partial U+\cdots+(-1)^s D_{i_1}\cdots D_{i_s}\partial U_{i_1\cdots i_s}. \tag{5.3}$$

Because the Viscoelastic equation depends on t , as a result, the multipliers of the local conservation law for Equation (1.1) are $\lambda=\xi(t,x,y,z,U,\partial_t U,\cdots\partial_t^l U)$ that $l=1,2,\cdots$ and we get all of its nontrivial local conservation laws from multipliers.

It can be concluded that $\Lambda=\Lambda(t,x,y,U,\partial_t U,\partial_x U,\partial_y U)$, is a multiplier of the law of conservation of Equation (1.1) iff

$$E_U\left[\Lambda(t,x,y,U,\partial U_t,\partial U_x,\partial U_y)\right.\\ \left.\left(\frac{\partial^2 U(x,y,t)}{\partial t^2}-\varepsilon\frac{\partial\Delta U(x,y,t)}{\partial t}-\gamma\Delta U(x,y,t)-f\right)\right]\equiv 0, \tag{5.4}$$

that $U(t,x,y)$ is an arbitrary function.
We search all multipliers $\Lambda=\Lambda(t,x,y,U,\partial U_t,\partial U_x,\partial U_y)$, for Equation (1.1). So, by splitting Equation (5.4) with respect to $U_x,U_{tx},\cdots,U_{xxxx}$, we get these equations

$$\Lambda_U x,y=0,\Lambda_{U,t,x}=0,\Lambda_{U_x,y,y}=-2\Lambda_{U,x},\Lambda_{U,t,y}=0,\Lambda_{U_x,t,y}=0,\Lambda_{U,t,t}=0,$$
$$\Lambda_{U_x,t,t}=-\frac{2\Lambda_{U,x}b}{-1+b},\Lambda_{U_y,t,t}=-\frac{2\Lambda U,yb}{-1+b},\Lambda_{x,x}=\frac{1}{b}(-2\Lambda_{U,y}bU_y$$
$$-2\Lambda_{U,x}bU_x-2U_t\Lambda_{U,t}b+\Lambda_{t,t}+2U_t\Lambda_{U,t}-\Lambda_{y,y}b-\Lambda_{t,t}b),\Lambda_{U_x,x}=2\Lambda_U,$$
$$\Lambda_{U_y,x}=-\Lambda_{U_x,y},\Lambda_{U_t,x}=-\frac{\Lambda_{U_x,t}(-1+b)}{b},$$
$$\Lambda_{U_y,y}=2\Lambda_U,\Lambda_{U_t,y}=-\frac{\Lambda_{U_y,t}(-1+b)}{b},\Lambda_{U_t,t}=2\Lambda_U,$$
$$\Lambda_{U,U}=0,\Lambda_{U,U_x}=0,\Lambda_{U,U_y}=0,\Lambda_{U,U_t}=0,\Lambda_{U_x,U_x}=0,\Lambda_{U_x,U_y}=0,$$
$$\Lambda_{U_t,U_x}=0,\Lambda_{U_y,U_y}=0,\Lambda_{U_t,U_y}=0,\Lambda_{U_t,U_t}=0,\epsilon=0.$$

Solving these equations leads to an infinite set of local multipliers:

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = (C_1t + C_4x + C_2y + C_3)\,U + \Big(C_lt^2 + 2(C_4x + C_2y + C_3)t + \frac{x^2C_1}{b} - x^2C_1 + C_7x + C_6 + C_5y + (-1 + \frac{1}{b})y^2C_1\,)U_t + \left(C_4x^2 + 2(C_1t + C_2y + C_3)x - \frac{bt(tC_4 + C_7)}{-1+b}) - C_4y^2 + C_8y + C_9\right)U_x + \frac{1}{(-1+b)e^{\sqrt{C_1}x}e^{\sqrt{C_2}y}}\,\Big(C_{14}C_{15}(e^{\sqrt{C_1}x})^2 + C_{16})((e^{\sqrt{C_2}y})^2C_{11} + C_{12})(-1+b)\cos(\frac{\sqrt{b}\sqrt{C_2 + C_1}t}{\sqrt{-1+b}}) + C_{13}(C_{15}(e^{\sqrt{C_1}x})^2 + C_{16}\Big)\Big((e^{\sqrt{C_2}y})^2C_{11} + C_{12})(b-1)\sin(\frac{\sqrt{b}\sqrt{C_2 + C_1}t}{\sqrt{-1+b}}) - e^{\sqrt{C_1}x}U_ye^{\sqrt{C_2}y}\Big(C_2(-1+b)x^2 - 2(-1+b)(C_4y - \frac{1}{2}C_8)x + ((-y^2 + t^2)b + y^2)C_2 + ((-2yC_1 + C_5)t - C_{10} - 2C_3y)b + 2C_3y + C_{10} + 2yC_1t\Big)\Big).$$

Hence, using the Bluman-Anco homotopy formula, we obtain the conservation components of ϕ^t, ϕ^x, ϕ^y with respect to Λ:

Case 1

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = \frac{1}{b}(tU_b + U_{tt^2b} + U_{tx^2} - U_{tx^2b} + U_{ty^2} - U_{ty^2b} + 2xtU_{xb} + 2U_{yytb}),$$

$$\begin{aligned}\phi^t =& \frac{1}{2}\frac{1}{b}(U^2b^2 + U_{t^2}U_{x^2} + U_{t^2}U_{y^2} - U^2b - 2U_{bx}U_x - 2U_bU_{yy}\\ &+ 2U_tU_{tb^2} + 2U_{b^2x}U_x + 2U_{b^2}U_{yy} - U_{b^2}U_{xxt^2} - U_bU_{xxx^2} + U_{b^2}U_{xxx^2}\\ &- U_bU_{xxy^2} + U_{b^2}U_{xxy^2} - U_{b^2}U_{yyt^2} - U_bU_{yyx^2} + U_{b^2}U_{yyx^2} - U_bU_{yyy^2}\\ &+ U_{b^2}U_{yyy^2} + U_{t^2t^2b} - 2U_{t^2x^2b} - 2U_{t^2y^2b} - U_{t^2t^2b^2} + U_{t^2x^2b^2}\\ &+ U_{t^2y^2b^2} - 2U_{xt}U_{txb} - 2U_{yt}U_{tyb} + 2U_{xt}U_{txb^2} + 2U_{yt}U_{tyb^2}\\ &+ 2U_{txt}U_{xb} - 2U_tU_{yytb} - 2U_{txt}U_{xb^2} - 2U_tU_{yytb^2}),\end{aligned}$$

$$\begin{aligned}\phi^x =&U_{tb}U_x + U_xU_t - U_xU_{tb} + \frac{1}{2}UU_{txt^2b} + \frac{1}{2}UU_{txx^2}\\ &- \frac{1}{2}UU_{txx^2b} + \frac{1}{2}UU_{txy^2} - \frac{1}{2}UU_{txy^2b} + U_{ytb}U_{xy}\\ &+ \frac{1}{2}U_xU_{tt^2b} - \frac{1}{2}U_xU_{tx^2} + \frac{1}{2}U_xU_{tx^2b} + \frac{1}{2}U_xU_{ty^2}\\ &+ \frac{1}{2}U_xU_{ty^2b} - xtU_{x^2b} - U_xU_{yytb} + U_{xt}U_{tt} - U_{xtb}U_{yy} - U_{xtb}U_{tt},\end{aligned}$$

$$\begin{aligned}\phi^y =&U_{tb}U_y + U_yU_t - U_yU_{tb} + \frac{1}{2}UU_{tyt^2b} + \frac{1}{2}UU_{tyx^2}\\ &- \frac{1}{2}UU_{tyx^2b} + \frac{1}{2}UU_{tyy^2} - \frac{1}{2}UU_{tyy^2b} + U_{xtb}U_{xy}\\ &- \frac{1}{2}U_yU_{tt^2b} - \frac{1}{2}U_yU_{tx^2} + dfrac12U_yU_{tx^2b} + \frac{1}{2}U_yU_{ty^2}\\ &+ \frac{1}{2}U_yU_{ty^2b} - U_{yxt}U_{xb} - U_{y^2ytb} + U_{yt}U_{tt} - U_{ytb}U_{xx} - U_{ytb}U_{tt}.\end{aligned}$$

Case 2

$$\begin{aligned}&\Lambda(x,y,t,U,U_x,U_y,U_t) = \frac{1}{-1+b}(-yU + yU_b + U_{tx^2} - 2ytU_t\\ &+ 2ytU_{tb} - 2yxU_x + 2yxU_{xb} + U_{yx^2} - U_{yx^2b} + U_{yy^2b} - U_{yt^2b} - U_{yy^2}),\end{aligned}$$

$$\begin{aligned}\phi^t =&ytU_{t^2} - U_{yx}U_{tx} - ytU_{t^2b} + U_{tyx}U_x + \frac{1}{2}U_yU_{tx^2b} - \frac{1}{2}U_yU_{ty^2b} - U_{tb}U_y -\\ &U_yU_{tb} - \frac{1}{2}UU_{tyt^2b} - \frac{1}{2}UU_{tyx^2b} + \frac{1}{2}UU_{tyy^2b} + \frac{1}{2}U_yU_{tt^2} - U_yU_t\\ &- \frac{1}{2}UU_{tyx^2} + \frac{1}{2}UU_{tyy^2} + \frac{1}{2}U_yU_{tx^2} - \frac{1}{2}U_yU_{ty^2} + U_{yx}U_{txb} - U_{tyx}U_{xb}\\ &- U_{ytb}U_{yy} + U_{tyb}U_{xx},\end{aligned}$$

$$\begin{aligned}\phi^x =&- \frac{1}{2}\frac{1}{-1+b}(2U_{by}U_x - 2U_{b^2y}U_x - 2U_bU_{yx} + 2U_{b^2}U_{yx} - U_bU_{xyx^2}\\ &+ U_{b^2}U_{xyx^2} - U_{b^2}U_{xyy^2} + U_{b^2}U_{xyt^2} + U_bU_{xyy^2} - 2U_{x^2byx} + 2U_{x^2b^2yx}\\ &+ U_{xb}U_{yx^2} - U_{xb^2}U_{yx^2} + U_{xb^2}U_{yy^2} - U_{xb^2}U_{yt^2} - U_{xb}U_{yy^2} + 2U_{yx}U_{tt}\\ &- 2U_{xbyt}U_t + 2U_{xb^2yt}U_t - 4U_{yxb}U_{tt} - 2U_{yxb}U_{yy} + 2U_{yx}U_{yyb^2} + 2U_{yx}U_{ttb^2}\\ &+ 2U_{byt}U_{tx} - 2U_{b^2yt}U_{tx}),\end{aligned}$$

$$\phi^y = -\frac{1}{2}\frac{1}{-1+b}(U_{b^2}^2 - U_b^2 - 2U_tU_{tb} - 2U_{bx}U_x - 2U_bU_{yy} - UU_{ttt^2b} - 2UU_{ttx^2b} + 2UU_{tty^2b} + 2U_tU_{tb^2} + 2U_{b^2x}U_x + 2U_{b^2}U_{yy} + UU_{ttt^2b^2} + UU_{ttx^2b^2} - UU_{tty^2b^2} - U_{y^2bx^2} + U_{y^2b^2x^2} - U_{y^2b^2y^2} + U_{y^2b^2t^2} + U_{y^2by^2} + U_{b^2}U_{xxt^2} - U_bU_{xxx^2} + U_{b^2}U_{xxx^2} + U_bU_{xxy^2} - U_{b^2}U_{xxy^2} + UU_{ttx^2} - UU_{tty^2} - 2U_{byx}U_{xy} + 2U_{b^2yx}U_{xy} + 2U_{ybyx}U_x - 2U_{yb^2yx}U_x - 2U_{yt}U_{tyb} + 2U_{yt}U_{tyb^2} + 2U_tU_{yytb} - 2U_tU_{yytb^2}).$$

Case 3

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = U + 2tU_t + 2xU_x + 2U_{yy},$$

$$\phi^t = -UU_t - U_xU_{tx} - U_yU_{ty} + U_bU_t + U_{bx}U_{tx} + U_{by}U_{ty} + tU_t + U_{tx}U_x + U_tU_{yy} - tU_{tb} - U_{tbx}U_x - U_{tb}U_{yy} - tU_bU_{xx} - tU_bU_{yy},$$

$$\phi^x = U_bU_x + U_{bt}Ut + U_{by}U_{xy} - U_{xbt}U_t - U_{x^2bx} - U_{xb}U_{yy} + U_xU_{tt} - U_{xb}U_{yy} - U_{xb}U_{tt},$$

$$\phi^y = U_bU_y + U_{bt}U_{ty} + U_{bx}U_{xy} - U_{ybt}U_t - U_{ybx}U_x - U_{yby} + yUU_{tt} - yU_bU_{xx} - yU_bU_t.$$

Case 4

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = \frac{-tbU_y - yU_t + yU_tb}{-1+b},$$

$$\phi^t = -\frac{1}{2}U_bU_y - \frac{1}{2}U_{bt}U_{ty} + \frac{1}{2}U_{ybt}U_t + \frac{1}{2}yU_{t^2} - \frac{1}{2}yU_{t^2b} - \frac{1}{2}yU_bU_{xx} - \frac{1}{2}U_{by}U_{yy},$$

$$\phi^x = \frac{1}{2}\frac{1}{-1+b}(b(-U_yU_{tx} + U_{yb}U_{tx} - U_{bt}U_y + U_{xtb}U_y + U_{xy}U_t - U_{xy}U_{tb})),$$

$$\phi^y = \frac{1}{2}\frac{1}{-1+b}(b(-UU_t + U_bU_t - U_yU_{ty} + U_{by}U_{ty} + tbU_{y^2} + U_tU_{yy} - U_{tb}U_{yy} - U_tU_{tt} + tU_bU_{xx} + U_{bt}U_{tt}.$$

Case 5

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = U_t,$$

$$\phi^t = \frac{1}{2}U_{r^2} - \frac{1}{2}U_{r^2}b - \frac{1}{2}U_bU_{xx} - \frac{1}{2}U_bU_{yy},$$

$$\phi^x = \frac{1}{2}U_bU_{tx} - \frac{1}{2}U_xU_{tb},$$

$$\phi^y = \frac{1}{2}U_bU_{ty} - \frac{1}{2}U_yU_{tb}.$$

Case 6

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = \frac{-tbU_x - xU_t + xU_tb}{-1+b},$$

$$\phi^t = -\frac{1}{2}U_bU_x - \frac{1}{2}U_{bt}U_{tx} + \frac{1}{2}U_{xbt}U_t + \frac{1}{2}xU_{t^2} - \frac{1}{2}xU_{t^2}b - \frac{1}{2}U_{bx}U_{xx} - \frac{1}{2}U_xU_{yy},$$

$$\phi^x = \frac{1}{2}\frac{1}{-1+b}(b(-UU_t + U_bU_t - U_xU_{tx} + U_{bx}U_{tx} + tbU_{x^2} + U_{tx}U_x - U_{tbx}U_x - U_tU_{tt} + tU_bU_{yy} + U_{bt}U_{tt})),$$

$$\phi^y = \frac{1}{2}\frac{1}{-1+b}(b(-U_xU_{ty} + U_{xb}U_{ty} - U_{bt}U_{xy} + U_{xtb}U_y + U_{yx}U_t - U_{yx}U_{tb})).$$

Case 7

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = yU_x - U_yx,$$

$$\phi^t = -\frac{1}{2}U_yU_{tx} + \frac{1}{2}U_{yb}U_{tx} + \frac{1}{2}U_xU_{ty} - \frac{1}{2}U_{xb}U_{ty} + \frac{1}{2}U_{xy}U_t - \frac{1}{2}U_{xy}U_{tb} - \frac{1}{2}U_{yx}U_t + \frac{1}{2}U_{yx}U_{tb},$$

$$\phi^x = -\frac{1}{2}U_bU_y - \frac{1}{2}U_{bx}U_{xy} - \frac{1}{2}U_{x^2by} + \frac{1}{2}U_{ybx}U_x + \frac{1}{2}yUU_{tt} - \frac{1}{2}U_{by}U_{yy} + \frac{1}{2}yU_bU_{tt},$$

$$\phi^y = \frac{1}{2}U_bU_x + \frac{1}{2}U_{by}U_{xy} - \frac{1}{2}U_{xb}U_{yy} - \frac{1}{2}U_{y^2bx} - \frac{1}{2}U_xU_{tt} + \frac{1}{2}U_{bx}U_{xx} + \frac{1}{2}U_{xb}U_{tt}.$$

Case 8

$$\Lambda(x,y,t,U,U_x,U_y,U_t) = U_x, \tag{5.5}$$

$$\phi^t = -\frac{1}{2}UU_{tx} + \frac{1}{2}U_bU_{tx} + \frac{1}{2}U_tU_x - \frac{1}{2}U_xU_{tb},$$

$$\phi^x = -\frac{1}{2}U_{x^2b} + \frac{1}{2}UU_{tt} - \frac{1}{2}U_bU_{yy} - \frac{1}{2}UU_{ttb},$$

$$\phi^y = -\frac{1}{2}U_bU_{xy} - \frac{1}{2}U_yU_{xb}.$$

Case 9

$$\Lambda(x, y, t, U, U_x, U_y, U_t) = U_y,$$

$$\phi^t = \frac{1}{2}(UU_{ty} - U_tU_y)(-1 + b),$$

$$\phi^x = \frac{1}{2}(UU_{xy} - U_xU_y)b,$$

$$\phi^y = -\frac{1}{2}U_{y^2b} + \frac{1}{2}UU_{tt} - \frac{1}{2}U_bU_{xx} - \frac{1}{2}UU_{ttb}.$$

Therefore, for all these cases we detected the local conservation law of Equation (1.1) as follows:

$$D_t\phi^t + D_x\phi^x + D_y\phi^y = 0.$$

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