Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **12**(1)(2023), 1-10 (Research paper)

Quasi-multipliers on $\ell-$algebras

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Abstract. In this paper we consider the notion of quasi-multipliers on an $\ell-$algebra. We prove that, for a Banach $\ell-$algebra A with an ultra approximate identity, the set $\ell QM(A)$ of all order continuous $\ell-$quasi-multipliers on A is a Banach $f-$algebra. Further, we establish the relationship between the space $\ell QM(A)$ and the space $\ell M(A)$ of all $\ell-$multipliers on A. It is shown that, for certain Banach $\ell-$algebra A, there exists a map $\varphi : \ell M(A) \to \ell QM(A)$ which is a positive, isometric and an algebraic lattice isomorphism.
Keywords: Multiplier, Quasi-multiplier, $\ell-$space, Banach $\ell-$algebra.

2000 Mathematics subject classification: 46A40; Secondary 46B42, 47B65.

1. Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [4] for C^*-algebras. McKennon [13] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m: A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

Let $QM(A)$ denote the set of all separately continuous quasi-multipliers on A. It is shown in [13] that $QM(A)$ is a Banach space for the norm

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Received: 27 March 2021
Revised: 05 May 2023
Accepted: 08 May 2023

$\|m\| = \sup\{\|m(a,b)\|; \quad a,b \in A, \|a\| = \|b\| = 1\}$. For some classical Banach algebras, the Banach space of quasi-multipliers may be identified with some other known space or algebras. For instance, by [13, Corollary of Theorem 22], one can identify $QM(L^1(G))$ with the measure algebra $M(G)$, where G is a locally compact Hausdorff group.

After McKennon's seminal paper the theory of quasi-multipliers on Banach algebras was developed further by many authors for example Kassem and Rowlands [10], Lin [12], Argün and Rowlands [7].

In [1, 3], we extended the notion of quasi-multipliers to the dual of a Banach algebra A whose second dual has a mixed identity. We considered algebras satisfying a weaker condition than Arens regularity.

In [2], we extended the notion of quasi-multipliers to complete k-normed algebras ($0 < k \leq 1$), and studied their bilinearity and joint continuity under some suitable conditions.

Multipliers on semilattices and lattices have been previously studied mainly from the point of view of interior operators by Szasz and Szendrei [16], Kolibiar [11] and Cornish [9]. Further developments have been made, among others, by Yilmaz and Rowlands [17] and Benamor[8].

The aim of this paper is to present a few new results for quasi-multipliers on $\ell-$algebras. We identify $\ell QM(A)$, where A is a Banach $\ell-$algebra, as a Banch $f-$algebra. For unexplained terminology and the basic results on $\ell-$spaces (vector lattices) we refer to [6], [14], [18].

Definition 1.1. Let A, B and C be ordered vector spaces. A mapping $q : A \times B \to C$ is said to be *positive* whenever $q(x,y) \in C^+$ holds for all $(x,y) \in A^+ \times B^+$.

Definition 1.2. Let A, B and C be ordered vector spaces. A subset D of $A \times B$ is called *order bounded* if there exist (a,b) and $(\tilde{a},\tilde{b})$ in $A \times B$ such that $(a,b) \leq (x,y) \leq (\tilde{a},\tilde{b})$ holds for all $(x,y) \in D$. A bilinear mapping $\varphi : A \times B \to C$ is said to be *order bounded* if φ maps order bounded subsets of $A \times B$ onto order bounded subsets of C. In other words, $\varphi : A \times B \to C$ is order bounded if there exist $u, v \in C$ such that

$$u \leq \varphi(x,y) \leq v$$

for all $(x,y) \in A \times B$ satisfying

$$(a,b) \leq (x,y) \leq (\tilde{a},\tilde{b})$$

for some $(a,b), (\tilde{a},\tilde{b}) \in A \times B$.

Let $\mathcal{B}(A \times B, C)$ denote the vector space of all bilinear mappings $\varphi : A \times B \to C$ and $\mathcal{B}_b(A \times B, C)$ the subset of $\mathcal{B}(A \times B, C)$ consisting of all order bounded mappings. It is not difficult to see that $\mathcal{B}_b(A \times B, C)$ is an ordered linear subspace of $\mathcal{B}(A \times B, C)$.

Definition 1.3. An $\ell-$space (or a *Riesz space* or a *vector lattice*) is an ordered vector space A with the additional property that for each pair of vectors $x, y \in A$ the supremum and the infimum of the set $\{x, y\}$ both exist in A. Following the classical notation, we shall write

$$x \vee y := \sup\{x, y\} \qquad \text{and} \qquad x \wedge y := \inf\{x, y\}.$$

Definition 1.4. A net $\{x_\alpha\}$ in an $\ell-$space A is *order convergent* to some vector $x \in A$, denoted $x_\alpha \xrightarrow{o} x$, whenever there exists another net $\{y_\alpha\} \subseteq A$ with the same index set satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$.

Definition 1.5. An operator $T : E \to F$ between two $\ell-$spaces is said to be *order continuous*, if $x_\alpha \xrightarrow{o} 0$ in E implies $T(x_\alpha) \xrightarrow{o} 0$ in F.

Definition 1.6. An $\ell-$space A is called *Dedekind complete* (or *order complete*) if for every nonempty subset D of A that is ordered bounded in A, $\sup D$ and $\inf D$ both exist and are elements of A.

Theorem 1.7. *Let A, B and C be $\ell-$spaces, with C Dedekind complete. For every $\varphi \in \mathcal{B}(A \times B, C)$ and $(x, y) \in A \times B$, the following statements hold:*
(1) *Every positive bilinear mapping $\varphi : A \times B \to C$ is order bounded.*
(2) *A bilinear mapping $\varphi : A \times B \to C$ is order bounded if and only if there exist positive bilinear mappings $\varphi_1, \varphi_2 : A \times B \to C$ such that $\varphi = \varphi_1 - \varphi_2$. In the usual notation, we write $\varphi_1 = \varphi^+$ and $\varphi_2 = \varphi^-$, and so $\varphi = \varphi^+ - \varphi^-$ holds in $\mathcal{B}_b(A \times B, C)$.*
(3) $\varphi^+(x, y) = \bigvee_{\substack{0 \leq a \leq x \\ 0 \leq b \leq y}} \varphi(a, b)$.
(4) $\varphi^-(x, y) = \bigvee_{\substack{0 \leq a \leq x \\ 0 \leq b \leq y}} -\varphi(a, b)$.
(5) $|\varphi(x, y)| \leq |\varphi|(x, y)$.
(6) $|\varphi(x, y)| \leq |\varphi|(|x|, |y|)$.
(7) $|\varphi|(x, y) = \bigvee_{\substack{|a| \leq x \\ |b| \leq y}} \varphi(a, b) = \bigvee_{\substack{|a| \leq x \\ |b| \leq y}} |\varphi(a, b)|$.

Definition 1.8.
(a) The real algebra A is called an $\ell-$*algebra* or (a *lattice-ordered algebra*) if A is an $\ell-$space such that $ab \in A$ whenever a, b are positive elements in A.
(b) An $\ell-$algebra A is called an $f-$*algebra* if A satisfies the condition that $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $0 \leq c \in A$.
(c) An $\ell-$algebra A is called a $d-$*algebra* if $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$ for all $a, b \in A$ and $c \in A^+$.

2. Main results

Definition 2.1. For an algebra A, a mapping $q : A \times A \to A$ is said to be a *quasi-multiplier* on A if it satisfies

$$q(xy, zt) = x\, q(y, z)\, t \quad \text{for all } x, y, z, t \in A.$$

Definition 2.2. For an $\ell-$algebra A, a quasi-multiplier $q : A \times A \to A$ is said to be an $\ell-$*quasi-multiplier* on A if it satisfies

$$q(x \wedge y, z \wedge t) = x \wedge q(y, z) \wedge t \quad \text{and} \quad q(x \vee y, z \vee t) = x \vee q(y, z) \vee t \quad (2.1)$$

for all $x, y, z, t \in A$.

Let $\ell QM(A)$ denote the set of all bilinear and order continuous $\ell-$quasi-multipliers on A.

Proposition 2.3. *If A is a $d-$algebra, then $\ell QM(A)$ is a lattice.*

Proof. We show that for all $p, q \in \ell QM(A)$, $p \vee q$ exists in $\ell QM(A)$ under the ordering

$$p \le q \iff \forall x, y \in A,\ p(x, y) \le q(x, y),$$

and

$$(p \vee q)(x, y) = p(x, y) \vee q(x, y) \quad , \quad (p \wedge q)(x, y) = p(x, y) \wedge q(x, y).$$

Let $x, t \in A^+$ and $y, z \in A$. Then

$$\begin{aligned} (p \vee q)(xy, zt) &= p(xy, zt) \vee q(xy, zt) \\ &= x\, p(y, z)\, t \ \vee \ x\, q(y, z)\, t \ (\text{since } p, q \in QM(A)) \\ &= x\, (p(y, z)\, t \ \vee \ q(y, z)\, t) \ (\text{since } A \text{ is a } d-\text{algebra}) \\ &= x\, (p(y, z) \ \vee \ q(y, z))\, t \\ &= x\, ((p \vee q)(y, z))\, t. \end{aligned}$$

i.e., $p \vee q \in QM(A)$.
Moreover,

$$\begin{aligned} (p \vee q)(x \wedge y, z \wedge t) &= p(x \wedge y, z \wedge t) \vee q(x \wedge y, z \wedge t) \\ &= (x \wedge p(y, z) \wedge t) \ \vee \ (x \wedge q(y, z) \wedge t) \ (\text{since } p, q \text{ satisfying } 2.1) \\ &= x \wedge [(p(y, z) \wedge t) \ \vee \ (q(y, z) \wedge t)] \ (\text{since } A \text{ is a distributive lattice}) \\ &= x \wedge [(p(y, z) \ \vee \ q(y, z)) \wedge t] \\ &= x \wedge [(p(y, z) \ \vee \ q(y, z)] \wedge t. \end{aligned}$$

It is easy to see that $p \vee q$ is order continuous as well. Hence $p \vee q \in \ell QM(A)$. A similar reasoning gives $p \wedge q \in \ell QM(A)$. □

Definition 2.4. Let A be a Banach algebra. A bounded approximate identity $\{e_\alpha : \alpha \in I\}$ in A is said to be an *ultra approximate identity* if, for all $m \in QM(A)$ and $a \in A$, the nets $\{m(a, e_\alpha) : \alpha \in I\}$ and $\{m(e_\alpha, a) : \alpha \in I\}$ are Cauchy ([13], p. 110).

If A is a Banach algebra with a minimal ultra approximate identity $\{e_\alpha : \alpha \in I\}$, then the equation

$$(p \odot q)(x, y) := p(x, \lim q(e_\alpha, y))$$

defines a multiplication in $QM(A)$ and $QM(A)$ becomes a Banach algebra ([7], p. 219).

Definition 2.5. Let A be an $\ell-$algebra. An ultra approximate identity $\{e_\alpha : \alpha \in I\}$ in A is said to be an *$\ell-$ultra approximate identity* if it satisfies

$$\lim_\alpha (e_\alpha \wedge x) = \lim_\alpha (x \wedge e_\alpha) = x \quad \text{for all } x \in A.$$

Proposition 2.6. *If A is a Banach $d-$algebra with an $\ell-$ultra approximate identity $\{e_\alpha : \alpha \in I\}$, then $\ell QM(A)$ is a Banach $f-$algebra.*

Proof. By the Proposition 2.3, $\ell QM(A)$ is a lattice and it is not diffucult to see that under the norm $|||p||| = \sup\{||p(x, y)|| : x, y \in A, ||x|| = ||y|| = 1\}$, $\ell QM(A)$ is a Banach $\ell-$algebra. In fact, let $p, q \in \ell QM(A)$. Then:

$$\begin{aligned}
(p \odot q)(x \wedge y, z \wedge t) &= p(x \wedge y, \lim_\alpha q(e_\alpha, z \wedge t)) \\
&= x \wedge p(y, \lim_\alpha q(e_\alpha, z \wedge t)) \text{ (since } p \in \ell QM(A)) \\
&= x \wedge p(y, \lim_\alpha (q(e_\alpha, z) \wedge t))) \text{ (since } q \in \ell QM(A)) \\
&= x \wedge p(y, \lim_\alpha q(e_\alpha, z) \wedge t) \text{ (since meet and join are continuous)} \\
&= x \wedge p(y, \lim_\alpha q(e_\alpha, z)) \wedge t \\
&= x \wedge (p \odot q)(y, z) \wedge t.
\end{aligned} \tag{2.2}$$

Now, we show that $\ell QM(A)$ is an $\ell-$algebra, as follows.
Let $p, q \in (\ell QM(A))^+$. We prove that $(p \odot q) \in (\ell QM(A))^+$. So let $(x, y) \in A^+ \times A^+$. As $q \in (\ell QM(A))^+$, we have $q(e_\alpha, y) \geq 0$. Also $p \in (\ell QM(A))^+$, then

$$(p \odot q)(x, y) = p(x, \lim_\alpha q(e_\alpha, y)) \geq 0.$$

To see that $\ell QM(A)$ is an $f-$algebra, let $p, q \in \ell QM(A)$ with $p \wedge q = 0$, and let $0 \leq h \in \ell QM(A)$. If $(x, y) \in (A \times A)^+$, then $p(x, y) \wedge q(x, y) = [p \wedge q](x, y) = 0$ implies $[(h \odot p) \wedge q](x, y) = (h \odot p)(x, y) \wedge q(x, y) = 0$, and so $(h \odot p) \wedge q = 0$. On the other hand, if $h_n = h \wedge nI$, then

$h_n(x,y) \uparrow h(x,y)$ holds for all $(x,y) \in (A\times A)^+$, and so (by the order continuity of p) it follows that $(p\odot h_n) \uparrow (p\odot h) \in \ell QM(A)$. Therefore, $(p\odot h_n)\wedge q \uparrow (p\odot h)\wedge q$ likewise holds in $\ell QM(A)$. Now since $p\wedge q = 0$ and $0 \leq (p\odot h_n) \leq np$, we see that $(p\odot h_n)\wedge q = 0$ holds for all n, and therefore $(p\odot h)\wedge q = 0$. □

Definition 2.7. For an algebra A, a map $T : A \to A$ is said to be a *multiplier* on A if it satisfies

$$T(ab) = aT(b) = T(a)b \quad \text{for all } a,b \in A.$$

Definition 2.8. For an $\ell-$algebra A, a multiplier $T : A \to A$ is said to be an $\ell-$*multiplier* on A if it satisfies

$$T(a\wedge b) = T(a)\wedge b \quad \text{for all } a,b \in A. \tag{2.3}$$

Clearly,

$$T(a\wedge b) = T(b\wedge a) = T(b)\wedge a = a\wedge T(b) \quad \text{for all } a,b \in A.$$

The space of all bilinear and order continuous $\ell-$multipliers on A is denoted by $\ell M(A)$. It is obvious that for each $a \in A$, the left multiplication operator $\xi_a(x) = a\wedge x$ is an $\ell-$multiplier on A. If A has an identity, then each $\ell-$multiplier on A is a left multiplication operator. Indeed, let e be an identity for A and $T \in \ell M(A)$ be arbitrary. Then equalities

$$T(x) = T(e\wedge x) = T(e)\wedge x = \xi_{T(e)}(x)$$

hold for $x \in A$, which means $T = \xi_{T(e)}$.
For arbitrary lattices, essentially nothing is known concerning $\ell-$multipliers; but for faithful lattices a considerable number of their properties are readily deduced. A lattice A is *faithful* if for all $x \in A$, $x\wedge A = \{0\}$ implies $x = 0$. Obviously if A has an identity, it is faithful.

Proposition 2.9. *Let A be a faithful $d-$algebra. Then each $\ell-$multiplier on A is a lattice homomorphism on A.*

Proof. Let $T \in \ell M(A)$. Then for any $x,y,z \in A$,

$$\begin{aligned} T(x\vee y)\wedge z &= (x\vee y)\wedge T(z) = T(z)\wedge(x\vee y) \text{ (since } T \text{ is an } \ell-\text{multiplier)}\\ &= (T(z)\wedge x)\vee(T(z)\wedge y)\\ &= (z\wedge T(x))\vee(z\wedge T(y))\\ &= (T(x)\wedge z)\vee(T(y)\wedge z)\\ &= (T(x)\vee T(y))\wedge z. \end{aligned} \tag{2.4}$$

By faithfulness of A, $T(x\vee y) = T(x)\vee T(y)$.
Now, let us show that $T(x\wedge y) = T(x)\wedge T(y)$. Indeed, let $T \in \ell M(A)$

and $x \in A$, then

$$\begin{aligned} T^2(x) &= T(T(x \wedge x)) = T(x \wedge T(x)) \\ &= T(x) \wedge T(x) = T(x), \end{aligned} \tag{2.5}$$

i.e., $T^2 = T$. So

$$\begin{aligned} T(x \wedge y) &= T(T(x \wedge y)) = T(x \wedge T(y)) \\ &= T(x) \wedge T(y). \end{aligned} \tag{2.6}$$

□

Proposition 2.10.
(i) *If A is a (distributive) $d-$algebra then $(\ell M(A), \vee, \wedge)$ is a distributive lattice with the identity function I as its unit and*

$$(T_1 \wedge T_2)(x) = T_1(x) \wedge T_2(x) \quad (x \in A),$$

$$(T_1 \vee T_2)(x) = T_1(x) \vee T_2(x) \quad (x \in A).$$

(ii) *If A is a faithful $d-$algebra, then*

$$\sigma(a) = \xi_a \quad (a \in A)$$

defines a lattice isomorphism $\sigma : A \to \ell M(A)$.

Proof.
(i) $\ell M(A)$ is evidently closed under $\vee$ and $\wedge$, and because these operations are defined pointwise, it follows that $\ell M(A)$ is itself a distributive lattice.
(ii) Each $\ell-$multiplier on A is a left multiplication operator, So σ is clearly onto.
Let $a \in A$ be arbitrary such that $\sigma(a) = 0$. Then for each $x \in A$,

$$\sigma(a)(x) = \xi_a(x) = a \wedge x = 0.$$

Faithfulness of A concludes that $a = 0$; that is σ is one to one.
Moreover, σ is a lattice homomorphism, as follows.

$$\sigma(a \wedge b)(x) = \xi_{(a\wedge b)}(x) = (a \wedge b) \wedge x = (a \wedge x) \wedge (b \wedge x) = \sigma(a)(x) \wedge \sigma(a)(x)$$

$$\sigma(a \vee b)(x) = \xi_{(a\vee b)}(x) = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = \sigma(a)(x) \vee \sigma(a)(x)$$

for all $a, b, x \in A$. □

Theorem 2.11. *If A is a $d-$algebra with a minimal $\ell-$ultra approximate identity $\{e_\alpha : \alpha \in I\}$, then*

$$\varphi_T(x, y) = x \wedge T(y) \qquad (T \in \ell M(A), x, y \in A)$$

defines an isometric and algebraic lattice isomorphism $\varphi : \ell M(A) \to \ell QM(A)$. Moreover, φ is positive if A is Dedekind complete.

Proof. A simple computation shows that $\varphi_T(x\wedge y, z\wedge t) = x\wedge\varphi_T(y,z)\wedge t$, so $\varphi_T \in \ell QM(A)$. Let $m \in \ell QM(A)$ be arbitrary. Define

$$T(x) = \lim_\alpha m(e_\alpha, x) \ (x \in A).$$

It is obvious that $T \in \ell M(A)$. Since m is continuous, for any $x, y \in A$, we have

$$\begin{aligned} m(x,y) &= m(\lim_\alpha (x\wedge e_\alpha), y) = \lim_\alpha m(x\wedge e_\alpha, y) \\ &= x\wedge \lim_\alpha m(e_\alpha, y) = x\wedge T(y) = \varphi_T(x,y). \end{aligned} \tag{2.7}$$

This means that φ is onto. Next, we prove that φ is isometry. Let $T \in \ell M(A)$ and $\epsilon > 0$ be arbitrary. If $x \in A$ is such that $||x|| \leq 1$ and $||T|| - \epsilon < ||T(x)||$, then

$$\begin{aligned} ||\varphi_T|| &\geq \lim_\alpha ||\varphi_T(e_\alpha \wedge x)|| = \lim_\alpha ||e_\alpha \wedge T(x)|| \\ &= ||T(x)|| > ||T|| - \epsilon. \end{aligned} \tag{2.8}$$

Thus φ is an isometry.
We check the multiplicativity of φ : for any $T_1, T_2 \in \ell M(A)$ and $x, y \in A$,

$$\begin{aligned} (\varphi_{T_1} \odot \varphi_{T_2})(x,y) &= \varphi_{T_1}(x, \lim_\alpha \varphi_{T_2}(e_\alpha, y)) \\ &= x\wedge T_1(\lim_\alpha \varphi_{T_2}(e_\alpha, y)) \\ &= T_1(x\wedge \lim_\alpha \varphi_{T_2}(e_\alpha, y)) \\ &= T_1(\lim_\alpha (x\wedge e_\alpha)\wedge T_2(y)) \\ &= T_1(x\wedge T_2(y)) \\ &= x\wedge T_1(T_2(y)) \\ &= \varphi_{T_1T_2}(x,y), \end{aligned} \tag{2.9}$$

which implies that φ is an algebraic homomorphism.
We note that φ is a lattice homomorphism, as follows.

$$\begin{aligned} \varphi_{T_1\vee T_2}(x,y) &= x\wedge (T_1\vee T_2)(y) = x\wedge (T_1(y)\vee T_2(y)) \\ &= (x\wedge T_1(y))\vee (x\wedge T_2(y)) = (\varphi_{T_1}\vee \varphi_{T_2})(x,y). \end{aligned} \tag{2.10}$$

In the similar way we have $\varphi_{T_1\wedge T_2}(x,y) = (\varphi_{T_1}\wedge \varphi_{T_2)}(x,y)$. Thus φ is a lattice homomorphism. Finally, if A is Dedekind complete, then by

Theorem 1.7

$$
\begin{aligned}
(\varphi_T)^+(x,y) &= \bigvee_{\substack{0\le a\le x\\ 0\le b\le y}} \varphi_T(a,b) = \bigvee_{\substack{0\le a\le x\\ 0\le b\le y}} (a\wedge T(b))\\
&= \bigvee_{0\le a\le x} a \ \wedge \bigvee_{0\le b\le y} T(b) = x\wedge T^+(y) \quad [[5], \text{Theorem } 3.3]\\
&= \varphi_{T^+}(x,y).
\end{aligned}
\tag{2.11}
$$

□

Remark. If A is a Dedekind complete lattice with a minimal $\ell-$ultra approximate identity, then Theorem 2.11 allows a natural definition of multiplication in $\ell QM(A)$. Namely, for arbitrary $m_1, m_2 \in \ell QM(A)$, let $T_1, T_2 \in \ell M(A)$ be uniquely determined multipliers satisfying $m_1 = \varphi_{T_1}$ and $m_2 = \varphi_{T_2}$. Then

$$m_1 \odot_\varphi \ m_2 = \varphi_{T_1} \odot_\varphi \varphi_{T_2} := \varphi_{T_1T_2}$$

gives a well defined multiplication in $\ell QM(A)$.

Acknowledgements: The author is very grateful to the referee for his helpful comments and suggestions.

References

[1] M. Adib, Arens regularity of quasi-multipliers, University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics **80** (2018), 199-206.

[2] M. Adib and A. Riazi and L. A. Khan, Quasi-multipliers on F-algebras, Abstr. Appl. Anal, Article ID 235273, (2011), 1-30.

[3] M. Adib and A. Riazi and J. Bracic, Quasi-multipliers of the dual of a Banach algebra, Banach J. Math. Anal. **5** (2011), 6-14.

[4] C. A. Akemann and G. K. Pedersen, *Complications of semicontinuity in C^*-algebra theory*, Duke Math. J. **40** (1973), 785–795.

[5] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces*, Academic Press, 1978.

[6] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, 1985.

[7] Z. Argün and K. Rowlands, *On quasi-multipliers*, Studia Math. **108** (1994), 217–245.

[8] F. Benamor, *On bi-orthomorphisms on a semiprime $f-$algebra*, Indagationes mathematica. **25** (2014), 44–48.

[9] W. H. Cornish, *The multiplier extension of a distributive lattice*, Journal Of Algebra. **32** (1974), 339–355.

[10] M. S. Kassem and K. Rowlands, *The quasi-strict topology on the space of quasi-multipliers of a B^*-algebra*, Math. Proc. Cambridge Philos. Soc. **101** (1987), 555–566.

[11] M. Kolibiar, *Bermerkungen uber translation der Verbande*, Acta Fac Rerum Natur. Univ. comenian Math. **5** (1961), 455-458.

[12] H. Lin, *The structure of quasi-multipliers of C^*-algebras*, Trans. Amer. Math. Soc. **315** (1987), 147–172.
[13] M. McKennon, *Quasi-multipliers*, Trans. Amer. Math. Soc. **233** (1977), 105–123.
[14] H. H. Schaefer, *Banach Lattices and Positive Operators*, , Springer, Berlin, 1974.
[15] G. Szasz, *Die Translationen der HalbverbZlnde*, Acta Sci. Math. (Szeged). **17** (1956), 165-169.
[16] G. Szasz and J. Szendrei, *Uber der Translation der Haibverbgnde*, Acta Sci. Math. (Szeged). **18** (1957), 44-47.
[17] R. Yilmaz and K. Rowlands, *On orthomorphisms, quasi-orthomorphisms and quasi-multipliers*, J. Math. Anal. Appl. **313** (2006), 120–131.
[18] A. C. Zaanen, *Riesz Spaces II*, North- Holland, Amsterdam (1983).