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On symmetries of the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. The purpose of this article is to study the symmetries of the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$. Specially, we study the existence of Ricci and matter collineations in this space.

Keywords: Pseudo-Riemannian metric, Killing and affine vector field, Ricci and matter collineation.

2000 Mathematics subject classification: 58D17, 53B30.

1. Introduction

The study of symmetries in general relativity has long been considered due to they are interesting both from the mathematical and the physical point of view (see for example [7]). The symmetry is a one-parameter group of diffeomorphisms of the pseudo-Riemannian manifold (M, g), which leaves a special mathematical or physical quantity invariant. This statement is equivalent to the Lie derivative of the geometry quantity under the vector field X vanishes, i.e., $\mathcal{L}_X \mathcal{S} = 0$. If $\mathcal{S}$ has geometrical or physical significance, then those special vector fields under which $\mathcal{S}$ is invariant will also be of significance. Isometries, homotheties, and conformal motions are well-known examples of symmetries. Recently, other types of symmetries including curvature collineations ($\mathcal{S} = \mathcal{R}$ being the

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curvature tensor), Ricci collineations ($\mathcal{S} = \rho$ being the Ricci tensor), and etc., have been studied. Some examples may be found in [1, 2].

On the pseudo-Riemannian manifold (M, g) a matter collineation is a vector field X, which preserves the energy-momentum tensor $\mathcal{S} = \rho - \frac{\tau}{2}g$, where τ displays the scalar curvature. Since the Ricci tensor is constructed from the connection of the metric tensor, Ricci collineations have geometrical importance [8]. However, matter collineations are more related to a physical viewpoint [4, 5]. These physical and geometric concepts give a single meaning in a particular case, for example, when the meter tensor has a zero scalar curvature.

The concept of product manifolds plays very important roles in differential geometry and mathematical physics. In the meantime, the product of the real line spaces of constant Gaussian curvature is of special importance. Hence, in this article, we study symmetries of the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$. We present a complete classification of its Ricci and matter collineations. Obviously, any Killing vector field (respectively, any affine vector field and curvature collineation) is an affine vector field (respectively, any curvature and Ricci collineation) but the inverse is always not true. Also, a homothetic vector field (i.e., a vector field that holds in relation $\mathcal{L}_X g = \kappa g$, where κ is a real number) is a Ricci collineation and so is a Yambe soliton with constant curvature [3]. Therefore, we analyze the existence of proper Ricci and curvature collineations, which are not Killing and homothetic. So, we also require to determine which are the killing, affine and homothetic vector fields, which is an interresting problem on its own, due to the natural geometric meaning of such symmetrics. $Maple16^{©}$ is used to check all computations.

2. The pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$

Assume $\mathbb{H}^2$ be expressed by the upper half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric $g_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$. Therefore, the left-invariant product metric on the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$ is given by

$$g = \frac{1}{y^2}(dx^2 + dy^2) - dz^2. \tag{2.1}$$

We will denote by ∇ the Levi-Civita connection of $(\mathbb{H}^2 \times \mathbb{R}, g)$, by $\mathcal{R}$ its curvature tensor, taken with the signed contract $\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and by ρ the Ricci tensor of $(\mathbb{H}^2 \times \mathbb{R}, g)$, which is defined by $\rho(X, Y) = \mathrm{tr}\{Z \mapsto \mathcal{R}(Z, X)Y\}$. The Ricci operator Ric is given by $\rho(X, Y) = g(Ric(X), Y)$ and the scalar curvature $\tau = \mathrm{tr}_g\, \rho$ is the metric trace of the Ricci tensor.

The non-zero components of the Levi-Civita connection ∇ of the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$ are given by

$$\nabla_{\partial_x}\partial_x = \tfrac{1}{y}\partial_y, \quad \nabla_{\partial_x}\partial_y = -\tfrac{1}{y}\partial_x, \quad \nabla_{\partial_y}\partial_x = -\tfrac{1}{y}\partial_x, \quad \nabla_{\partial_y}\partial_y = -\tfrac{1}{y}\partial_y. \tag{2.2}$$

The non-zero component of the curvature tensor $\mathcal{R}$ is given by

$$\mathcal{R}(\partial_x, \partial_y)\partial_x = \tfrac{1}{y^2}\partial_y, \quad \mathcal{R}(\partial_x, \partial_y)\partial_y = -\tfrac{1}{y^2}\partial_x, \tag{2.3}$$

and the non-zero components of the Ricci tensor are $\rho_{11} = \rho_{22} = -\frac{1}{y^2}$.

3. Symmetries of $\mathbb{H}^2 \times \mathbb{R}$

A conformal Killing vector field X with the conformal function ψ satisfying $\mathcal{L}_X g = 2\psi g$ which reduces to homothetic or Killing vector field when ψ is non-zero constant or zero constant respectively. A vector field X is called proper conformal Killing vector field if ψ is non-constant. We begin by examining the existence of the Killing, homothetic, and a proper conformal Killing vector fields on $(\mathbb{H}^2 \times \mathbb{R}, g)$. In general, we have the following theorem.

Theorem 3.1. *Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary vector field and ψ be a smooth function on the pseudo-Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R}, g)$. Then*

(i) *X is a Killing vector field if and only if*

$$X^1 = \frac{1}{2}c_1(x^2 - y^2) + c_2 x + c_3, \; X^2 = (c_1 x + c_2)y, \; X^3 = c_4.$$

(ii) *X is a homothetic, non-Killing vector field if and only if*

$$X^1 = \frac{1}{2}\kappa(1 + \ln(y))x + c_1 y^2 z + f_1(x) + f_2(y),$$
$$X^2 = (\frac{1}{2}\kappa\ln(y) + f_1^{'}(x))y, \; X^3 = \frac{1}{2}\kappa z + c_1 x + c_2,$$

where $\kappa \neq 0$ is a real constant and f_1, f_2 are smooth functions on $\mathbb{H}^2 \times \mathbb{R}$, satisfying

$$(2f_1^{''}(x) + 4c_1 z)y^2 + 2y f_2^{'}(y) + \kappa x = 0.$$

(iii) *X is a proper conformal Killing vector field if and only if*

$$X^1 = y(2c_1 x + c_2)e^z - y(2c_3 x + c_4)e^{-z} + \frac{1}{2}c_5(x^2 - y^2) + c_6 x + c_7,$$
$$X^2 = (c_1(y^2 - x^2) - c_2 x - c_8)e^z + (c_3(x^2 - y^2) + c_4 x + c_9)e^{-z} + y(c_5 x + c_6),$$
$$X^3 = \frac{1}{y}(c_1(x^2 + y^2) + c_2 x + c_8)e^z + (c_3(x^2 + y^2) + c_4 x + c_9)e^{-z} + c_{10} y,$$

and ψ is given by

$$\psi(x,y,z)=\frac{1}{y}((c_1(x^2+y^2)+c_2x+c_8)e^z-(c_3(x^2+y^2)-c_4x-c_9)e^{-z}).$$

In the above expressions, c_i is an arbitrary real number, for any indices i.

Proof. A straightforward computation shows that the Lie derivative of g is given by

$$\begin{aligned}\mathcal{L}_Xg&=\frac{2}{y^3}(y\partial_xX^1-X^2)dxdx+\frac{2}{y^2}(\partial_xX^2+\partial_yX^1)dxdy\\&-\frac{2}{y^2}(y^2\partial_xX^3-\partial_zX^1)dxdz+\frac{2}{y^3}(y\partial_yX^2-X^2)dydy\\&-\frac{2}{y^2}(y^2\partial_yX^3-\partial_zX^2)dydz-2\partial_zX^3dzdz.\end{aligned}$$

By putting all the coefficients of the $\mathcal{L}_Xg$ equivalent to zero and solving the system of partial differential equations, the Killing vector fields are obtained which gives the case (i).

Now, we need to put $\mathcal{L}_Xg=\kappa g$, for any real constant $\kappa\neq 0$. Solving the corresponding system of partial differential equations gives the homothetic vector fields as in the case (ii).

Next, assume ψ be a smooth function on the pseudo-Riemannian manifold $(\mathbb{H}^2\times\mathbb{R},g)$. Then, X holds $\mathcal{L}_Xg=2\psi g$ if and only if:

$$\begin{aligned}&y^2\partial_xX^3-\partial_zX^1=0,\ y^2\partial_yX^3-\partial_zX^2=0,\ \partial_xX^2+\partial_yX^1=0,\\&\psi(x,y,z)y-y\partial_xX^1+X^2=0,\ \psi(x,y,z)y-y\partial_yX^2+X^2=0,\\&\psi(x,y,z)-\partial_zX^3=0.\end{aligned}$$

The solutions to the above system give case (iii) and end the proof. □

Note that the set of all isometries of M forms a group under the composition of mapping. In relativity, groups of isometries are known as isometric symmetry groups. In general, symmetry groups arise as groups of transformations of M or local G-transformation groups acting on M which have some special property respect to a geometric object on M. A physical example is the isometry of a spacetime such that g is invariant in time, that is, $\mathcal{L}_Tg=0$, where we take a time coordinate t for which $T=\partial_t$. Such spaces are called static spacetimes [7].

Now, we classify affine vector fields of $(\mathbb{H}^2\times\mathbb{R},g)$. The results are reported in the following theorem.

Theorem 3.2. *Assume $X=X^1\partial_x+X^2\partial_y+X^3\partial_z$ be an arbitrary vector field on the pseudo-Riemannian manifold $(\mathbb{H}^2\times\mathbb{R},g)$. Then X is* *an affine, non-Killing vector field if and only if*

$$X^1 = c_1(y^2 - x^2) + c_2 x + c_3,\ \ X^2 = (-2c_1 x + c_2)y,\ \ X^3 = c_4 z + c_5.$$

where c_i is an arbitrary real number, for any indices i.

Proof. To obtain the affine vector fields, we require to compute the Lie derivative of the Levi-Civita connection ∇. Using

$$\mathcal{L}_X \nabla(Y, Z) = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + \mathcal{R}(X, Y)Z,$$

and (2.2) we prove that the components of the $\mathcal{L}_X \nabla$ are given by

$$\begin{aligned}
\mathcal{L}_X \nabla(\partial_x, \partial_x) &= -\frac{1}{y}(2\partial_x X^2 - y\partial^2_{xx} X^1 + \partial_y X^1)\partial_x \\
&\quad + \frac{1}{y^2}(2y\partial_x X^1 + y^2\partial^2_{xx} X^2 - y\partial_y X^2 - X^2)\partial_y \\
&\quad + \frac{1}{y}(y\partial^2_{xx} X^3 - \partial_y X^3)\partial_z, \\
\mathcal{L}_X \nabla(\partial_x, \partial_y) &= \frac{1}{y^2}(-y\partial_y X^2 + y^2\partial^2_{yx} X^1 + X^2)\partial_x + \frac{1}{y}(\partial_y X^1 + y\partial^2_{yx} X^2)\partial_y \\
&\quad + \frac{1}{y}(\partial_x X^3 + y\partial^2_{yx} X^3)\partial_z, \\
\mathcal{L}_X \nabla(\partial_x, \partial_z) &= -\frac{1}{y}(\partial_z X^2 - y\partial^2_{yx} X^1)\partial_x + \frac{1}{y}(\partial_z X^1 + y\partial^2_{zx} X^2)\partial_y + \partial^2_{zx} X^3 \partial_z, \\
\mathcal{L}_X \nabla(\partial_y, \partial_y) &= \frac{1}{y}(-\partial_y X^1 + y\partial^2_{yy} X^1)\partial_x + \frac{1}{y^2}(-y\partial_y X^2 + y^2\partial^2_{yy} X^2 + X^2)\partial_y \\
&\quad + \frac{1}{y}(y\partial^2_{yy} X^3 + \partial_y X^3)\partial_z, \\
\mathcal{L}_X \nabla(\partial_y, \partial_z) &= \frac{1}{y}(-\partial_z X^1 + y\partial^2_{zy} X^1)\partial_x - \frac{1}{y}(\partial_z X^2 + y\partial^2_{zy} X^2)\partial_y + \partial^2_{zy} X^3 \partial_z, \\
\mathcal{L}_X \nabla(\partial_z, \partial_z) &= \partial^2_{zz} X^1 \partial_x + \partial^2_{zz} X^2 \partial_y + \partial^2_{zz} X^3 \partial_z.
\end{aligned}$$

Now, it suffices to put the coefficients of the above Lie derivative of the Levi-Civita connection ∇ and solve the corresponding system of partial differential equations to obtain the affine vector fields.

□

Next, we will focus on symmetries of $(\mathbb{H}^2 \times \mathbb{R}, g)$ relative to curvature. The results are reported in the following theorem.

Theorem 3.3. *Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary vector field on the pseudo-Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R}, g)$. Then*

(i) *X is a Ricci collineation if and only if X^3 is arbitrary and*

$$X^1 = \frac{1}{2}c_1(x^2 - y^2) + c_2 x + c_3,\ \ X^2 = (c_1 x + c_2)y.$$

(ii) *X is a curvature collineation if and only if*

$$X^1 = \frac{1}{2}c_1(x^2 - y^2) + c_2 x + c_3, \;\; X^2 = (c_1 x + c_2)y, \;\; X^3 = f(z),$$

where $f(z)$ is an arbitrary smooth function on $\mathbb{H}^2 \times \mathbb{R}$.

Proof. The Lie derivative of the Ricci tensor in the direction X is determined by

$$(\mathcal{L}_X \rho) = -\frac{2}{y^3}(y\partial_x X^1 - X^2)dxdx - \frac{2}{y^2}(\partial_x X^2 + \partial_y X^1)dxdy$$
$$- \frac{2}{y^2}\partial_z X^1 dxdz - \frac{2}{y^3}(y\partial_y X^2 - X^2)dydy - \frac{2}{y^2}\partial_z X^2 dydz.$$

To determine the Ricci collineations, we need to put the coefficients of the $\mathcal{L}_X\rho$ equivalent to zero and solve the system of partial differential equations. The solutions to this system give case (i).

Next, we investigate curvature collineations, beginning from an arbitrary Ricci collineation and apply the extra condition $\mathcal{L}_X\mathcal{R} = 0$. Thus

$$X = \frac{1}{2}c_1(x^2 - y^2)\partial_x + (c_1 x + c_2)y\partial_y + X^3\partial_z,$$

is also a curvature collineation if and only if

$$\partial_x X^3 = \partial_y X^3 = 0,$$

which gives the result case (ii). □

Now, we classify matter collineations on the pseudo-Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R}, g)$.

Theorem 3.4. *Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary smooth vector field on the pseudo-Riemannian manifold $(\mathbb{H}^2 \times \mathbb{R}, g)$. Then, X is a matter collineation if and only if X^1, X^2 are arbitrary and $X^3 = c$, where c is a real constant.*

Proof. A straightforward computation displays that only the non-zero component of the tensor field $\mathcal{S}$ is $\mathcal{S}(\partial_z, \partial_z) = -1$. Now, we compute the Lie derivative of the tensor field $\mathcal{S}$. We have

$$\mathcal{L}_X\mathcal{S} = -2\partial_x X^3 dxdz - 2\partial_y X^3 dydz - 2\partial_z X^3 dzdz.$$

Requiring that $\mathcal{L}_X\mathcal{S} = 0$. So, we attain the system of partial differential equations, which solutions specify the matter collineations of $(\mathbb{H}^2 \times \mathbb{R}, g)$. Thus, X is a matter collineation if and only if X^1, X^2 are arbitrary and X^3 is a real constant and this completes the proof. □

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