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Extensions of a common fixed point theorem of Jungck in probabilistic Banach spaces

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Abstract. In this paper, we extend and improve a common fixed point theorem of G. Jungck. We utilize the notions of weakly commuting and compatible mappings in probabilistic Banach spaces to prove some common fixed point theorems for improved type Jungck contractions. In addition, we present some examples which support our theorems.

Keywords: Common fixed point, probabilistic Banach space, weakly commuting mapping, compatible mapping.

2000 Mathematics subject classification: Primary 47H10, 47H09; Secondary 46S50.

1. Introduction and preliminaries

G. Jungck in 1976 [9], by using the concept of commuting mappings as a tool extended the well-known Banach contraction principle. Jungck proved one of the most classical theorems in common fixed point theory. He showed that for two commuting self-mappings T and S on complete metric space (X,d), if there exists $\kappa \in [0,1)$ such that

$$d(Sx, Sy) \leq \kappa d(Tx, Ty).$$

Then under certain conditions, T and S have a unique common fixed point. Note that, Banach contraction principle can be derived if $T = I$.

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In later years, many authors have generalized this theorem, for instance see [5, 11, 12, 17]. Sessa in 1982 [21] introduced weakly commuting mappings and Jungck in 1986 [10] generalized concept of weakly commuting mappings, he introduced compatible mappings. Then they have generalized above theorem. Recently, Marchiş [14] has generalized above theorem and proved some common fixed theorems for two weakly commuting and compatible mappings.

The study of common fixed point theorems in many spaces such as metric spaces, partial metric spaces, partial b-metric spaces,... is one of the most active research areas in fixed point theory. Many authors established new common fixed point theorems for some mappings in this spaces, for example, one can refer to [1, 3, 13, 22].

Menger in 1942 [15] introduced probabilistic metric space. He used distribution functions instead of nonnegative real numbers as values of the metric. Probabilistic metric spaces are widely used in probabilistic functional analysis, ε^{∞} theory, quantum particle physics, nonlinear analysis and applications, for example see [2, 4]. Following that, many researchers like as Schweizer and Sklar [19] became interested in studying probabilistic metric spaces. Lately, many researchers have studied fixed point theorems in probabilistic metric spaces. Sehgal and Bharucha-Reid [20] were the first researchers to take this direction. After that, several researchers have studied fixed point theorems, common fixed point theorems and recently best proximity point in probabilistic metric spaces, for example see [16, 23, 24].

For well-known definitions (such as probabilistic metric space, t-norm, t-norm of H-type, probabilistic Menger space (abbreviated, PM-space), complete PM-space, probabilistic normed (abbreviated,PN-space), probabilistic Banach space, etc.) and known results, we refer to [6] and [19]. Now, we state some definitions, and known results.

Proposition 1.1. [2, 2.5.3] *If (X, F, Δ) is a PM-space, then probabilistic distance function F is a low semi continuous function of points, i.e. for every fixed point $t \geq 0$, if $x_n \to x$ and $y_n \to y$, then*

$$\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Lemma 1.2. [7] *Let $F_n : \mathbb{R} \to [0,1] (n \in \mathbb{N})$ be a sequence of nondecreasing functions, $g_n : [0,\infty) \to [0,\infty) (n \in \mathbb{N})$ be a sequence of functions such that for any $t > 0$, $\lim_{n\to\infty} g_n(t) = 0$ and $F : \mathbb{R} \to [0,1]$ be a function such that $\sup\{F(t) : t > 0\} = 1$. If*

$$F_n(g_n(t)) \geq F(t) \qquad (\forall n \in \mathbb{N}, t > 0),$$

then $\lim_{n\to\infty} F_n(t) = 1$ for any $t > 0$.

Lemma 1.3. *Let* (X, F, Δ) *be a PM-space and* $\varphi : [0, \infty) \to [0, \infty)$ *be a mapping that* $\varphi(0) = 0$, $\varphi(x) < x$ *and* $\lim_{n\to\infty} \varphi^n(x) = 0$. *If*

$$F_{p,q}(\varphi(t)) \geq F_{p,q}(t) \qquad (\forall p, q \in X, t > 0),$$

then $p = q$.

Proof. By using the above lemma, the result follows. □

Lemma 1.4. ([6]) *Let* $\{x_n\}$ *be a sequence in a PM-space* (X, F, Δ). *If*

$$F_{x_{n+1},x_n}(kt) \geq F_{x_n,x_{n-1}}(t) \qquad (\forall n \in \mathbb{N}, t > 0),$$

for some $k \in (0, 1)$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Definition 1.5. Let (X, F, Δ) be a PM-space, $A, B \subseteq X$ and $T : A \to B$ be a mapping. Then the mapping T is said to be continuous at a point $x \in A$ if for every sequence $\{x_n\}$ in A, which converges to x, the sequence $\{Tx_n\}$ in B converges to Tx.

A mapping T is said to be continuous on A if T is continuous at every point in A.

Definition 1.6. Let (X, ν, Δ) be a PN-space and $T, S : X \to X$ be two mappings. The mappings T, S are said to be

(i) commuting if $TSx = STx$ for all $x \in X$,

(ii) weakly commuting (R-weakly commuting) if for all $x \in X$ and $t \geq 0$,

$$\nu_{TSx-STx}(t) \geq \nu_{Tx-Sx}(t) \ (\nu_{TSx-STx}(Rt) \geq \nu_{Tx-Sx}(t)),$$

where R is a positive real number.

(iii) compatible if for any $t \geq 0$, $\lim_{n\to\infty} \nu_{TSx_n-STx_n}(t) = \varepsilon_0(t)$, for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_0$, where $x_0 \in X$, where ε_0, defined by

$$\varepsilon_0(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

In the section 2, we extend and improve a common fixed point theorem of G. Jungck for weakly commuting and compatible mappings in probabilistic Banach spaces. For example if T and S are two weakly commuting or compatible mappings in probabilistic Banach space (X, ν, Δ) such that

$$\nu_{\alpha(x-y)+Tx-Ty}(\beta t) \geq \nu_{Sx-Sy}(t) \qquad (\forall x, y \in X,\ t \geq 0),$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \alpha + 1$, then under certain conditions T and S have a unique common fixed point. Also, we will present some examples which support our theorems.

2. Main results

Now we state and prove our main results about extensions of a common fixed point theorem of Jungck in probabilistic Banach spaces. In 1976 Jungck [9] proved the following theorem. Throughout this section, all t-norms are assumed to be H-type.

Theorem 2.1. *Let S be a continuous mapping of a complete metric space (X, d) into itself and $T : X \to X$ be a map that satisfy the following conditions:*

(i) $T(X) \subseteq S(X)$,
(ii) *two mappings T and S are commuting mappings,*
(iii) $d(Tx, Ty) \leq \kappa d(Sx, Sy)$ *for all* $x, y \in X$ *and for some* $0 < \kappa < 1$.

Then T and S have a unique common fixed point.

In the following theorem, O'Regan et al. [16] proved the probabilistic version of the above theorem.

Theorem 2.2. *Let S be a continuous mapping of a complete PM-space (X, F, Δ) into itself and $T : X \to X$ be a map that satisfy the following conditions:*

(i) $T(X) \subseteq S(X)$,
(ii) *two mappings T and S are commuting mappings,*
(iii) $F_{Tx,Ty}(\phi(t)) \geq F_{Sx,Sy}(t)$ *for all* $x, y \in X$ *where, the function* $\phi : [0, \infty) \to [0, \infty)$ *is onto, strictly increasing and* $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ *for all* $t > 0$ *. In addition assume there exists* $x_0 \in X$ *with*

$$\sup\{\inf\{t > 0 : \ F_{Tx_0,Sx_0}(t) > 1 - \gamma\} : \ \gamma \in (0, 1)\} < \infty.$$

Then T and S have a unique common fixed point.

In 2008, Ješić et al. [8] extend the above theorem for R-weakly (weakly) commuting mappings.

The following theorem is the first main result.

Theorem 2.3. *Let S be a continuous mapping of a probabilistic Banach space (X, ν, Δ) into itself and $T : X \to X$ be a map that satisfy the following conditions:*

(i) *there exist two real numbers* $0 \leq \alpha < \infty$ *and* $0 < \beta < \alpha + 1$ *such that*

$$\nu_{\alpha(x-y)+Tx-Ty}(\beta t) \geq \nu_{Sx-Sy}(t) \qquad (\forall x, y \in X, \ t \geq 0), \tag{2.1}$$

(ii) *two mappings T_λ and S are weakly commuting mappings, where* $T_\lambda x = (1 - \lambda)x + \lambda Tx$ *and* $\lambda = 1/(\alpha + 1)$,
(iii) $T_\lambda(X) \subseteq S(X)$.

Then there exists a unique common fixed point $x_0 \in X$ for two mappings T, S and for arbitrary $x \in X$, the iterative sequence $\{Sx_{n+1} = T_\lambda x_n\}$ converges to x_0.

Proof. By the contraction condition (2.1), for all $x, y \in X$ and $t \geq 0$, we have

$$\begin{aligned}\nu_{(1-\lambda)(x-y)+\lambda(Tx-Ty)}(\lambda\beta t) =&\nu_{\frac{(1-\lambda)(x-y)+\lambda(Tx-Ty)}{\lambda}}(\beta t)\\ \geq&\nu_{Sx-Sy}(t).\end{aligned}$$

Therefore

$$\nu_{T_\lambda x - T_\lambda y}(\kappa t) \geq \nu_{Sx-Sy}(t) \qquad (\forall x, y \in X,\ t \geq 0), \tag{2.2}$$

where $0 < \kappa = \lambda\beta < 1$. It is easy to see that from (2.2) that the continuity of S implies that of T_λ. Let x be an arbitrary element in X, now from condition (iii) we can define the iterative sequence $\{x_n\}$ in X by

$$Sx_{n+1} = T_\lambda x_n = y_n \qquad (\forall n \in \mathbb{N}). \tag{2.3}$$

From (2.2), we have

$$\nu_{T_\lambda x_n - T_\lambda x_{n-1}}(\kappa t) \geq \nu_{Sx_n - Sx_{n-1}}(t) = \nu_{T_\lambda x_{n-1} - T_\lambda x_{n-2}}(t) \qquad (\forall t \geq 0),$$

by Lemma (1.4), the sequence $\{T_\lambda x_n\}$ is a Cauchy sequence and hence converges to some $x_0 \in X$. Therefore by (2.3), we have

$$\lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} T_\lambda x_n = \lim_{n\to\infty} y_n = x_0,$$

by continuity of T_λ and S we get

$$\lim_{n\to\infty} T_\lambda Sx_{n+1} = \lim_{n\to\infty} T_\lambda y_n = T_\lambda x_0, \quad \lim_{n\to\infty} SSx_{n+1} = \lim_{n\to\infty} Sy_n = Sx_0.$$

Now we show that $T_\lambda x_0 = \lim_{n\to\infty} T_\lambda Sx_{n+1} = \lim_{n\to\infty} T_\lambda y_n = Sx_0$, to this end, from condition (ii) for any $t \geq 0$, we obtain

$$\begin{aligned}\nu_{T_\lambda y_n - Sx_0}(t) \geq&\Delta(\nu_{T_\lambda y_n - Sy_{n+1}}(\frac{t}{2}), \nu_{Sy_{n+1}-Sx_0}(\frac{t}{2}))\\ =&\Delta(\nu_{T_\lambda Sx_{n+1} - ST_\lambda x_{n+1}}(\frac{t}{2}), \nu_{Sy_{n+1}-Sx_0}(\frac{t}{2}))\\ \geq&\Delta(\nu_{T_\lambda x_{n+1} - Sx_{n+1}}(\frac{t}{2}), \nu_{Sy_{n+1}-x_0}(\frac{t}{2}))\\ \geq&\Delta(\Delta(\nu_{T_\lambda x_{n+1}-x_0}(\frac{t}{4}), \nu_{x_0-Sx_{n+1}}(\frac{t}{4})), \nu_{Sy_{n+1}-Sx_0}(\frac{t}{2})),\end{aligned}$$

letting $n \to \infty$, show that $T_\lambda x_0 = \lim_{n\to\infty} T_\lambda Sx_{n+1} = \lim_{n\to\infty} T_\lambda y_n = Sx_0$. On the other hand, from (2.2), we have

$$\nu_{T_\lambda Sx_{n+1} - T_\lambda x_n}(\kappa t) \geq \nu_{SSx_{n+1}-Sx_n}(t) \qquad (\forall t \geq 0).$$

Since

$$\lim_{n\to\infty} T_\lambda Sx_{n+1} = \lim_{n\to\infty} SSx_{n+1} = Sx_0, \quad \lim_{n\to\infty} T_\lambda x_n = \lim_{n\to\infty} Sx_n = x_0,$$

then by Proposition 1.1 we get

$$\nu_{Sx_0-x_0}(\kappa t) \geq \nu_{Sx_0-x_0}(t) \qquad (\forall t \geq 0).$$

By Lemma 1.3 we have $T_\lambda x_0 = Sx_0 = x_0$, so x_0 is a common fixed point for T_λ and S. If y_0 is another common fixed point for T_λ and S, then from (2.2) we have

$$\nu_{x_0-y_0}(\kappa t) = \nu_{T_\lambda x_0 - T_\lambda y_0}(\kappa t) \geq \nu_{Sx_0-Sy_0}(t) = \nu_{x_0-y_0}(t) \qquad (\forall t \geq 0),$$

again by Lemma 1.3 we have $x_0 = y_0$. Hence T_λ and S have a unique common fixed point. Also it is easy to check that $Fix(T) = Fix(T_\lambda)$ and so the result follows.

□

Remark 2.4. Note that, by using a similar argument as in the proof of Theorem 2.3, we can show that the conclusion of Theorem 2.3 remain valid if in condition (ii) we replace weakly commuting by R-weakly commuting.

As the first immediate consequence of Theorem 2.3, if $\alpha = 0$ and $S = I$ (identity mapping), then we get the probabilistic version of the classical Banach contraction principle in probabilistic Banach space which has been proved by Sehgal et al. [20] in 1972.

Corollary 2.5. *Let (X, ν, Δ_m) be a probabilistic Banach space. If T is a contraction mapping of X into itself, that is, there exists a constant $0 < \beta < 1$ such that*

$$\nu_{Tx-Ty}(\beta t) \geq \nu_{x-y}(t) \qquad (\forall x, y \in X, t \geq 0).$$

Then there is a unique $x_0 \in X$ such that $Tx_0 = x_0$. Moreover, $\{T^n x\}$ converges to x_0 for each $x \in X$.

As the second immediate consequence of Theorem 2.3, if $\alpha = 0$, then we obtain Theorem 2.2 in probabilistic Banach space for the case that $\phi(t) = \beta t$.

Corollary 2.6. *Let S be a continuous mapping of a probabilistic Banach space (X, ν, Δ) into itself and $T : X \to X$ be a map that satisfy the following conditions:*

(i) *there exists a constant $0 < \beta < 1$ such that*

$$\nu_{Tx-Ty}(\beta t) \geq \nu_{Sx-Sy}(t) \qquad (\forall x, y \in X,\ t \geq 0),$$

(ii) *two mappings T and S are commuting mappings,*

(iii) *$T(X) \subseteq S(X)$.*

Then there exists a unique common fixed point $x_0 \in X$ for two mappings T, S and for arbitrary $x \in X$, the iterative sequence $\{Sx_{n+1} = T_\lambda x_n\}$ converges to x_0.

We now give an example that satisfies all assumptions of the statement of Theorem 2.3, but does not satisfy some assumptions of the statement of Corollary 2.6.

Example 2.7. Let $X = \mathbb{R}$ and $\nu_x(t) = \frac{t}{t+|x|}$, it is easy to see that (X, ν, Δ_m) is a probabilistic Banach space. Suppose that $T, S : X \to X$ defined by

$$Tx = 1 - x, \quad Sx = \begin{cases} (4x^2 - x + 2)/5, & x \in [\frac{1}{2}, 1], \\ x, & \text{otherwise.} \end{cases}$$

If T and S satisfy in Jungck's contraction in Corollary 2.6, then for all $x, y \in [\frac{1}{2}, 1]$, for $t > 0$ we have

$$\frac{\kappa t}{\kappa t + \mid x - y \mid} = \nu_{Tx-Ty}(\kappa t) \geq \nu_{Sx-Sy}(t) = \frac{t}{t + \mid (4x^2 - x - 4y^2 + y)/5 \mid},$$

so for $x \neq y$ and $t > 0$,

$$1 \leq \kappa \mid (4x + 4y - 1)/5 \mid,$$

since

$$\inf_{x,y \in [\frac{1}{2},1]} \mid (4x + 4y - 1)/5 \mid = \frac{3}{5},$$

then

$$1 \leq \kappa(3/5),$$

a contradiction, hence T and S don't satisfy in Jungck's contraction condition of Corollary 2.6. To verify contraction condition (2.1) in Theorem 2.3, we need to consider several possible cases.
Case 1. Let $x, y \in (-\infty, \frac{1}{2}] \cup [1, \infty)$. Then for $t > 0$ we have

$$\frac{\beta t}{\beta t + \mid (\alpha - 1)(x - y) \mid} = \nu_{\alpha(x-y)+Tx-Ty}(\beta t) \geq \nu_{Sx-Sy}(t) = \frac{t}{t + \mid x - y \mid},$$

so for $x \neq y$ and $t > 0$,

$$\mid \alpha - 1 \mid \leq \beta,$$

if $\beta = \frac{1}{2}$, then $\alpha \in (\frac{1}{2}, \frac{3}{2})$.
Case 2. Let $x, y \in [\frac{1}{2}, 1]$. Then for $t > 0$, we have

$$\begin{aligned} \frac{\beta t}{\beta t + \mid (\alpha - 1)(x - y) \mid} &= \nu_{\alpha(x-y)+Tx-Ty}(\beta t) \geq \nu_{Sx-Sy}(t) \\ &= \frac{t}{t + \mid (4x^2 - x - 4y^2 + y)/5 \mid}, \end{aligned}$$

so for $x \neq y$ and $t > 0$,

$$\mid \alpha - 1 \mid \leq \beta \mid (4x + 4y - 1)/5 \mid,$$

if $\beta = \frac{1}{2}$, then $\alpha \in (\frac{7}{10}, \frac{13}{10})$, since

$$\inf_{x,y\in[\frac{1}{2},1]} \mid (4x + 4y - 1)/5 \mid= \frac{3}{5}.$$

Case 3. Let $x \in (-\infty, \frac{1}{2}] \cup [1, \infty)$ and $y \in [\frac{1}{2}, 1]$. Then for $t > 0$, we have

$$\frac{\beta t}{\beta t+ \mid (\alpha - 1)(x - y) \mid} = \nu_{\alpha(x-y)+Tx-Ty}(\beta t)$$
$$\geq \nu_{Sx-Sy}(t) = \frac{t}{t+ \mid (5x - 4y^2 + y - 2)/5 \mid},$$

so for $x \neq y$ and $t > 0$, we obtain

$$\mid \alpha - 1 \mid\leq \beta \mid \frac{5x - 4y^2 + y - 2}{5(x - y)} \mid,$$

if $\beta = \frac{1}{2}$, then $\alpha \in (\frac{7}{10}, \frac{13}{10})$, since

$$\inf_{x\in(-\infty,\frac{1}{2}]\cup[1,\infty),y\in[\frac{1}{2},1],x\neq y} \mid \frac{5x - 4y^2 + y - 2}{5(x - y)} \mid= \frac{3}{5}.$$

Hence, for every $\alpha \in (\frac{7}{10}, \frac{13}{10})$ and $\beta \in (\frac{1}{2}, \alpha+1)$, the contraction condition (2.1) in Theorem 2.3 holds. Next, we show that T_λ ($\alpha = \frac{4}{5}, \lambda = \frac{5}{9}$) and S are weakly commuting mappings, to do this, we need to consider two possible cases.

Case 1. If $x \in (-\infty, \frac{1}{2}] \cup [1, \infty)$ and $T_\lambda x \in [\frac{1}{2}, 1]$, then we have

$$T_\lambda Sx = T_\lambda x = \frac{5 - x}{9}, \quad ST_\lambda x = \frac{4x^2 - 31x + 217}{405},$$

it is easy to see that

$$\mid T_\lambda Sx - ST_\lambda x \mid=\mid \frac{4x^2 + 14x - 8}{405} \mid\leq\mid \frac{5(1 - 2x)}{9} \mid=\mid T_\lambda x - Sx \mid .$$

If $x \in (-\infty, \frac{1}{2}] \cup [1, \infty)$ and $T_\lambda x \in (-\infty, \frac{1}{2}] \cup [1, \infty)$, then we have

$$ST_\lambda x = T_\lambda Sx = T_\lambda x = \frac{5 - x}{9}, \quad \mid T_\lambda Sx - ST_\lambda x \mid= 0 \leq\mid T_\lambda x - Sx \mid .$$

Therefore $\nu_{TSx-STx}(t) \geq \nu_{Tx-Sx}(t)$ for all $x \in [0, \frac{1}{2}]$ and $t \geq 0$.

Case 2. If $x \in [\frac{1}{2}, 1]$, then we have $T_\lambda x \in [0, \frac{1}{2}]$, and

$$T_\lambda Sx = \frac{-4x^2 + x + 23}{45}, \quad ST_\lambda x = T_\lambda x = \frac{5 - x}{9},$$

it is easy to see that

$$\mid T_\lambda Sx - ST_\lambda x \mid=\mid \frac{-4x^2 + 6x - 2}{45} \mid\leq\mid \frac{-324x^2 + 36x + 63}{405} \mid=\mid T_\lambda x - Sx \mid .$$

Therefore $\nu_{TSx-STx}(t) \geq \nu_{Tx-Sx}(t)$ for all $x \in [\frac{1}{2}, 1]$ and $t \geq 0$. Hence T and S are weakly commuting mappings. Clearly S is continuous, $T_\lambda X \subseteq SX$ and $Fix(T) \cap Fix(S) = \{\frac{1}{2}\}$.

Next, we bring the following lemma, which we will use later.

Lemma 2.8. *Let (X, ν, Δ) be a probabilistic Banach space and $T, S : X \to X$ be two compatible mappings. If $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_0 \qquad (x_0 \in X),$$

and T is continuous in x_0. Then $\lim_{n\to\infty} STx_n = Tx_0$.

Proof. Since T is continuous in x_0 and $\lim_{n\to\infty} Sx_n = x_0$, then

$$\lim_{n\to\infty} TSx_n = Tx_0.$$

By the hypotheses we have

$$\nu_{STx_n-Tx_0}(t) \geq \Delta(\nu_{STx_n-TSx_n}(\frac{t}{2}), \nu_{TSx_n-Tx_0}(\frac{t}{2})) \qquad (\forall t \geq 0),$$

now, letting $n \to \infty$, so the result follows. ☐

Our next theorem is the extension of Theorem 2.3 for two compatible mappings.

Theorem 2.9. *Let S be a continuous mapping of a probabilistic Banach space (X, ν, Δ) into itself and $T : X \to X$ be a map that satisfy the following conditions:*

(i) *there exist two real numbers $0 \leq \alpha < \infty$ and $0 < \beta < \alpha + 1$ such that*

$$\nu_{\alpha(x-y)+Tx-Ty}(\beta t) \geq \nu_{Sx-Sy}(t) \qquad (\forall x, y \in X,\ t \geq 0), \tag{2.4}$$

(ii) *two mappings T_λ and S are compatible mappings, where $T_\lambda x = (1-\lambda)x + \lambda Tx$ and $\lambda = 1/(\alpha+1)$,*

(iii) *$T_\lambda(X) \subseteq S(X)$.*

Then there exists a unique common fixed point $x_0 \in X$ for two mappings T, S and for arbitrary $x \in X$, the iterative sequence $\{Sx_{n+1} = T_\lambda x_n\}$ converges to x_0.

Proof. By using a similar argument as in the proof of Theorem 2.3 we can show that

$$\nu_{T_\lambda x - T_\lambda y}(\kappa t) \geq \nu_{Sx-Sy}(t) \qquad (\forall x, y \in X,\ t \geq 0), \tag{2.5}$$

where $0 < \kappa = \lambda\beta < 1$ and for arbitrary element $x \in X$, the iterative sequence define by $Sx_{n+1} = T_\lambda x_n$ converges to a point $x_0 \in X$. Now by Lemma 2.8 we have

$$\lim_{n\to\infty} T_\lambda Sx_n = Sx_0.$$

Since S is continuous mapping we obtain $\lim_{n\to\infty} SSx_n = Sx_0$, so by (2.5) we have

$$\nu_{T_\lambda Sx_n - T_\lambda x_n}(\kappa t) \geq \nu_{SSx_n - Sx_n}(t) \qquad (\forall t \geq 0),$$

Since

$$\lim_{n\to\infty} T_\lambda Sx_n = \lim_{n\to\infty} SSx_n = Sx_0, \quad \lim_{n\to\infty} T_\lambda x_n = \lim_{n\to\infty} Sx_n = x_0,$$

then by Proposition 1.1 we get

$$\nu_{Sx_0 - x_0}(\kappa t) \geq \nu_{Sx_0 - x_0}(t) \qquad (\forall t \geq 0).$$

Since $0 < \kappa < 1$, then by Lemma 1.3 we get $Sx_0 = x_0$. It is easy to see that from (2.5) that the continuity of S implies that of T_λ. By using Lemma 2.8 we have

$$\lim_{n\to\infty} ST_\lambda x_n = T_\lambda x_0, \tag{2.6}$$

also by continuity of S we obtain

$$\lim_{n\to\infty} ST_\lambda x_n = Sx_0. \tag{2.7}$$

Now by (2.6), (2.7) we conclude

$$T_\lambda x_0 = Sx_0 = x_0.$$

Finally, by using a similar argument as in the proof of Theorem 2.3 we can show that x_0 is a unique common fixed point for T and S, as required. □

Now, we give a example concerning Theorem 2.9, also in this example we show that two compatible mappings are not necessarily weakly commuting mappings.

Example 2.10. Let $X = \mathbb{R}$ and $\nu_x(t) = \frac{t}{t+|x|}$, it is easy to see that (X, ν, Δ_m) is a probabilistic Banach space. Suppose that $T, S : X \to X$ defined by

$$Tx = \begin{cases} 6x^2 - x, & x \geq \frac{1}{3}, \\ x, & x \leq \frac{1}{3}, \end{cases} \qquad Sx = \begin{cases} 4x^2 - \frac{1}{3}x, & x \geq \frac{1}{3}, \\ 2x - \frac{1}{3}, & x \leq \frac{1}{3}. \end{cases}$$

If T and S satisfy in Jungck's contraction in Corollary 2.6, then for $t > 0$ and $x \geq \frac{1}{3}$, we have

$$\begin{aligned} \frac{\kappa t}{\kappa t + \mid x - y \mid\mid 6(x+y) - 1 \mid} &= \nu_{Tx-Ty}(\kappa t) \\ &\geq \nu_{Sx-Sy}(t) = \frac{t}{t + \mid x - y \mid\mid 4(x+y) - \frac{1}{3} \mid}, \end{aligned}$$

so for $x \neq y$ and $t > 0$,

$$\mid 6(x+y) - 1 \mid \leq \kappa \mid 4(x+y) - \frac{1}{3} \mid .$$

If $x = 1$ and $y = 0.9$, then $\frac{312}{218} \leq \kappa$, a contradiction, hence T and S don't satisfy in Jungck's contraction in Corollary 2.6. To verify contraction condition (2.4) in Theorem 2.9, we need to consider several possible cases.

Case 1. Let $x, y \in [\frac{1}{3}, \infty)$. Then we have

$$\begin{aligned}\frac{\beta t}{\beta t + \mid ((\alpha-1)+6(x+y))(x-y) \mid} &= \nu_{\alpha(x-y)+Tx-Ty}(\beta t)\\ &\geq \nu_{Sx-Sy}(t)\\ &= \frac{t}{t + \mid (4(x+y)-\frac{1}{3})(x-y) \mid},\end{aligned}$$

so for $x \neq y$,

$$\mid (\alpha-1)+6(x+y) \mid \leq \beta \mid 4(x+y) - \frac{1}{3} \mid,$$

if $\alpha = 1$, then $\beta \in (\frac{12}{7}, 2)$, since

$$\sup_{x,y\in[\frac{1}{3},\infty)} \frac{\mid 6x+6y \mid}{\mid 4(x+y)-\frac{1}{3} \mid} = \frac{12}{7}.$$

Case 2. Let $x, y \in (-\infty, \frac{1}{3})$. Then we have

$$\begin{aligned}\frac{\beta t}{\beta t + \mid (\alpha+1)(x-y) \mid} &= \nu_{\alpha(x-y)+Tx-Ty}(\beta t)\\ &\geq \nu_{Sx-Sy}(t)\\ &= \frac{t}{t + \mid 2(x-y) \mid},\end{aligned}$$

so for $x \neq y$,

$$\mid (\alpha+1) \mid \leq 2\beta,$$

if $\alpha = 1$, then $\alpha \in (1, 2)$.

Case 3. Let $x \in [\frac{1}{3}, \infty)$ and $y \in (-\infty, \frac{1}{3})$(similarly if $x \in (-\infty, \frac{1}{3})$ and $y \in [\frac{1}{3}\infty)$). Then we have

$$\begin{aligned}\frac{\beta t}{\beta t + \mid \alpha(x-y)+6x^2-x-y \mid} &= \nu_{\alpha(x-y)+Tx-Ty}(\beta t)\\ &\geq \nu_{Sx-Sy}(t)\\ &= \frac{t}{t + \mid 4x^2-\frac{1}{3}x-2y+\frac{1}{3} \mid},\end{aligned}$$

so

$$\mid \alpha(x-y)+6x^2-x-y \mid \leq \beta \mid 4x^2 - \frac{1}{3}x - 2y + \frac{1}{3} \mid,$$

if $\alpha = 1$, then $\beta \in (\frac{12}{7}, 2)$, since

$$\sup_{x\in[\frac{1}{3},\infty),y\in(-\infty,\frac{1}{3})} \frac{\mid 6x^2 - 2y \mid}{\mid 4x^2 - \frac{1}{3}x - 2y + \frac{1}{3} \mid} = \frac{12}{7}.$$

Hence, for every $\alpha = 1$ and $\beta \in (\frac{12}{7}, 2)$, the contraction condition (2.4) in Theorem 2.9 holds. It is easy to check that T_λ $(\alpha = 1, \lambda = \frac{1}{2})$ and S are not commuting mappings. We also show that T_λ and S are not weakly commuting mappings. To do this, for $x \geq \frac{1}{3}$ we have

$$T_\lambda x = 3x^2,\ T_\lambda Sx = 48x^4 - 8x^3 + \frac{1}{3}x^2,\ ST_\lambda x = 36x^4 - x^2,$$

if T_λ and S are weakly commuting mappings, then for $t > 0$,

$$\frac{t}{t+\mid 12x^4 - 8x^3 + \frac{4}{3}x^2 \mid} = \nu_{T_\lambda Sx - ST_\lambda x}(t) \geq \nu_{T_\lambda x - Sx}(t) = \frac{t}{t+\mid x^2 - \frac{1}{3}x \mid},$$

so

$$\mid 12x^4 - 8x^3 + \frac{4}{3}x^2 \mid \leq \mid x^2 - \frac{1}{3}x \mid,$$

obviously this inequality is never true for $x = 1$, therefore T_λ and S are not weakly commuting mappings. Finally, we show that T and S are compatible mappings. If $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} T_\lambda x_n = \lim_{n\to\infty} Sx_n = x_0 \quad (x_0 \in X),$$

then it is easy to see that $\lim_{n\to\infty} x_n = x_0 = \{\frac{1}{3}\}$. So we obtain

$$\lim_{n\to\infty} \mid T_\lambda Sx_n - ST_\lambda x_n \mid = \mid T_\lambda x_0 - Sx_0 \mid = 0,$$

therefore $\lim_{n\to\infty} \nu_{T_\lambda Sx_n - ST_\lambda x_n}(t) = \varepsilon_0(t)$, for all $t \geq 0$, hence T_λ and S are compatible mappings. Clearly S is continuous, $T_\lambda X \subseteq SX$ and $Fix(T) \cap Fix(S) = \{\frac{1}{3}\}$.

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