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A new application of q- homotopy analysis method to solve non-linear optimal control problems

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This paper advances a new application of q-homotopy analysis method (q-HAM) to solve non-linear optimal control problems(NOCPs). First, the NOCP was transformed into a non-linear two-point boundary value problem by using the Pontryagins maximum principle (PMP). Then, we applied the q-HAM to solve this system. The proposed method is based on the HAM but the q-HAM, has an increased interval of convergence than the HAM. Three examples are provided to demonstrate the reliability and efficiency of the method. Next, the numerical results of the proposed method are compared with those of other methods. As can be seen from the tables, the maximum error in the second and third examples is much better than other methods.

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1. Introduction

Optimal control problems are widely used in various fields such as biomedicine [1], robotics [2], aircraft systems [3], physics [4], and economics [5]. Optimal control of non-linear systems has been one of the most challenging issues which have been extensively studied by researchers over the years. Since there is no analytical solution for non-linear optimal control problems (NOCPs), considerable attention has been devoted to such problems. Many approaches have been proposed to solve these equations. The so-called measure theory is one example proposed by Rubio [6]. This technique transforms NOCPs to linear programming problems and provides a piecewise constant control law. In [7], optimal control problems were solved using the spectral homotopy analysis method. More recently, a hybrid parametrization approach was introduced by [8] for solving NOCPs. In this paper, an effective technique is developed for control parametrization and state variables are computed using homotopy analysis method (HAM). Effati et al. [9] used HAM to present an analytic-approximate solution for these equations. The optimal homotopy perturbation method was used by Jajarmi et al.[10] for answering a class of NOCPs. Hwang et al. [11] constructed a computational approach to unravel optimal control problems using differential transformation. Legendre approximations were suggested by El-Kady et al. [12] to respond to optimal control problems governed by higher-order ordinary differential equations. Jia et al. [13] introduced a new method based on optimal HAM and Pontryagins maximum principle (PMP) for solving linear optimal control problems. A numerical approach based on the Boubaker polynomials expansion scheme was suggested by Kafash et al. [14] to make answer optimal control problems. Lin et al. [15] introduced the fundamentals of the control parameterization method and examined its various applications to non-standard optimal control problems. The radial basis function (RBF) collocation method was proposed by Mirinejad et al. [16] to deal with OCP, robotics, and autonomous systems. Nazemi et al. [17] developed an algorithm capable of solving a class of NOCPs. A combination of the hybrid spectral collocation technique and HAM was used by Saberi et al. [18] to construct an iteration algorithm for solving a class of NOCPs. He's variational iteration method was proposed by Shirazian et al. [19] to investigate a class of NOCPs. The modified variational iteration method (MVIM) and the advantages of using this method are discussed in that paper. Chen et al. [20] provided a new spectral method based on Galerkin approximation solutions of the non-linear optimal control systems. Overwhelming research has been done on developing applications of fractional iteration algorithm in engineering fields and physical models [21, 22, 23]. The authors studied new modified variational iteration algorithmsI, modified variational iteration algorithmsII and Riccati transformation methods in [24, 25, 26, 27, 28] to approximate the solutions of kind of the problems.

HAM is a semi-exact method for solving non-linear equations which does not need small/large parameters proposed by Liao [29]. This is an efficient method to solve highly linear and non-linear problems in various fields [30]. One of the strengths of HAM, compared to other analytical methods, concerns its great freedom to choose the initial approximation. Besides, it contains a specific auxiliary parameter which facilitates adjusting and controlling the convergence region and the rate of convergence of the series solution.

El-Tawil and Huseen [31] formulated a method called q-HAM which is a more general method of HAM. This method has been successfully applied to solve many types of non-linear problems [32, 33, 34, 35, 36, 37, 38, 39, 40].

In this work, an indirect method based on q-HAM is introduced to solve NOCPs. One of the main preference of q-HAM is that it contains an auxiliary parameter n same as h such that the case of $n = 1$ (q-HAM; $n = 1$) the standard HAM can be reached. But, this fraction factor n has much flexibility to adjust and control the convergence region and rate of convergence of the series of solution compared to auxiliary parameter h in HAM. To show the efficiency of the method and its effectiveness, three practical examples have been solved and compared with other methods.

The paper is organized as follows: Section2 introduces q-HAM to solve optimal control problems; In Section 3, three examples are presented to show the efficiency of the method; In the last Section, some conclusions are drawn based on the findings.

1.1. **The basic concepts of the q-HAM.** In order to understand the fundamentals of q-HAM, consider the following differential equations:

$$N[w(t)] = 0, \tag{1.1}$$

where N represents a non-linear operator and $w(t)$ is stands for an unknown function[41]. It is possible to formulate the zero-order deformation equation thus:

$$(1 - nq)L[\phi(t; q) - w_0(t)] = F(n)qN[\phi(t; q)], \tag{1.2}$$

$q \in [0, \frac{1}{n}]$ signifies an embedding parameter, n denotes a non-zero auxiliary parameter, $F(n)$ represents a non-zero auxiliary function $n \geq 1$, L is an auxiliary linear operator with the property $L(c_1) = 0$ where c_1 denotes an integral constant, $w_0(t)$ shows an initial guess of $w(t)$, and $\phi(t;q)$ stands for an unknown function. We have great freedom to take auxiliary objects such as q and L in HAM. Obviously, when $q = 0$ and $q = \frac{1}{n}$,

$$\phi(t;0) = w_0(t), \qquad \phi(t;1) = w(t), \tag{1.3}$$

respectively. Thus, as q rises from 0 to 1, the solutions $\phi(t;q)$ alter from the initial guess $w_0(t)$ to the solution $w(t)$. By expanding $\phi(t;q)$ in Taylor series with regard to q, we will have

$$\phi(t;q) = w_0(t) + \sum_{m=1}^{+\infty} w_m(t) q^m, \tag{1.4}$$

where

$$w_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{1.5}$$

Let's suppose that $F(n)$, $w_0(t)$, and L are accurately selected such that the series converges at $q = \frac{1}{n}$ and we have

$$w(t) = \phi(t; \frac{1}{n}) = w_0(t) + \sum_{m=1}^{+\infty} w_m(t) (\frac{1}{n})^m. \tag{1.6}$$

Defining the vector $w_r(t) = \{w_0(t), w_1(t), w_2(t), ..., w_r(t)\}$. Differentiating equation (1.2) m times with regard to q and then setting $q = 0$ and eventually dividing them by $m!$, we will obtain the m^{th} order deformation equation:

$$L[w_m(t) - k_m w_{m-1}(t)] = F(n) R_m(\vec{w_{m-1}}), \tag{1.7}$$

where

$$R_m(\vec{w_{m-1}}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{1.8}$$

and

$$k_m = \begin{cases} 0, & m \leqslant 1, \\ n, & \text{otherwise} \end{cases} \tag{1.9}$$

It needs to be stressed that $w_m(t)$ for $m \geqslant 1$ is governed by the linear equation (1.7) with linear boundary conditions which are related to the original problem. Let

$$\triangle_m = \int_\Omega (N[w_m(t)])^2 d\Omega, \tag{1.10}$$

where

$$w(t) = w_0(t) + \sum_{m=1}^{+\infty} w_m(t)(\frac{1}{n})^m \tag{1.11}$$

represents the square residual error of the m^{th}-order approximation of the equation (1.1) which is integrated into the whole domain Ω. Theoretically, if the square residual error tends $\triangle_m$ to zero, then

$$w(t) = \sum_{m=1}^{+\infty} w_m(t)(\frac{1}{n})^m \tag{1.12}$$

will serve as a series solution for the original equation (1.1). Besides, the optimal value of the auxiliary parameter n is given by solving the following non-linear algebraic equation

$$\frac{d\triangle_m}{dn} = 0. \tag{1.13}$$

1.2. **Non-linear optimal control problems.** Consider the following non-linear dynamical system

$$\begin{aligned} &\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), \qquad t \in [t_0, t_f], \\ &x(t_0) = x_0, \qquad x(t_f) = x_f, \end{aligned} \tag{1.14}$$

where $u(t) \in \Re^m$ is the control variable, $x(t) \in \Re^s$ indicates the state variable, x_0 and x_f are the given initial and final states at t_0 and t_f, respectively. Moreover, $f(t, x(t)) \in \Re^s$ and $g(t, x(t))u(t) \in \Re^{s\times m}$ are two continuously differentiable functions in all arguments. We have the quadratic objective functional as follows:

$$J[x, u] = \frac{1}{2}\int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \tag{1.15}$$

which should be minimized subject to the non-linear system (1.14), for $Q \in \Re^{s\times s}$ and $R \in \Re^{m\times m}$ positive semi-definite and positive definite matrices, respectively. Since the performance index (1.15) is convex, the following extreme necessary conditions are also sufficient for optimality:

$$\begin{aligned} &\dot{x} = f(t, x(t)) + g(t, x(t))u^*, \\ &\dot{\lambda} = -H_x(x, u^*, \lambda), \\ &u^* = arg\, min_u H_x(x, u, \lambda), \\ &x(t_0) = x_0, \qquad x(t_f) = x_f, \end{aligned} \tag{1.16}$$

where $H_x(x, u, \lambda) = \frac{1}{2}[x^TQx + u^TRu] + \lambda^T[f(t, x) + g(t, x)u]$ is referred to as the Hamiltonian. Equivalently, (1.16) can be written in the form of:

$$\dot{x} = f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda],$$
$$\dot{\lambda} = -(Qx + (\frac{\partial f(t,x)}{\partial x})^T\lambda + \sum_{i=1}^{s}\lambda_i[-R^{-1}g^T(t,x)\lambda]^T\frac{\partial g_i(t,x)}{\partial x}), \quad (1.17)$$
$$x(t_0) = x_0, \qquad x(t_f) = x_f,$$

where $\lambda(t) \in \Re^s$ is the co-state vector with the i^{th} component, $\lambda_i(t), i = 1,...,s$ and $g(t,x) = [g_1(t,x),...,g_s(t,x)]^T$, with $g_i(t,x) \in \Re^m, i = 1,...,s$. Also, the optimal control law is obtained by

$$u^* = -R^{-1}g^T(t,x)\lambda. \quad (1.18)$$

For solving the system (1.17), we can solve the following system with q-HAM method

$$\dot{x} = f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda],$$
$$\dot{\lambda} = -(Qx + (\frac{\partial f(t,x)}{\partial x})^T\lambda + \sum_{i=1}^{s}\lambda_i[-R^{-1}g^T(t,x)\lambda]^T\frac{\partial g_i(t,x)}{\partial x}), \quad (1.19)$$
$$x(t_0) = x_0, \qquad \lambda(t_0) = \alpha,$$

where $\alpha \in \Re$ is an unknown parameter. Which is obtained after sufficient iterations of q-HAM, as discussed hereinafter. We can change the equation (1.17) as follows:

$$\dot{x} = \theta_1(t,x,\lambda) \quad (1.20)$$
$$\dot{\lambda} = \theta_2(t,x,\lambda) \qquad t \in [t_0,t_f]$$
$$x(t_0) = x_0, \qquad \lambda(t_0) = \alpha,$$

1.3. **The basic idea of q-HAM for NOCPs.** To solve the non-linear system (1.19) by means of q-HAM method, consider these following initial approximations:

$$x_0(t) = 0,$$
$$\lambda_0(t) = \alpha,$$

Besides, we consider the linear operators as follows:

$$L[\phi_i(t;q)] = \frac{\partial \phi_i(t;q)}{\partial t},$$

based on equation (1.20), the following non - linear operators will obtain:

$$N_1[\phi_i(t;q)] = \frac{\partial \phi_1(t,q)}{\partial t} - \theta_1(t,\phi_1(t,q),\phi_2(t,q)),$$
$$N_2[\phi_i(t;q)] = \frac{\partial \phi_2(t,q)}{\partial t} - \theta_2(t,\phi_1(t,q),\phi_2(t,q)).$$

We also constructed the following zero-order deformation equations:

$$(1-nq)L[\phi_1(t;q)-x_0(t)] = F(n)qN_1[\phi_i(t;q)], \qquad (1.21)$$
$$(1-nq)L[\phi_2(t;q)-\lambda_0(t)] = F(n)qN_2[\phi_i(t;q)].$$

Obviously, when $q=0$ and $q=\frac{1}{n}$, we will have

$$\phi_1(t;0) = x_0(t), \qquad \phi_1(t;\frac{1}{n}) = x(t),$$
$$\phi_2(t;0) = \lambda_0(t), \qquad \phi_2(t;\frac{1}{n}) = \lambda(t),$$

respectively. Therefore, as q rises from 0 to $\frac{1}{n}$, the solutions $\phi_i(t;q)$ changes from the initial guesses $x_0(t), \lambda_0(t)$ to the solutions $x(t), \lambda(t)$.

$$x(t) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t)(\frac{1}{n})^m, \qquad (1.22)$$
$$\lambda(t) = \lambda_0(t) + \sum_{m=1}^{+\infty} \lambda_m(t)(\frac{1}{n})^m,$$

where

$$x_m(t) = D_m[\phi_1(t;q)] = \frac{1}{m!}\frac{\partial^m \phi_1(t;q)}{\partial q^m}|_{q=0}, \qquad (1.23)$$
$$\lambda_m(t) = D_m[\phi_2(t;q)] = \frac{1}{m!}\frac{\partial^m \phi_2(t;q)}{\partial q^m}|_{q=0},$$

in which D_m denotes the m^{th}-order q-homotopy-derivative of ϕ_i.

$$\vec{x}_k = \{x_0(t), x_1(t), ..., x_k(t)\},$$
$$\vec{\lambda}_k = \{\lambda_0(t), \lambda_1(t), ..., \lambda_k(t)\}.$$

By differentiating equation(1.21) m times with regard to the embedding parameter q and setting $q=0$ and ultimately dividing them by $m!$, the so-called m^{th}-order deformation equations will obtain as follows

$$L[x_m(t) - k_m x_{m-1}(t)] = F(n)R_m(\vec{x}_{m-1}), \qquad (1.24)$$

$$L[\lambda_m(t) - k_m \lambda_{m-1}(t)] = F(n)R_m(\vec{\lambda}_{m-1}), \qquad (1.25)$$

where

$$R_m(\vec{x}_{m-1}) = x'_{m-1} - \theta_{1m-1}(t, x_{m-1}, \lambda_{m-1}), \qquad (1.26)$$
$$R_m(\vec{\lambda}_{m-1}) = \lambda'_{m-1} - \theta_{2m-1}(t, x_{m-1}, \lambda_{m-1}), \qquad (1.27)$$

and

$$k_m = \begin{cases} 0, & m \leqslant 1, \\ n, & \text{otherwise} \end{cases} \qquad (1.28)$$

Theorem 1.1. *If the series solution defined in (1.22) is convergent, then it converges to an exact solution of the non-linear problem (1.20).*

As the series is hypothetically convergent, it holds:

$$V(t)=\sum_{m=0}^{+\infty}x_m(t)(\frac{1}{n})^m,$$
$$W(t)=\sum_{m=0}^{+\infty}\lambda_m(t)(\frac{1}{n})^m.$$

Consequently, the necessary condition for the convergence of the series is valid; in other words:

$$\lim_{m\to\infty}x_m(t)(\frac{1}{n})^m=0, \tag{1.29}$$
$$\lim_{m\to\infty}\lambda_m(t)(\frac{1}{n})^m=0.$$

Using (1.24), (1.25), and L (i.e., linear operator), we have:

$$F(n)\sum_{m=1}^{+\infty}R_m(\vec{x}_{m-1})=\sum_{m=1}^{+\infty}L[x_m(t)-k_mx_{m-1}(t)], \tag{1.30}$$
$$F(n)\sum_{m=1}^{+\infty}R_m(\vec{\lambda}_{m-1})=\sum_{m=1}^{+\infty}L[\lambda_m(t)-k_m\lambda_{m-1}(t)]. \tag{1.31}$$

We multiply the sides of equation (1.30) in $\sum_{m=1}^{+\infty}(\frac{1}{n})^m$

$$F(n)\sum_{m=1}^{\infty}(\frac{1}{n})^mR_m(\vec{x}_{m-1})=\sum_{m=1}^{\infty}L\left[(\frac{1}{n})^mx_m(t)-(\frac{1}{n})^mk_mx_{m-1}(t)\right]=$$
$$L\left[\lim_{i\to\infty}\sum_{m=1}^{i}\left((\frac{1}{n})^mx_m(t)-(\frac{1}{n})^mk_mx_{m-1}(t)\right)\right]=L[\lim_{i\to\infty}(((\frac{1}{n})x_1(t)-(\frac{1}{n})k_1x_0(t))+$$
$$((\frac{1}{n})^2x_2(t)-(\frac{1}{n})^2k_2x_1(t))+((\frac{1}{n})^3x_3(t)-(\frac{1}{n})^3k_3x_2(t))+...+((\frac{1}{n})^{i-1}x_{i-1}(t)$$
$$-(\frac{1}{n})^{i-1}k_{i-1}x_{i-2}(t))+((\frac{1}{n})^ix_i(t)-(\frac{1}{n})^ik_ix_{i-1}(t)))].$$

By considering equation (1.28), we have

$$L[\lim_{i\to\infty}(\frac{1}{n})^ix_i(t)],$$

and based on equation (1.29) we have

$$L[\lim_{i\to\infty}(\frac{1}{n})^ix_i(t)]=0.$$

Hence

$$F(n)\sum_{m=1}^{+\infty}(\frac{1}{n})^m R_m(\vec{x}_{m-1}) = 0.$$

On the other hand, according to the definition:

$$\sum_{m=1}^{+\infty}(\frac{1}{n})^m R_m(\vec{x}_{m-1}) = 0,$$

based on equation (1.26), we have

$$\sum_{m=1}^{+\infty}(\frac{1}{n})^m (x'_{m-1} - \theta_{1m-1}(t, x_{m-1}, \lambda_{m-1})) = 0;$$

subsequently,

$$\sum_{m=1}^{+\infty}(\frac{1}{n})^m x'_{m-1} = \sum_{m=1}^{+\infty}(\frac{1}{n})^m \theta_{1m-1}(t, x_{m-1}, \lambda_{m-1}),$$

and finally

$$x' = \theta_1(t, x, \lambda).$$

In the case of $\lambda' = \theta_2(t, x, \lambda)$, the proof is similar. This completes the proof.

2. Application

In this part, we solve three examples of NOCPs to demonstrate the efficiency and accuracy of the proposed method.

Example 2.1. We consider the following NOCP:

$$\begin{aligned} &minJ = \int_0^1 u^2(t)dt, \\ &s.t. \quad \dot{x} = \frac{1}{2}x^2(t)\sin x(t) + u(t), \qquad t \in [0, 1] \\ &x(0) = 0, \quad x(1) = 0.5. \end{aligned} \tag{2.1}$$

By using PMP, we obtain the following equations:

$$\begin{aligned} &\dot{x} = \frac{1}{2}x^2(t)\sin x(t) - \frac{1}{2}\lambda(t), \qquad t \in [0, 1] \\ &\dot{\lambda} = -\lambda(t)x(t)\sin x(t) - \frac{1}{2}\lambda(t)x^2(t)\cos x(t), \\ &x(0) = 0, \quad \lambda(0) = \alpha, \end{aligned} \tag{2.2}$$

such that $u(t) = -\frac{1}{2}\lambda(t)$. Based on equation (1.26), we have

$$R_m(\vec{x}_{m-1}) = \dot{x}_{m-1} - \frac{1}{2}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N_1(\sum_{k=0}^{m-1} q^k x_k, \sum_{k=0}^{m-1} q^k \lambda_k)_{q=0} + \frac{1}{2}\lambda_{m-1},$$
$$R_m(\vec{\lambda}_{m-1}) = \dot{\lambda}_{m-1} + \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N_2(\sum_{k=0}^{m-1} q^k x_k, \sum_{k=0}^{m-1} q^k \lambda_k)_{q=0}$$
$$+\frac{1}{2}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N_3(\sum_{k=0}^{m-1} q^k x_k, \sum_{k=0}^{m-1} q^k \lambda_k)_{q=0}.$$

By getting $n = 3.695$ and according to the final state condition, we must have $\sum_{i=0}^{4} x_i(1) = 0.5$, which yields $\alpha = -1.0053935951332145$. Now, we successively calculate the following:

$$x_1(t) = \frac{1}{2}(1-n)t\alpha,$$
$$x_2(t) = -(\frac{1}{2})(1-n)(-1+n)t\alpha + \frac{1}{2}(1-n)nt\alpha,$$
$$x_3(t) = -(\frac{1}{2})(1-n)(-1+n)t\alpha + n(-\frac{1}{2}(1-n)(-1+n)t\alpha + \frac{1}{2}(1-n)nt\alpha),$$
$$\vdots$$
$$\lambda_1(t) = 0,$$
$$\lambda_2(t) = 0,$$
$$\lambda_3(t) = \frac{1}{8}(1-n)(-1+n)^2t^3\alpha^3,$$
$$\vdots$$

By applying the proposed method, the numerical results of J_N and the relative error of objective value will be as given in Table 1. Table 2 compares the maximum absolute error of the proposed method with that of other methods, which shows the proposed method has achieved similar results with other methods. The obtained numerical solutions for $m = 4$ and $n = 3.695$ are provided in Figure 1.

Example 2.2. In this example, we consider the optimal maneuvers of a rigid asymmetric spacecraft [44]. The Euler's equations for the angular

TABLE 1. Numerical results of various iterations (Example 2.1)

N	J_N	$\frac{J_N - J_{N-1}}{J_N}$
1	0.252704	-
2	0.252704	0
3	0.246596	$2.47692582 \times 10^{-2}$
4	0.237523	3.8198406×10^{-2}

TABLE 2. Minimum objective value of J and the final state error of the proposed method and other methods (Example 2.1)

Method	Objective value	Max error
Proposed method(m=4)	0.237523	$3.5793 * 10^{-6}$
Measure theory method [6]	0.2425	$4.3 * 10^{-3}$
HAM (h = -1, m=5) [9]	0.23533	$4.2 * 10^{-6}$
VIM (m=5) [19]	0.2353	$4.2 * 10^{-6}$
HPM [42]	0.23533	$4.2 * 10^{-6}$
Modal series method [43] (m=5)	0.2353	$2.8 * 10^{-5}$

velocities of the spacecraft are given by:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{I_3 - I_2}{I_1} x_2(t)x_3(t) \\ -\frac{I_1 - I_3}{I_2} x_1(t)x_3(t) \\ -\frac{I_2 - I_1}{I_3} x_1(t)x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad (2.3)$$

where x_1, x_2, and x_3 are the angular velocities of the spacecraft, u_1, u_2, and u_3 represent the control torques, and $I_1 = 86.24$, $I_2 = 85.07$ and $I_3 = 113.59\ kgm^2$ are the spacecraft principal inertia. The purpose of optimal control is to find that control $u(t)$ which minimizes the cost function

$$J[x,u] = \frac{1}{2}\int_0^{100} (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \quad (2.4)$$

where $\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In addition, the following boundary conditions should be satisfied:

$$\begin{cases} x_1(0) = 0.01r/s, x_2(0) = .005r/s, x_3(0) = .001r/s, \\ x_1(100) = x_2(100) = x_3(100) = 0r/s. \end{cases} \quad (2.5)$$

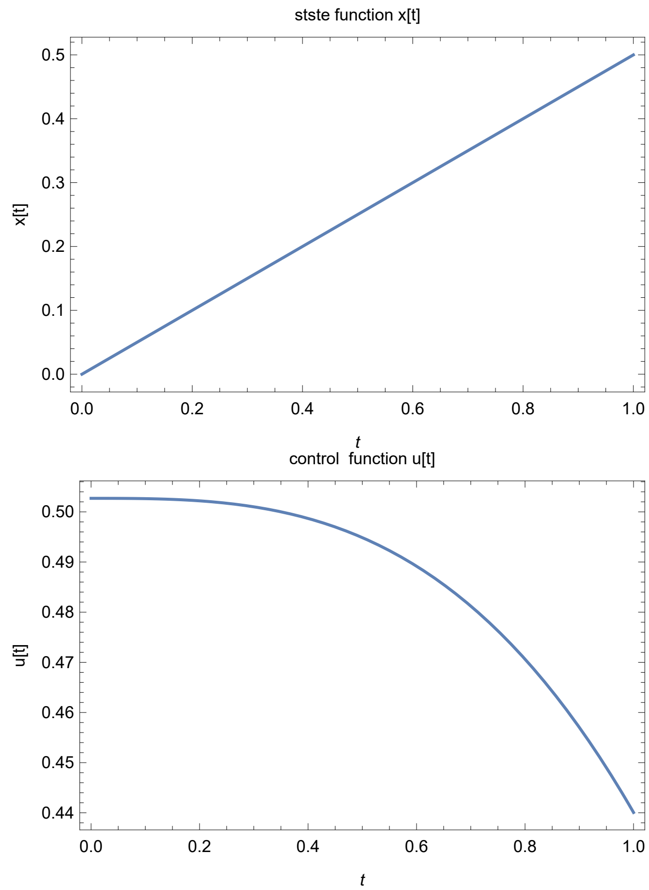


FIGURE 1. Suboptimal state $x(t)$ and control $u(t)$ (Example 2.1).

By using PMP, we obtain the following equations:

$$\begin{cases} \dot{x}_1(t) = -\dfrac{\lambda_1(t)}{I_1^2} - \dfrac{I_3 - I_2}{I_1} x_2(t)x_3(t), \\ \dot{x}_2(t) = -\dfrac{\lambda_2(t)}{I_2^2} - \dfrac{I_1 - I_3}{I_2} x_1(t)x_3(t), \\ \dot{x}_3(t) = -\dfrac{\lambda_3(t)}{I_3^2} - \dfrac{I_2 - I_1}{I_3} x_1(t)x_2(t), \\ \dot{\lambda}_1(t) = \dfrac{I_1 - I_3}{I_2} x_3(t)\lambda_2(t) + \dfrac{I_2 - I_1}{I_3} x_2(t)\lambda_3(t), \\ \dot{\lambda}_2(t) = \dfrac{I_3 - I_2}{I_1} x_3(t)\lambda_1(t) + \dfrac{I_2 - I_1}{I_3} x_1(t)\lambda_3(t), \\ \dot{\lambda}_3(t) = \dfrac{I_3 - I_2}{I_1} x_2(t)\lambda_1(t) + \dfrac{I_1 - I_3}{I_2} x_1(t)\lambda_2(t), \end{cases} \tag{2.6}$$

and the optimal control law is obtained by:

$$\begin{cases} {u^*}_1(t) = -\dfrac{\lambda_1(t)}{I_1}, & t \in [0, 100], \\ {u^*}_2(t) = -\dfrac{\lambda_2(t)}{I_2}, & t \in [0, 100], \\ {u^*}_3(t) = -\dfrac{\lambda_3(t)}{I_3}, & t \in [0, 100]. \end{cases} \tag{2.7}$$

To solve equation (2.6) using q-HAM, consider these initial conditions $x_1(0) = -0.0001t + 0.01 \quad r/s, \quad x_2(0) = -0.00005t + 0.005 \quad r/s, \quad x_3(0) = -0.00001t + 0.001 \quad r/s$, $\lambda_1(t) = \alpha_1$, $\lambda_2(t) = \alpha_2$, and $\lambda_3(t) = \alpha_3$. We have used equation (1.7) to obtain the mth-order deformation equations for $m \geq 1$. Based on the equation (1.8), we have

$$\begin{cases} R_m(\vec{x}_{1,m-1}) = \dot{x}_{1,m-1} + \dfrac{\lambda_{1,m-1}(t)}{I_1^2} + \dfrac{I_3 - I_2}{I_1} \sum_{k=0}^{m-1} x_{2,k} x_{3,m-1-k}, \\ R_m(\vec{x}_{2,m-1}) = \dot{x}_{2,m-1} + \dfrac{\lambda_{2,m-1}(t)}{I_2^2} + \dfrac{I_1 - I_3}{I_2} \sum_{k=0}^{m-1} x_{1,k} x_{3,m-1-k}, \\ R_m(\vec{x}_{3,m-1}) = \dot{x}_{3,m-1} + \dfrac{\lambda_{3,m-1}(t)}{I_3^2} + \dfrac{I_2 - I_1}{I_3} \sum_{k=0}^{m-1} x_{1,k} x_{2,m-1-k}, \\ R_m(\vec{\lambda}_{1,m-1}) = \dot{\lambda}_{1,m-1} - \dfrac{I_1 - I_3}{I_2} \sum_{k=0}^{m-1} x_{3,k} \lambda_{2,m-1-k} - \dfrac{I_2 - I_1}{I_3} \sum_{k=0}^{m-1} x_{2,k} \lambda_{3,m-1-k}, \\ R_m(\vec{\lambda}_{2,m-1}) = \dot{\lambda}_{2,m-1} - \dfrac{I_3 - I_2}{I_1} \sum_{k=0}^{m-1} x_{3,k} \lambda_{1,m-1-k} - \dfrac{I_2 - I_1}{I_3} \sum_{k=0}^{m-1} x_{1,k} \lambda_{3,m-1-k}, \\ R_m(\vec{\lambda}_{3,m-1}) = \dot{\lambda}_{3,m-1} - \dfrac{I_3 - I_2}{I_1} \sum_{k=0}^{m-1} x_{2,k} \lambda_{1,m-1-k} - \dfrac{I_1 - I_3}{I_2} \sum_{k=0}^{m-1} x_{1,k} \lambda_{2,m-1-k}. \end{cases} \tag{2.8}$$

Similar to the previous example and getting $n = 27.515$, we gained $\alpha_1 = 0.7437367602150743$, $\alpha_2 = 0.361839861949632$ and $\alpha_3 = 0.1289813901352318$.

Then

$$
\begin{aligned}
x_{1,1}(t) &= (1-n)(0-1.65353\times 10^{-8}t^2+5.51175\times 10^{-11}t^3+t(-0.0000983465+0.000134457\alpha_1)),\\
x_{1,2}(t) &= (1-n)n(0.-1.65353\times 10^{-8}t^2+5.51175\times 10^{-11}t^3+t(-0.0000983465+0.000134457\alpha_1))+...\\
x_{2,1}(t) &= (1-n)(0.+3.215\times 10^{-8}t^2-1.07167\times 10^{-10}t^3+t(-0.000053215+0.000138181\alpha_2)),\\
x_{2,2}(t) &= (1-n)n(0.+3.215\times 10^{-8}t^2-1.07167\times 10^{-10}t^3+t(-0.000053215+0.000138181\alpha_2))+...,\\
x_{3,1}(t) &= (1-n)(0+5.1501\times 10^{-9}t^2-1.7167\times 10^{-11}t^3+t(-0.000010515+0.0000775032\alpha_3)),\\
x_{3,2}(t) &= (1-n)n(0.+5.1501\times 10^{-9}t^2-1.7167\times 10^{-11}t^3+t(-0.000010515+0.0000775032\alpha_3))+...,\\
&\vdots\\
\lambda_{1,1}(t) &= (1-n)t((0.0003215-1.6075\times 10^{-6}t)\alpha_2+(0.000051501-2.57505\times 10^{-7}t)\alpha_3),\\
\lambda_{1,2}(t) &= (1-n)nt((0.0003215-1.6075\times 10^{-6}t)\alpha_2+(0.000051501-2.57505\times 10^{-7}t)\alpha_3)+...,\\
\lambda_{2,1}(t) &= (1-n)t((-0.000330705+1.65353\times 10^{-6}t)\alpha_1+(0.000103002-5.1501\times 10^{-7}t)\alpha_3),\\
\lambda_{2,2}(t) &= (1-n)nt((-0.000330705+1.65353\times 10^{-6}t)\alpha_1+(0.000103002-5.1501\times 10^{-7}t)\alpha_3)+...,\\
\lambda_{3,1}(t) &= (1-n)t((-0.00165353+8.26763\times 10^{-6}t)\alpha_1+(0.003215-0.000016075t)\alpha_2),\\
\lambda_{3,2}(t) &= (1-n)nt((-0.00165353+8.26763\times 10^{-6}t)\alpha_1+(0.003215-0.000016075t)\alpha_2)+...,\\
&\vdots
\end{aligned}
$$

By applying the proposed method, the numerical results of J_N and the relative error of objective value will be as given in Table 3. The maximum absolute error of the proposed method is compared with that of other methods in Table 4. It is noteworthy that the given method improves the maximum absolute error which indicates the efficiency of the method. The obtained numerical solutions for $m = 4$ and $n = 27.515$ are presented in Figures 2 to 4.

TABLE 3. Numerical results for various iterations (Example 2.2)

N	J_N	$\lvert\frac{J_N-J_{N-1}}{J_N}\rvert$
1	0.00468836	-
2	0.00468813	4.90601×10^{-5}
3	0.00468779	7.25288×10^{-5}
4	0.00468776	6.3996×10^{-6}

Example 2.3. Consider the nonlinear system described by

$$
\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -x_1+x_2(1-x_1^2)+u,\\
x_1(0) &= 1, \qquad x_2(0)=0,
\end{aligned}
$$

TABLE 4. Minimum performance index value of J and final state error of the proposed method and other methods (Example 2.2)

Method	Objective value	Max error
Proposed method (m=4)	0.00468776	$2.20699 * 10^{-17}$
Modified VIM[45]	0.004678	$2.40484 * 10^{-14}$
SHAM Chebyshev (m=6, n=50, h=-1.2) [46]	0.004687	$1.0586 * 10^{-9}$
SHAM Legendre (m=6, n=50, h=-1.2) [46]	0.004687	$1.0589 * 10^{-9}$
HPM (m=3) [10]	0.00468779	——-

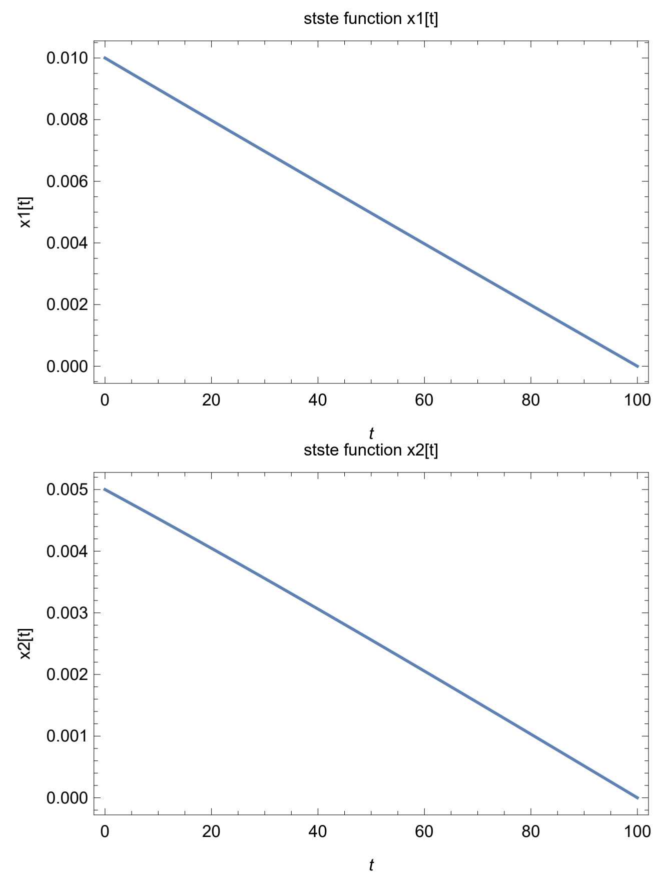


FIGURE 2. Suboptimal state $x1(t)$ and $x2(t)$ (Example 2.2)

and the cost functional

$$J = \frac{1}{2} \int_0^1 (x_1^2 + x_2^2 + u^2) dt.$$

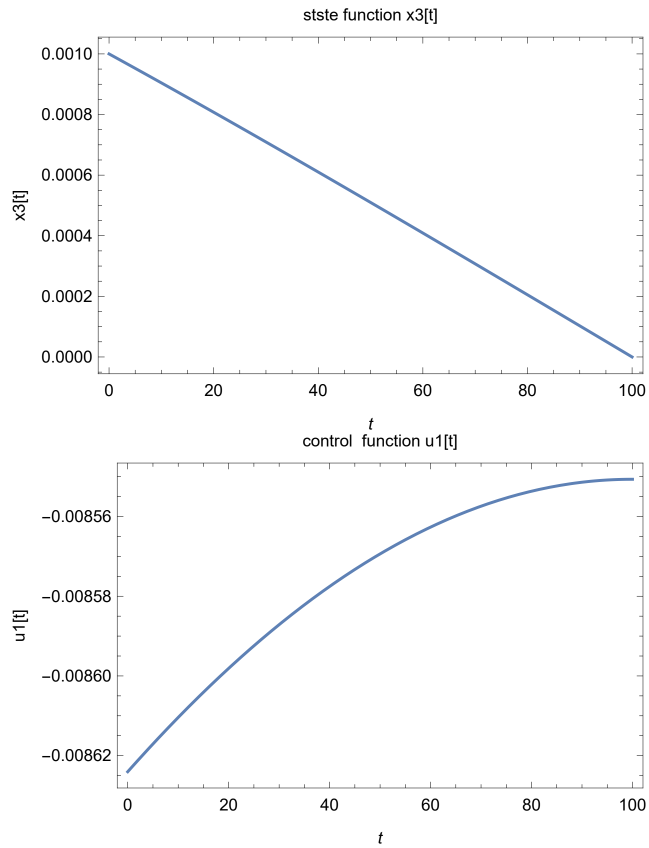
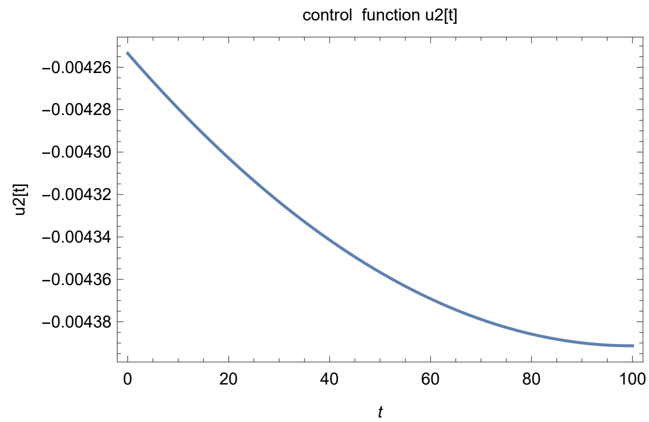


FIGURE 3. Suboptimal state $x3(t)$ and control $u1(t)$ (Example 2.2)

By using PMP, we obtain the following equations:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + x_2(1 - x_1^2) - \lambda_2, \\
\dot{\lambda}_1 &= -x_1 + \lambda_2 + 2x_1x_2\lambda_2, \\
\dot{\lambda}_2 &= -x_2 - \lambda_1 - \lambda_2 + \lambda_2 x_1^2,
\end{aligned}$$

we consider the initial condition $x_1(0) = 1$, $x_2(0) = 0$, $\lambda_1(1) = \lambda_2(1) = 0$, and the optimal control is $u = -\lambda_2$. Similar to the previous examples, we gained $\alpha_1 = 2.352709654003546$ and $\alpha_2 = 0.48235132085840954$.



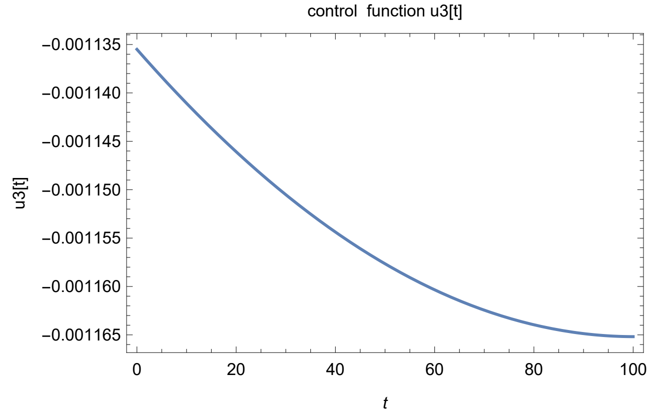


FIGURE 4. Suboptimal control $u2(t)$ and $u3(t)$ (Example 2.2)

Then

$$
\begin{aligned}
&x_{1,1}(t) = 0,\\
&x_{1,2}(t) = \frac{1}{2}(1-n)(-1+n)\,t^2(1+\alpha_2),\\
&x_{2,1}(t) = (1-n)\,t(1+\alpha_2),\\
&x_{2,2}(t) = (1-n)\,n\,t(1+\alpha_2) - \frac{1}{2}(1-n)(-1+n)\,t(2+t\alpha_1+2\alpha_2),\\
&\vdots\\
&\lambda_{1,1}(t) = (1-n)(t-t\alpha_2),\\
&\lambda_{1,2}(t) = (1-n)n(t-t\alpha_2) + \frac{1}{2}(1-n)(-1+n)t(2(-1+\alpha_2)+t(\alpha_1+2\alpha_2(1+\alpha_2))),\\
&\lambda_{2,1}(t) = (1-n)t\,\alpha_1,\\
&\lambda_{2,2}(t) = (1-n)n\,t\,\alpha_1 - (1-n)(-1+n)t(t+\alpha_1),\\
&\vdots
\end{aligned}
$$

Table 5 presents the numerical results for J_N and the relative error of objective value as obtained by applying the proposed method. In Table 6, the maximum absolute error of the proposed method is compared with that of other methods. The obtained numerical solutions for $m = 8$ and $n = 2.7385$ are depicted in Figures 5 to 6.

TABLE 5. Numerical results for various iterations (Example 2.3)

N	J_N	$\frac{J_N - J_{N-1}}{J_N}$
1	3.94696	-
2	2.53263	$5.584431994 \times 10^{-1}$
3	1.56757	$6.156407688 \times 10^{-1}$
4	1.24656	$2.575166859 \times 10^{-1}$
5	1.27434	$2.17995198 \times 10^{-2}$
6	1.02859	$2.389192973 \times 10^{-1}$
7	1.06416	$3.34254247 \times 10^{-2}$
8	1.18127	$9.91390622 \times 10^{-2}$

TABLE 6. Minimum performance index value of J and final state error of the proposed method and other methods (Example 2.3)

Method	Objective value	Max error
Proposed method (m=8)	1.18127	$2.17384 * 10^{-6}$
SHAM Chebyshev(m=15, N=50, h=−0.5)	1.0472	$4.2749 * 10^{-4}$
SHAM Legendre (m=15, N=50, h=−0.5)	1.0472	$4.2749 * 10^{-4}$
DT (m=15)	1.0478	$4.4380 * 10^{-4}$

3. Conclusion

In this paper, we implemented the q-HAM to solve NOCPs. This method has been applied for the first time for NOCPs and can also be adapted to solving OCP with integral constraints and fractional problems. The q-HAM is a semi-exact method for solving linear and nonlinear equations which does not need small/large parameters. The q-HAM contains an auxiliary parameter n that provides us a convenient way to guarantee the convergence of solution series so that it is valid even if non-linearity becomes rather strong. The accuracy and efficiency of q-HAM are presented with some examples and the results of solving OCP by q-HAM are compared with other methods. Comparing the proposed method reveals that the accuracy of the q-HAM is better than

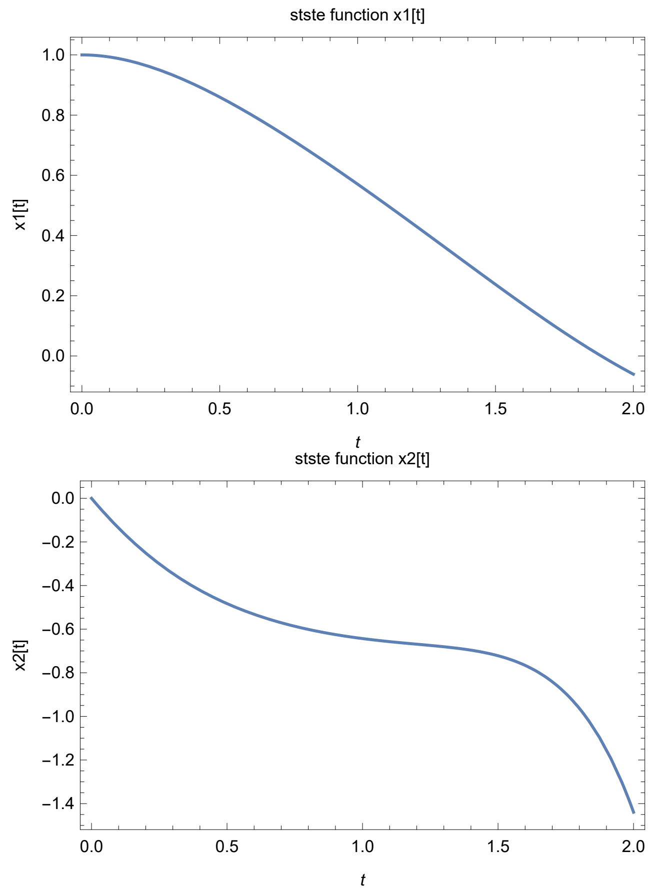


FIGURE 5. Suboptimal state $x1(t)$ and $x2(t)$ (Example 2.3)

other methods. For example, as you can see in the table of example 3, the q-HAM has performed better than the HAM and has a faster convergence. Also, the q-HAM software programs are easily written with any software. Mathematica software was used for the current study.

References

[1] Itik M., Salamci M. U., Banksa S. P. (2009). ”Optimal control of drug therapy in cancer treatment”, Nonlinear Analysis, 71, 1473 - 1486.

[2] Wei S., Zefran M., Decarlo R. A. (2008). ”Optimal control of robotic system with logical constraints, application to UAV path planning”, Proceeding (s) of the IEEE International Conference on Robotic and Automation, Pasadena, CA, USA.

[3] Garrard W. L., Jordan J. M. (1997). ”Design of nonlinear automatic flight control systems”, Automatica, 13, 497 - 505.

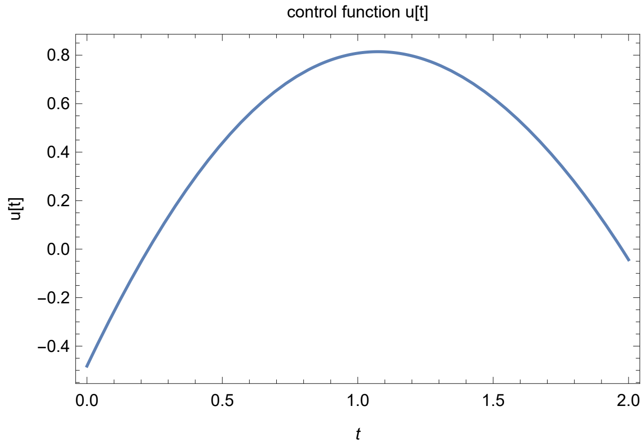


FIGURE 6. Suboptimal control $u(t)$ (Example 2.3)

[4] Nolsu T., Konishi M., Imai J. (2008). "Optimal water cooling control for plate rolling", International Journal of Innovative Computing, 4, 3169 - 3181.

[5] Manousiouthakis V., Chmielewski D. J. (2002). "On constrained infinite - time nonlinear optimal control", Chemical Engineering Science, 57, 105 - 114.

[6] Rubio J. E. (1986). "Control and optimization, the linear treatment of nonlinear problems", Manchester University Press, Manchester, UK.

[7] Saberi Nik H., Effati S., Motsa S. S., Shirazian M. (2014). "Spectral homotopy analysis method and its convergence for solving a class of nonlinear optimal control problems", Numerical Algorithms, 65, 171 - 194.

[8] Alipour M., Vali M. A., Borzabadi A. H. (2019). " A hybrid parametrization approach for a class of nonlinear optimal control problems", Numerical Algebra, Control and Optimization, 9 (4), 493 - 506.

[9] Effati S., Saberi Nik H., Shirazian M. (2013). "Analytic-approximate solution for a class of nonlinear optimal control problems by homotopy analysis method", Asian-European Journal of Mathematics, 6, 1 - 22.

[10] Jajarmi A., Pariz N., Vahidian Kamyad A., Effati S. (2012). "A highly computational efficient method to solve nonlinear optimal control problems", Scientia Iranica D, 19 (3), 759 - 766.

[11] Hwang I. (2007). "A computational approach to solve optimal control problems using differential transformation", In Proceedings of the 2007 American Control Conference, Marriott Marquis Hotel at Times Square, New York City, USA, 11-13.

[12] El-Kady M. (2012). "Legendre approximations for solving optimal control problems governed by ordinary differential equations", International Journal of Control Science and Engineering, 2, 54 - 59.

[13] Jia W., He X., Guo L. (2017). "The optimal homotopy analysis method for solving linear optimal control problems", Applied Mathematical Modelling, 45, 865 - 880.

[14] Kafash B., Delavarkhalafi A., Karbassi S. M., Boubaker K. (2014). "A numerical approach for solving optimal control problems using the Boubaker polynomials expansion scheme", Journal of Interpolation and Approximation in Scientific Computing, 2014, 1 - 18.

[15] Lin Q., Loxton R., Teo K. L. (2014). "The control parameterization method for nonlinear optimal control: A survey", Journal of Industrial and Managment Optimization, 10, 275 - 309.

[16] Mirinejad H., Inanc T. (2017). "An RBF collocation method for solving optimal control problems", Robotics and Autonomous Systems, 87, 219 - 225.

[17] Nazemi A., Hesam S., Haghbin A. (2015). "An application of differential transform method for solving nonlinear optimal control problems", Computational Methods for Differential Equations, 3, 200 - 217.

[18] Saberi Nik H., Effati S., Motsa S. S., Shirazian M. (2014). "Spectral homotopy analysis method and its convergence for solving a class of nonlinear optimal control problems", Numerical Algorithms, 65, 171 - 194.

[19] Shirazian M., Effati S. (2012). "Solving a class of nonlinear optimal control problems via Hes variational iteration method", International Journal of Control, Automation, and Systems, 10, 249 - 256.

[20] Chen X. S., Li X. K., Zhang L. L., and Cai S. T. (2017). "A new spectral method for the nonlinear optimal control, Proceedings of the 36th Chinese Control Conference", July 26-28, Dalian, China.

[21] Ahmad H., Akg¨ul A., Khan T. A., Stanimirovi'c P. S., and Chu Y. M. (2020). "New perspective on the conventional solutions of the nonlinear time-fractional partial differential equations", Complexity, Volume 2020, Article ID 8829017, https://doi.org/10.1155/2020/8829017.

[22] Inca M., Khanc M. N., Ahmadd I., Yaoe S. V., Ahmadc H., and Thounthonge P. (2020). "Analysing time-fractional exotic options via efficient local meshless method", Results in Physics, Volume 19, Article ID 103385, https://doi.org/10.1016/j.rinp.2020.103385.

[23] Ahmad H., Khan T. A., Ahmadd I., Stanimirovi'c P. S., and Chu Y. M. (2020). "A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations", Results in Physics, Volume 19, Article ID 103462, https://doi.org/10.1016/j.rinp.2020.103462.

[24] Ahmad H., Khan T. A. (2019). "Variational iteration algorithm-I with an auxiliary parameter for wave-like vibration equations", Journal of Low Frequency Noise, Vibration and Active Control, Volume 38, Issue 3-4, https://doi.org/10.1177/1461348418823.

[25] Ahmad H., Khan T. A., Cesarano C. (2019). "Numerical solutions of coupled Burgers' equations", Axioms, 8(4), 119, https://doi.org/10.3390/axioms8040119.

[26] Ahmad H., Khan T. A., Stanimirovi'c P. S., Chu Y. M., and Ahmadd I. (2020). "Modified variational iteration algorithm-II: convergence and applications to diffusion models", Complexity, Volume 2020, Article ID 8841718, https://doi.org/10.1155/2020/8841718.

[27] Bazighifan O., Ahmad H., and Yao S. W. (2020). "New Oscillation Criteria for Advanced Differential Equations of Fourth Order", Mathematics, 8(5), 728, https://doi.org/10.3390/math8050728.

[28] Ahmad H., Seadawy A. R., Khan T. A. (2020). "Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm", Mathematics and Computers in Simulation, 177, 13-23.

[29] Liao S. J. (1992). "The proposed homotopy analysis technique for the solution of nonlinear problems", China: Shanghai Jiao Tong University-Ph.D. Thesis.

[30] Jafari H., Alipour M. (2010). "Solution of the Davey–Stewartson equation using homotopy analysis method", Nonlinear Analysis: Modelling and Control, 15, 423 - 433.

[31] El-Tawil M. A., Huseen S. N. (2012). "The q-Homotopy Analysis Method (q-HAM)", International Journal of Applied mathematics and mechanics, 8, 51 - 75.

[32] Mohamed S. M., Hamed S. Y. (2016). "Solving the convection–diffusion equation by means of the optimal q-homotopy analysis method (Oq-HAM)", Results in Physics, 6, 20 - 25.

[33] El-Tawil M. A., Huseen S. N. (2013). "On convergence of the q-homotopy analysis method", International Journal of Contemporary Mathematical Sciences, 8, 481 - 497.

[34] Huseen S. N., Grace S. R. (2013). "Approximate solutions of nonlinear partial differential equations by modified q-homotopy analysis method (mq-HAM)", Hindawi Publishing Corporation, Journal of Applied Mathematics, Article ID 569674 9.

[35] Huseen S. N., Grace S. R., El-Tawil M. A. (2013). "The optimal q-homotopy analysis method (Oq-HAM)", International Journal of Computers and Technology, 11, 2859 - 2866.

[36] Iyiola O. S. (2013). "q-Homotopy analysis method and application to fingero-imbibition phenomena in double phase flow through porous media", Asian Journal of Current Engineering and Maths, 2, 283 - 286.

[37] Iyiola O. S. (2013). "A numerical study of Ito equation and Sawada–Kotera equation both of time-fractional type", Advances in Mathematics: Scientific Journal, 2, 71 - 79.

[38] Iyiola O. S., Soh M. E., Enyi C. D. (2013). "Generalized homotopy analysis method (q-HAM) for solving foam drainage equation of time fractional type", Mathematics in Engineering, Science and Aerospace, 4(4), 229 - 440.

[39] Mohamed M. S., Elagan S. K., Gepreel K. A., Elosimy A. (2014). "Optimal $q$homotopy analysis method for nonlinear fractional gas dynamics equation", Jökull Journal, 64 (9), 317 - 326.

[40] Mohamed S. M., Hamed Y. S. (2016). "Solving the fractional Fornberg–Whitham equation by means of the optimal q-homotopy analysis method (Oq-HAM)", Italian Journal of Pure and Applied Mathematics, 36, 345 - 358.

[41] Liao S. J. (2003). "Beyond perturbation: Introduction to the homotopy analysis method", CRC Press, Boca Raton: Chapman Hall.

[42] Effati S., Saberi Nik H. (2011). "Solving a class of linear and non-linear optimal control problems by homotopy perturbation method", IMA Journal of Mathematical Control and Information, 28, 539 - 553.

[43] Jajarmi A., Pariz N., Vahidian Kamyad A., Effati S. (2011). "A novel modal series representation approach to solve a class of nonlinear optimal control problems", International Journal of Innovative Computing, Information and Control, 1413 - 1425.

[44] Junkins J. L., Turner J. D. (1986). "Optimal Spaceraft Rotaional Maneuvers", Elsevier, Amsterdam.

[45] Omidiniya P., Alipour M. (2020). "A Semi-Analytic Method for Solving a Class of Non-Linear Optimal Control Problems", International Journal of Industrial Electronics, Control and Optimization, 3, 407 - 414.

[46] Saberi Nik H., Effati S., Motsa S. S., Shirazian M. (2014). "Spectral homotopy analysis method and its convergence for solving a class of nonlinear optimal control problems", Numerical Algorithms, 65, 171 - 194.