A new application of q- homotopy analysis method to solve non-linear optimal control problems

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This paper advances a new application of q-homotopy analysis method (q-HAM) to solve non-linear optimal control problems(NOCPs). First, the NOCP was transformed into a non-linear two-point boundary value problem by using the Pontryagins maximum principle (PMP). Then, we applied the q-HAM to solve this system. The proposed method is based on the HAM but the q-HAM, has an increased interval of convergence than the HAM. Three examples are provided to demonstrate the reliability and efficiency of the method. Next, the numerical results of the proposed method are compared with those of other methods. As can be seen from the tables, the maximum error in the second and third examples is much better than other methods.

Keywords: q- homotopy analysis method; Non-linear optimal control problems; Pontryagins maximum principle; Analytic - approximate solution.

1. Introduction

Optimal control problems are widely used in various fields such as biomedicine [1], robotics [2], aircraft systems [3], physics [4], and economics [5]. Optimal control of non-linear systems has been one of the most challenging issues which have been extensively studied by researchers over the years. Since there is no analytical solution for non-linear optimal control problems (NOCPs), considerable attention has been devoted to such problems. Many approaches have been proposed to solve these equations. The so-called measure theory is one example proposed by Rubio [6]. This technique transforms NOCPs to linear programming problems and provides a piecewise constant control law. In [7], optimal control problems were solved using the spectral homotopy analysis method. More recently, a hybrid parametrization approach was introduced by [8] for solving NOCPs. In this paper, an effective technique is developed for control parametrization and state variables are computed using homotopy analysis method (HAM). Effati et al. [9] used HAM to present an analytic-approximate solution for these equations. The optimal homotopy perturbation method was used by Jajarmi et al.[10] for answering a class of NOCPs. Hwang et al. [11] constructed a computational approach to unravel optimal control problems using differential transformation. Legendre approximations were suggested by El-Kady et al. [12] to respond to optimal control problems governed by higher-order ordinary differential equations. Jia et al. [13] introduced a new method based on optimal HAM and Pontryagins maximum principle (PMP) for solving linear optimal control problems. A numerical approach based on the Boubaker polynomials expansion scheme was suggested by Kafash et al. [14] to make answer optimal control problems. Lin et al. [15] introduced the fundamentals of the control parameterization method and examined its various applications to non-standard optimal control problems. The radial basis function (RBF) collocation method was proposed by Mirinejad et al. [16] to deal with OCP, robotics, and autonomous systems. Nazemi et al. [17] developed an algorithm capable of solving a class of NOCPs. A combination of the hybrid spectral collocation technique and HAM was used by Saberi et al. [18] to construct an iteration algorithm for solving a class of NOCPs. He's variational iteration method was proposed by Shirazian et al. [19] to investigate a class of NOCPs. The modified variational iteration method (MVIM) and the advantages of using this method are discussed in that paper. Chen et al. [20] provided a new spectral method based on Galerkin approximation solutions of the non-linear optimal control systems. Overwhelming research has been done

on developing applications of fractional iteration algorithm in engineering fields and physical models [21, 22, 23]. The authors studied new modified variational iteration algorithmsI, modified variational iteration algorithmsII and Riccati transformation methods in [24, 25, 26, 27, 28] to approximate the solutions of kind of the problems.

HAM is a semi-exact method for solving non-linear equations which does not need small/large parameters proposed by Liao [29]. This is an efficient method to solve highly linear and non-linear problems in various fields [30]. One of the strengths of HAM, compared to other analytical methods, concerns its great freedom to choose the initial approximation. Besides, it contains a specific auxiliary parameter which facilitates adjusting and controlling the convergence region and the rate of convergence of the series solution.

El-Tawil and Huseen [31] formulated a method called q-HAM which is a more general method of HAM. This method has been successfully applied to solve many types of non-linear problems [32, 33, 34, 35, 36, 37, 38, 39, 40].

In this work, an indirect method based on q-HAM is introduced to solve NOCPs. One of the main preference of q-HAM is that it contains an auxiliary parameter n same as h such that the case of $n = 1$ (q-HAM; $n = 1$) the standard HAM can be reached. But, this fraction factor n has much flexibility to adjust and control the convergence region and rate of convergence of the series of solution compared to auxiliary parameter h in HAM. To show the efficiency of the method and its effectiveness, three practical examples have been solved and compared with other methods.

The paper is organized as follows: Section2 introduces q-HAM to solve optimal control problems; In Section 3, three examples are presented to show the efficiency of the method; In the last Section, some conclusions are drawn based on the findings.

1.1. **The basic concepts of the q-HAM.** In order to understand the fundamentals of q-HAM, consider the following differential equations:

$$N[w(t)] = 0, \tag{1.1}$$

where N represents a non-linear operator and $w(t)$ is stands for an unknown function[41]. It is possible to formulate the zero-order deformation equation thus:

$$(1 - nq)L[\phi(t; q) - w_0(t)] = F(n)qN[\phi(t; q)], \tag{1.2}$$

$q \in [0, \dfrac{1}{n}]$ signifies an embedding parameter, n denotes a non-zero auxiliary parameter, $F(n)$ represents a non-zero auxiliary function $n \geq 1$, L is an auxiliary linear operator with the property $L(c_1) = 0$ where c_1 denotes an integral constant, $w_0(t)$ shows an initial guess of $w(t)$, and

$\phi(t; q)$ stands for an unknown function. We have great freedom to take auxiliary objects such as q and L in HAM. Obviously, when $q = 0$ and $q = \dfrac{1}{n}$,

$$\phi(t; 0) = w_0(t), \qquad \phi(t; 1) = w(t), \tag{1.3}$$

respectively. Thus, as q rises from 0 to 1, the solutions $\phi(t; q)$ alter from the initial guess $w_0(t)$ to the solution $w(t)$. By expanding $\phi(t; q)$ in Taylor series with regard to q, we will have

$$\phi(t; q) = w_0(t) + \sum_{m=1}^{+\infty} w_m(t) q^m, \tag{1.4}$$

where

$$w_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t; q)}{\partial q^m} \Big|_{q=0}. \tag{1.5}$$

Let's suppose that $F(n)$, $w_0(t)$, and L are accurately selected such that the series converges at $q = \dfrac{1}{n}$ and we have

$$w(t) = \phi(t; \frac{1}{n}) = w_0(t) + \sum_{m=1}^{+\infty} w_m(t)(\frac{1}{n})^m. \tag{1.6}$$

Defining the vector $w_r(t) = \{w_0(t), w_1(t), w_2(t), ..., w_r(t)\}$. Differentiating equation (1.2) m times with regard to q and then setting $q = 0$ and eventually dividing them by $m!$, we will obtain the mth order deformation equation:

$$L[w_m(t) - k_m w_{m-1}(t)] = F(n) R_m(\vec{w}_{m-1}), \tag{1.7}$$

where

$$R_m(\vec{w}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t; q)]}{\partial q^{m-1}} \Big|_{q=0}, \tag{1.8}$$

and

$$k_m = \begin{cases} 0, & m \leqslant 1, \\ n, & \text{otherwise} \end{cases} \tag{1.9}$$

It needs to be stressed that $w_m(t)$ for $m \geqslant 1$ is governed by the linear equation (1.7) with linear boundary conditions which are related to the original problem. Let

$$\triangle_m = \int_\Omega (N[w_m(t)])^2 d\Omega, \tag{1.10}$$

where

$$w(t) = w_0(t) + \sum_{m=1}^{+\infty} w_m(t)(\frac{1}{n})^m \tag{1.11}$$

represents the square residual error of the mth-order approximation of the equation (1.1) which is integrated into the whole domain Ω. Theoretically, if the square residual error tends $\triangle_m$ to zero, then

$$w(t) = \sum_{m=1}^{+\infty} w_m(t)(\frac{1}{n})^m \qquad (1.12)$$

will serve as a series solution for the original equation (1.1). Besides, the optimal value of the auxiliary parameter n is given by solving the following non-linear algebraic equation

$$\frac{d\triangle_m}{dn} = 0. \qquad (1.13)$$

1.2. **Non-linear optimal control problems.** Consider the following non-linear dynamical system

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), \qquad t \in [t_0, t_f],$$
$$x(t_0) = x_0, \qquad x(t_f) = x_f, \qquad (1.14)$$

where $u(t) \in \Re^m$ is the control variable, $x(t) \in \Re^s$ indicates the state variable, x_0 and x_f are the given initial and final states at t_0 and t_f, respectively. Moreover, $f(t, x(t)) \in \Re^s$ and $g(t, x(t))u(t) \in \Re^{s \times m}$ are two continuously differentiable functions in all arguments. We have the quadratic objective functional as follows:

$$J[x, u] = \frac{1}{2} \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \qquad (1.15)$$

which should be minimized subject to the non-linear system (1.14), for $Q \in \Re^{s \times s}$ and $R \in \Re^{m \times m}$ positive semi-definite and positive definite matrices, respectively. Since the performance index (1.15) is convex, the following extreme necessary conditions are also sufficient for optimality:

$$\dot{x} = f(t, x(t)) + g(t, x(t))u^*,$$
$$\dot{\lambda} = -H_x(x, u^*, \lambda),$$
$$u^* = arg\, min_u H_x(x, u, \lambda), \qquad (1.16)$$
$$x(t_0) = x_0, \qquad x(t_f) = x_f,$$

where $H_x(x, u, \lambda) = \frac{1}{2}[x^T Qx + u^T Ru] + \lambda^T[f(t, x) + g(t, x)u]$ is referred to as the Hamiltonian. Equivalently, (1.16) can be written in the form of:

$$\dot{x} = f(t, x) + g(t, x)[-R^{-1}g^T(t, x)\lambda],$$
$$\dot{\lambda} = -(Qx + (\frac{\partial f(t, x)}{\partial x})^T\lambda + \sum_{i=1}^{s} \lambda_i[-R^{-1}g^T(t, x)\lambda]^T \frac{\partial g_i(t, x)}{\partial x} \quad (1.17)$$
$$x(t_0) = x_0, \qquad x(t_f) = x_f,$$

where $\lambda(t) \in \Re^s$ is the co-state vector with the ith component, $\lambda_i(t), i = 1, ..., s$ and $g(t, x) = [g_1(t, x), ..., g_s(t, x)]^T$, with $g_i(t, x) \in \Re^m, i = 1, ..., s$. Also, the optimal control law is obtained by

$$u^* = -R^{-1}g^T(t, x)\lambda. \qquad (1.18)$$

For solving the system (1.17), we can solve the following system with q-HAM method

$$\dot{x} = f(t, x) + g(t, x)[-R^{-1}g^T(t, x)\lambda],$$

$$\dot{\lambda} = -(Qx + (\frac{\partial f(t, x)}{\partial x})^T\lambda + \sum_{i=1}^{s} \lambda_i[-R^{-1}g^T(t, x)\lambda]^T\frac{\partial g_i(t, x)}{\partial x} \qquad (1.19)$$

$$x(t_0) = x_0, \qquad \lambda(t_0) = \alpha,$$

where $\alpha \in \Re$ is an unknown parameter. Which is obtained after sufficient iterations of q-HAM, as discussed hereinafter. We can change the equation (1.17) as follows:

$$\dot{x} = \theta_1(t, x, \lambda) \qquad (1.20)$$
$$\dot{\lambda} = \theta_2(t, x, \lambda) \qquad t \in [t_0, t_f]$$
$$x(t_0) = x_0, \qquad \lambda(t_0) = \alpha,$$

1.3. The basic idea of q-HAM for NOCPs. To solve the non-linear system (1.19) by means of q-HAM method, consider these following initial approximations:

$$x_0(t) = 0,$$
$$\lambda_0(t) = \alpha,$$

Besides, we consider the linear operators as follows:

$$L[\phi_i(t; q)] = \frac{\partial \phi_i(t; q)}{\partial t},$$

based on equation (1.20), the following non - linear operators will obtain:

$$N_1[\phi_i(t; q)] = \frac{\partial \phi_1(t, q)}{\partial t} - \theta_1(t, \phi_1(t, q), \phi_2(t, q)),$$

$$N_2[\phi_i(t; q)] = \frac{\partial \phi_2(t, q)}{\partial t} - \theta_2(t, \phi_1(t, q), \phi_2(t, q)).$$

We also constructed the following zero-order deformation equations:

$$(1 - nq)L[\phi_1(t; q) - x_0(t)] = F(n)qN_1[\phi_i(t; q)], \qquad (1.21)$$
$$(1 - nq)L[\phi_2(t; q) - \lambda_0(t)] = F(n)qN_2[\phi_i(t; q)].$$

Obviously, when $q = 0$ and $q = \frac{1}{n}$, we will have

$$\phi_1(t; 0) = x_0(t), \qquad \phi_1(t; \frac{1}{n}) = x(t),$$

$$\phi_2(t; 0) = \lambda_0(t), \qquad \phi_2(t; \frac{1}{n}) = \lambda(t),$$

respectively. Therefore, as q rises from 0 to $\frac{1}{n}$, the solutions $\phi_i(t; q)$ changes from the initial guesses $x_0(t), \lambda_0(t)$ to the solutions $x(t), \lambda(t)$.

$$x(t) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t)(\frac{1}{n})^m, \qquad (1.22)$$

$$\lambda(t) = \lambda_0(t) + \sum_{m=1}^{+\infty} \lambda_m(t)(\frac{1}{n})^m,$$

where

$$x_m(t) = D_m[\phi_1(t; q)] = \frac{1}{m!} \frac{\partial^m \phi_1(t; q)}{\partial q^m}|_{q=0}, \qquad (1.23)$$

$$\lambda_m(t) = D_m[\phi_2(t; q)] = \frac{1}{m!} \frac{\partial^m \phi_2(t; q)}{\partial q^m}|_{q=0},$$

in which D_m denotes the mth-order q-homotopy-derivative of ϕ_i.

$$\vec{x}_k = \{x_0(t), x_1(t), ..., x_k(t)\},$$

$$\vec{\lambda}_k = \{\lambda_0(t), \lambda_1(t), ..., \lambda_k(t)\}.$$

By differentiating equation(1.21) m times with regard to the embedding parameter q and setting $q = 0$ and ultimately dividing them by $m!$, the so-called m^{th}-order deformation equations will obtain as follows

$$L[x_m(t) - k_m x_{m-1}(t)] = F(n) R_m(\vec{x}_{m-1}), \qquad (1.24)$$

$$L[\lambda_m(t) - k_m \lambda_{m-1}(t)] = F(n) R_m(\vec{\lambda}_{m-1}), \qquad (1.25)$$

where

$$R_m(\vec{x}_{m-1}) = x'_{m-1} - \theta_{1m-1}(t, x_{m-1}, \lambda_{m-1}), \qquad (1.26)$$

$$R_m(\vec{\lambda}_{m-1}) = \lambda'_{m-1} - \theta_{2m-1}(t, x_{m-1}, \lambda_{m-1}), \qquad (1.27)$$

and

$$k_m = \begin{cases} 0, & m \leqslant 1, \\ n, & \text{otherwise} \end{cases} \qquad (1.28)$$

Theorem 1.1. *If the series solution defined in (1.22) is convergent, then it converges to an exact solution of the non-linear problem (1.20).*

As the series is hypothetically convergent, it holds:

$$V(t) = \sum_{m=0}^{+\infty} x_m(t)(\frac{1}{n})^m,$$

$$W(t) = \sum_{m=0}^{+\infty} \lambda_m(t)(\frac{1}{n})^m.$$

Consequently, the necessary condition for the convergence of the series is valid; in other words:

$$\lim_{m\to\infty} x_m(t)(\frac{1}{n})^m = 0, \tag{1.29}$$

$$\lim_{m\to\infty} \lambda_m(t)(\frac{1}{n})^m = 0.$$

Using (1.24), (1.25), and L (i.e., linear operator), we have:

$$F(n)\sum_{m=1}^{+\infty} R_m(\vec{x}_{m-1}) = \sum_{m=1}^{+\infty} L[x_m(t) - k_m x_{m-1}(t)], \tag{1.30}$$

$$F(n)\sum_{m=1}^{+\infty} R_m(\vec{\lambda}_{m-1}) = \sum_{m=1}^{+\infty} L[\lambda_m(t) - k_m \lambda_{m-1}(t)]. \tag{1.31}$$

We multiply the sides of equation (1.30) in $\sum_{m=1}^{+\infty}(\frac{1}{n})^m$

$$F(n)\sum_{m=1}^{\infty}(\frac{1}{n})^m R_m(\vec{x}_{m-1}) = \sum_{m=1}^{\infty} L\left[(\frac{1}{n})^m x_m(t) - (\frac{1}{n})^m k_m x_{m-1}(t)\right] =$$

$$L\left[\lim_{i\to\infty}\sum_{m=1}^{i}\left((\frac{1}{n})^m x_m(t) - (\frac{1}{n})^m k_m x_{m-1}(t)\right)\right] = L[\lim_{i\to\infty}(((\frac{1}{n})x_1(t) - (\frac{1}{n})k_1 x_0(t)) +$$

$$((\frac{1}{n})^2 x_2(t) - (\frac{1}{n})^2 k_2 x_1(t)) + ((\frac{1}{n})^3 x_3(t) - (\frac{1}{n})^3 k_3 x_2(t)) + ... + ((\frac{1}{n})^{i-1} x_{i-1}(t)$$

$$-(\frac{1}{n})^{i-1} k_{i-1} x_{i-2}(t)) + ((\frac{1}{n})^i x_i(t) - (\frac{1}{n})^i k_i x_{i-1}(t)))].$$

By considering equation (1.28), we have

$$L[\lim_{i\to\infty}(\frac{1}{n})^i x_i(t)],$$

and based on equation (1.29) we have

$$L[\lim_{i\to\infty}(\frac{1}{n})^i x_i(t)] = 0.$$

Hence

$$F(n) \sum_{m=1}^{+\infty} (\frac{1}{n})^m R_m(\vec{x}_{m-1}) = 0.$$

On the other hand, according to the definition:

$$\sum_{m=1}^{+\infty} (\frac{1}{n})^m R_m(\vec{x}_{m-1}) = 0,$$

based on equation (1.26), we have

$$\sum_{m=1}^{+\infty} (\frac{1}{n})^m (x'_{m-1} - \theta_{1m-1}(t, x_{m-1}, \lambda_{m-1})) = 0;$$

subsequently,

$$\sum_{m=1}^{+\infty} (\frac{1}{n})^m x'_{m-1} = \sum_{m=1}^{+\infty} (\frac{1}{n})^m \theta_{1m-1}(t, x_{m-1}, \lambda_{m-1}),$$

and finally

$$x' = \theta_1(t, x, \lambda).$$

In the case of $\lambda' = \theta_2(t, x, \lambda)$, the proof is similar. This completes the proof.

2. APPLICATION

In this part, we solve three examples of NOCPs to demonstrate the efficiency and accuracy of the proposed method.

Example 2.1. We consider the following NOCP:

$$min J = \int_0^1 u^2(t) dt,$$

$$s.t. \quad \dot{x} = \frac{1}{2} x^2(t) \sin x(t) + u(t), \qquad t \in [0,1] \qquad (2.1)$$

$$x(0) = 0, \quad x(1) = 0.5.$$

By using PMP, we obtain the following equations:

$$\dot{x} = \frac{1}{2} x^2(t) \sin x(t) - \frac{1}{2} \lambda(t), \qquad t \in [0,1]$$

$$\dot{\lambda} = -\lambda(t) x(t) \sin x(t) - \frac{1}{2} \lambda(t) x^2(t) \cos x(t), \qquad (2.2)$$

$$x(0) = 0, \quad \lambda(0) = \alpha,$$

such that $u(t) = -\frac{1}{2}\lambda(t)$. Based on equation (1.26), we have

$$R_m(\vec{x}_{m-1}) = \dot{x}_{m-1} - \frac{1}{2}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N_1(\sum_{k=0}^{m-1}q^k x_k, \sum_{k=0}^{m-1}q^k \lambda_k)_{q=0} + \frac{1}{2}\lambda_{m-1},$$

$$R_m(\vec{\lambda}_{m-1}) = \dot{\lambda}_{m-1} + \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N_2(\sum_{k=0}^{m-1}q^k x_k, \sum_{k=0}^{m-1}q^k \lambda_k)_{q=0}$$

$$+\frac{1}{2}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N_3(\sum_{k=0}^{m-1}q^k x_k, \sum_{k=0}^{m-1}q^k \lambda_k)_{q=0}.$$

By getting $n = 3.695$ and according to the final state condition, we must have $\sum_{i=0}^{4} x_i(1) = 0.5$, which yields $\alpha = -1.0053935951332145$. Now, we successively calculate the following:

$$x_1(t) = \frac{1}{2}(1-n)t\alpha,$$

$$x_2(t) = -(\frac{1}{2})(1-n)(-1+n)t\alpha + \frac{1}{2}(1-n)nt\alpha,$$

$$x_3(t) = -(\frac{1}{2})(1-n)(-1+n)t\alpha + n(-\frac{1}{2}(1-n)(-1+n)t\alpha + \frac{1}{2}(1-n)nt\alpha),$$

$$\vdots$$

$$\lambda_1(t) = 0,$$

$$\lambda_2(t) = 0,$$

$$\lambda_3(t) = \frac{1}{8}(1-n)(-1+n)^2 t^3 \alpha^3,$$

$$\vdots$$

By applying the proposed method, the numerical results of J_N and the relative error of objective value will be as given in Table 1. Table 2 compares the maximum absolute error of the proposed method with that of other methods, which shows the proposed method has achieved similar results with other methods. The obtained numerical solutions for $m = 4$ and $n = 3.695$ are provided in Figure 1.

Example 2.2. In this example, we consider the optimal maneuvers of a rigid asymmetric spacecraft [44]. The Euler's equations for the angular

TABLE 1. Numerical results of various iterations (Example 2.1)

N	J_N	$\frac{J_N - J_{N-1}}{J_N}$
1	0.252704	-
2	0.252704	0
3	0.246596	$2.47692582 \times 10^{-2}$
4	0.237523	3.8198406×10^{-2}

TABLE 2. Minimum objective value of J and the final state error of the proposed method and other methods (Example 2.1)

Method	Objective value	Max error
Proposed method(m=4)	0.237523	$3.5793 * 10^{-6}$
Measure theory method [6]	0.2425	$4.3 * 10^{-3}$
HAM ($h = -1$, m=5) [9]	0.23533	$4.2 * 10^{-6}$
VIM (m=5) [19]	0.2353	$4.2 * 10^{-6}$
HPM [42]	0.23533	$4.2 * 10^{-6}$
Modal series method [43] (m=5)	0.2353	$2.8 * 10^{-5}$

velocities of the spacecraft are given by:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{I_3 - I_2}{I_1} x_2(t) x_3(t) \\ -\dfrac{I_1 - I_3}{I_2} x_1(t) x_3(t) \\ -\dfrac{I_2 - I_1}{I_3} x_1(t) x_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{I_1} & 0 & 0 \\ 0 & \dfrac{1}{I_2} & 0 \\ 0 & 0 & \dfrac{1}{I_3} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad (2.3)$$

where x_1, x_2, and x_3 are the angular velocities of the spacecraft, u_1, u_2, and u_3 represent the control torques, and $I_1 = 86.24$, $I_2 = 85.07$ and $I_3 = 113.59 \; kgm^2$ are the spacecraft principal inertia. The purpose of optimal control is to find that control $u(t)$ which minimizes the cost function

$$J[x, u] = \frac{1}{2} \int_0^{100} (x^T(t) Q x(t) + u^T(t) R u(t)) dt, \quad (2.4)$$

where $\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In addition, the following boundary conditions should be satisfied:

$$\begin{cases} x_1(0) = 0.01r/s, x_2(0) = .005r/s, x_3(0) = .001r/s, \\ x_1(100) = x_2(100) = x_3(100) = 0r/s. \end{cases} \quad (2.5)$$
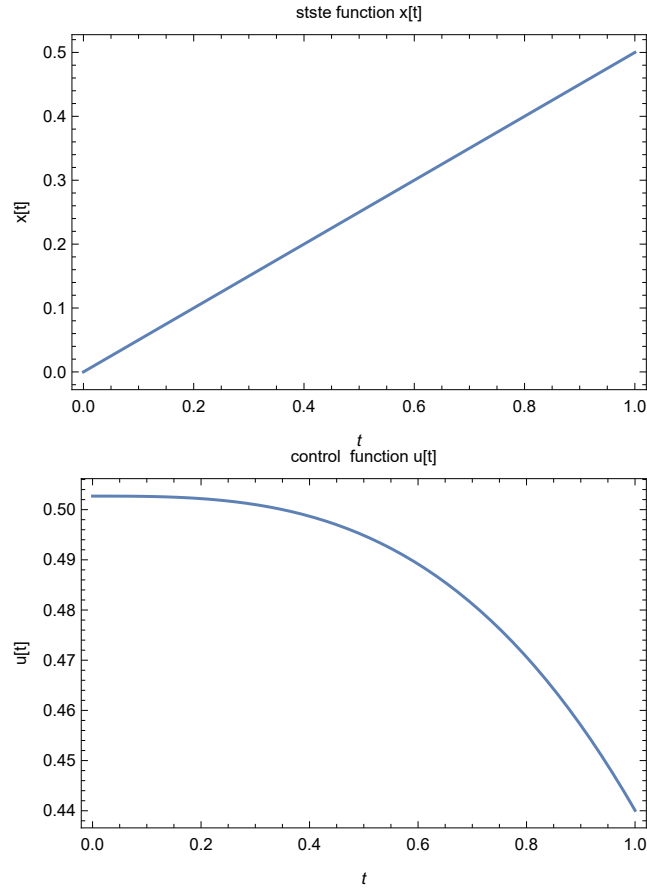
FIGURE 1. Suboptimal state $x(t)$ and control $u(t)$ (Example 2.1).

By using PMP, we obtain the following equations:

$$
\begin{cases}
\dot{x}_1(t) = -\dfrac{\lambda_1(t)}{I_1^2} - \dfrac{I_3 - I_2}{I_1} x_2(t) x_3(t), \\[2mm]
\dot{x}_2(t) = -\dfrac{\lambda_2(t)}{I_2^2} - \dfrac{I_1 - I_3}{I_2} x_1(t) x_3(t), \\[2mm]
\dot{x}_3(t) = -\dfrac{\lambda_3(t)}{I_3^2} - \dfrac{I_2 - I_1}{I_3} x_1(t) x_2(t), \\[2mm]
\dot{\lambda}_1(t) = \dfrac{I_1 - I_3}{I_2} x_3(t) \lambda_2(t) + \dfrac{I_2 - I_1}{I_3} x_2(t) \lambda_3(t), \\[2mm]
\dot{\lambda}_2(t) = \dfrac{I_3 - I_2}{I_1} x_3(t) \lambda_1(t) + \dfrac{I_2 - I_1}{I_3} x_1(t) \lambda_3(t), \\[2mm]
\dot{\lambda}_3(t) = \dfrac{I_3 - I_2}{I_1} x_2(t) \lambda_1(t) + \dfrac{I_1 - I_3}{I_2} x_1(t) \lambda_2(t),
\end{cases}
\tag{2.6}
$$

and the optimal control law is obtained by:

$$\begin{cases} u^*_1(t) = -\dfrac{\lambda_1(t)}{I_1}, & t \in [0, 100], \\[2mm] u^*_2(t) = -\dfrac{\lambda_2(t)}{I_2}, & t \in [0, 100], \\[2mm] u^*_3(t) = -\dfrac{\lambda_3(t)}{I_3}, & t \in [0, 100]. \end{cases} \qquad (2.7)$$

To solve equation (2.6) using q-HAM, consider these initial conditions $x_1(0) = -0.0001t + 0.01 \quad r/s, \quad x_2(0) = -0.00005t + 0.005 \quad r/s, \quad x_3(0) = -0.00001t + 0.001 \quad r/s, \; \lambda_1(t) = \alpha_1, \; \lambda_2(t) = \alpha_2,$ and $\lambda_3(t) = \alpha_3$. We have used equation (1.7) to obtain the mth-order deformation equations for $m \geq 1$. Based on the equation (1.8), we have

$$\begin{cases} R_m(\vec{x}_{1,m-1}) = \dot{x}_{1,m-1} + \dfrac{\lambda_{1,m-1}(t)}{I_1^2} + \dfrac{I_3 - I_2}{I_1} \sum_{k=0}^{m-1} x_{2,k} x_{3,m-1-k}, \\[3mm] R_m(\vec{x}_{2,m-1}) = \dot{x}_{2,m-1} + \dfrac{\lambda_{2,m-1}(t)}{I_2^2} + \dfrac{I_1 - I_3}{I_2} \sum_{k=0}^{m-1} x_{1,k} x_{3,m-1-k}, \\[3mm] R_m(\vec{x}_{3,m-1}) = \dot{x}_{3,m-1} + \dfrac{\lambda_{3,m-1}(t)}{I_3^2} + \dfrac{I_2 - I_1}{I_3} \sum_{k=0}^{m-1} x_{1,k} x_{2,m-1-k}, \\[3mm] R_m(\vec{\lambda}_{1,m-1}) = \dot{\lambda}_{1,m-1} - \dfrac{I_1 - I_3}{I_2} \sum_{k=0}^{m-1} x_{3,k} \lambda_{2,m-1-k} - \dfrac{I_2 - I_1}{I_3} \sum_{k=0}^{m-1} x_{2,k} \lambda_{3,m-1-k}, \\[3mm] R_m(\vec{\lambda}_{2,m-1}) = \dot{\lambda}_{2,m-1} - \dfrac{I_3 - I_2}{I_1} \sum_{k=0}^{m-1} x_{3,k} \lambda_{1,m-1-k} - \dfrac{I_2 - I_1}{I_3} \sum_{k=0}^{m-1} x_{1,k} \lambda_{3,m-1-k}, \\[3mm] R_m(\vec{\lambda}_{3,m-1}) = \dot{\lambda}_{3,m-1} - \dfrac{I_3 - I_2}{I_1} \sum_{k=0}^{m-1} x_{2,k} \lambda_{1,m-1-k} - \dfrac{I_1 - I_3}{I_2} \sum_{k=0}^{m-1} x_{1,k} \lambda_{2,m-1-k}. \end{cases}$$

$$(2.8)$$

Similar to the previous example and getting $n = 27.515$, we gained $\alpha_1 = 0.7437367602150743$, $\alpha_2 = 0.361839861949632$ and $\alpha_3 = 0.1289813901352318$.

Then

$$x_{1,1}(t) = (1-n)(0 - 1.65353 \times 10^{-8}t^2 + 5.51175 \times 10^{-11}t^3 + t(-0.0000983465 + 0.000134457\alpha_1)),$$

$$x_{1,2}(t) = (1-n)n(0. - 1.65353 \times 10^{-8}t^2 + 5.51175 \times 10^{-11}t^3 + t(-0.0000983465 + 0.000134457\alpha_1)) + ...$$

$$x_{2,1}(t) = (1-n)(0. + 3.215 \times 10^{-8}t^2 - 1.07167 \times 10^{-10}t^3 + t(-0.000053215 + 0.000138181\alpha_2)),$$

$$x_{2,2}(t) = (1-n)n(0. + 3.215 \times 10^{-8}t^2 - 1.07167 \times 10^{-10}t^3 + t(-0.000053215 + 0.000138181\alpha_2)) + ...,$$

$$x_{3,1}(t) = (1-n)(0 + 5.1501 \times 10^{-9}t^2 - 1.7167 \times 10^{-11}t^3 + t(-0.000010515 + 0.0000775032\alpha_3)),$$

$$x_{3,2}(t) = (1-n)n(0. + 5.1501 \times 10^{-9}t^2 - 1.7167 \times 10^{-11}t^3 + t(-0.000010515 + 0.0000775032\alpha_3)) + ...,$$

$$\vdots$$

$$\lambda_{1,1}(t) = (1-n)t((0.0003215 - 1.6075 \times 10^{-6}t)\alpha_2 + (0.000051501 - 2.57505 \times 10^{-7}t)\alpha_3),$$

$$\lambda_{1,2}(t) = (1-n)nt((0.0003215 - 1.6075 \times 10^{-6}t)\alpha_2 + (0.000051501 - 2.57505 \times 10^{-7}t)\alpha_3) + ...,$$

$$\lambda_{2,1}(t) = (1-n)t((-0.000330705 + 1.65353 \times 10^{-6}t)\alpha_1 + (0.000103002 - 5.1501 \times 10^{-7}t)\alpha_3),$$

$$\lambda_{2,2}(t) = (1-n)nt((-0.000330705 + 1.65353 \times 10^{-6}t)\alpha_1 + (0.000103002 - 5.1501 \times 10^{-7}t)\alpha_3) + ...,$$

$$\lambda_{3,1}(t) = (1-n)t((-0.00165353 + 8.26763 \times 10^{-6}t)\alpha_1 + (0.003215 - 0.000016075t)\alpha_2),$$

$$\lambda_{3,2}(t) = (1-n)nt((-0.00165353 + 8.26763 \times 10^{-6}t)\alpha_1 + (0.003215 - 0.000016075t)\alpha_2) + ...,$$

$$\vdots$$

By applying the proposed method, the numerical results of J_N and the relative error of objective value will be as given in Table 3. The maximum absolute error of the proposed method is compared with that of other methods in Table 4. It is noteworthy that the given method improves the maximum absolute error which indicates the efficiency of the method. The obtained numerical solutions for $m = 4$ and $n = 27.515$ are presented in Figures 2 to 4.

TABLE 3. Numerical results for various iterations (Example 2.2)

N	J_N	$\left|\frac{J_N - J_{N-1}}{J_N}\right|$
1	0.00468836	-
2	0.00468813	4.90601×10^{-5}
3	0.00468779	7.25288×10^{-5}
4	0.00468776	6.3996×10^{-6}

Example 2.3. Consider the nonlinear system described by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2) + u,$$
$$x_1(0) = 1, \qquad x_2(0) = 0,$$

TABLE 4. Minimum performance index value of J and final state error of the proposed method and other methods (Example 2.2)

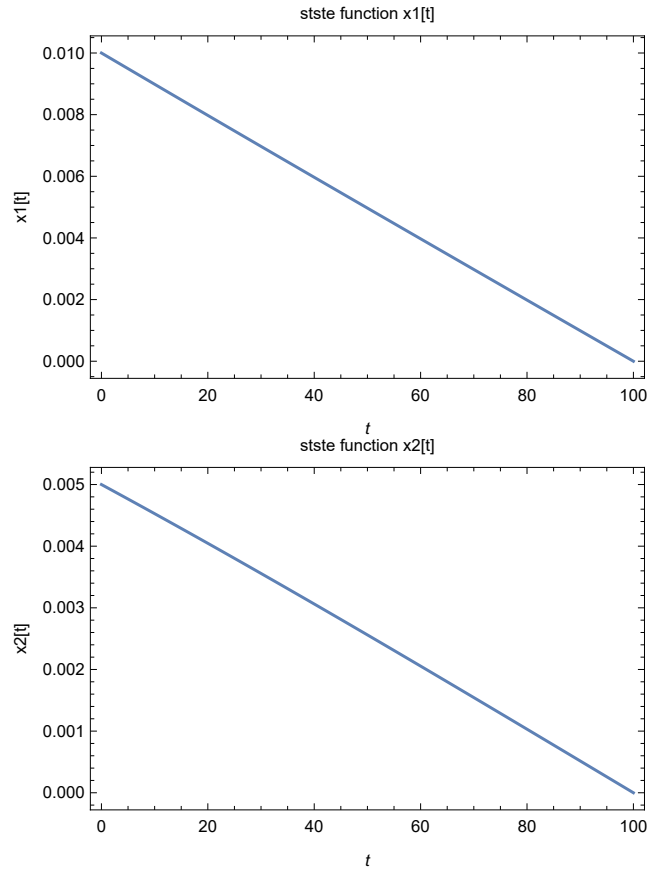| Method	Objective value	Max error
Proposed method (m=4)	0.00468776	$2.20699 * 10^{-17}$
Modified VIM[45]	0.004678	$2.40484 * 10^{-14}$
SHAM Chebyshev (m=6, n=50, h=-1.2) [46]	0.004687	$1.0586 * 10^{-9}$
SHAM Legendre (m=6, n=50, h=-1.2) [46]	0.004687	$1.0589 * 10^{-9}$
HPM (m=3) [10]	0.00468779	———-

FIGURE 2.   Suboptimal state $x1(t)$ and $x2(t)$ (Example 2.2)

and the cost functional
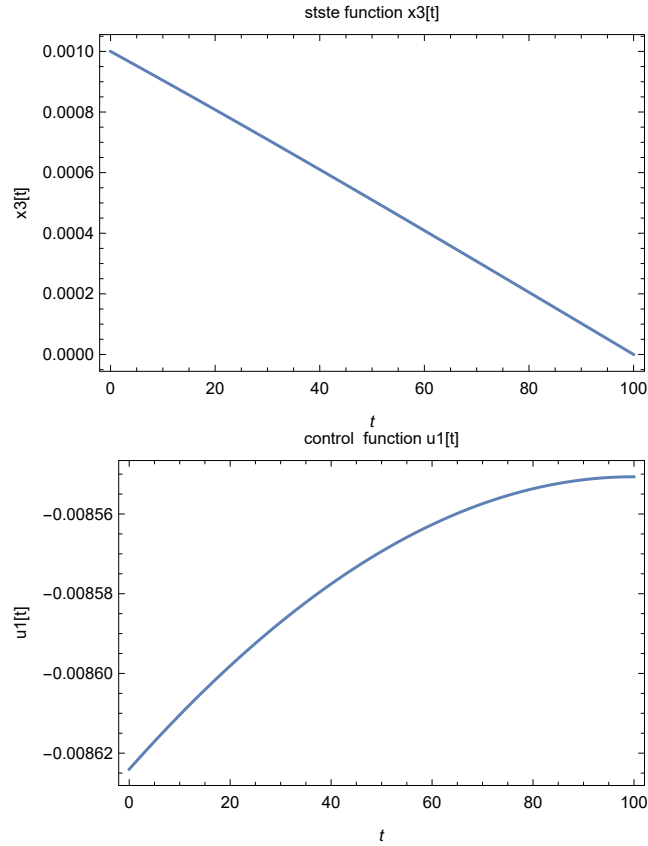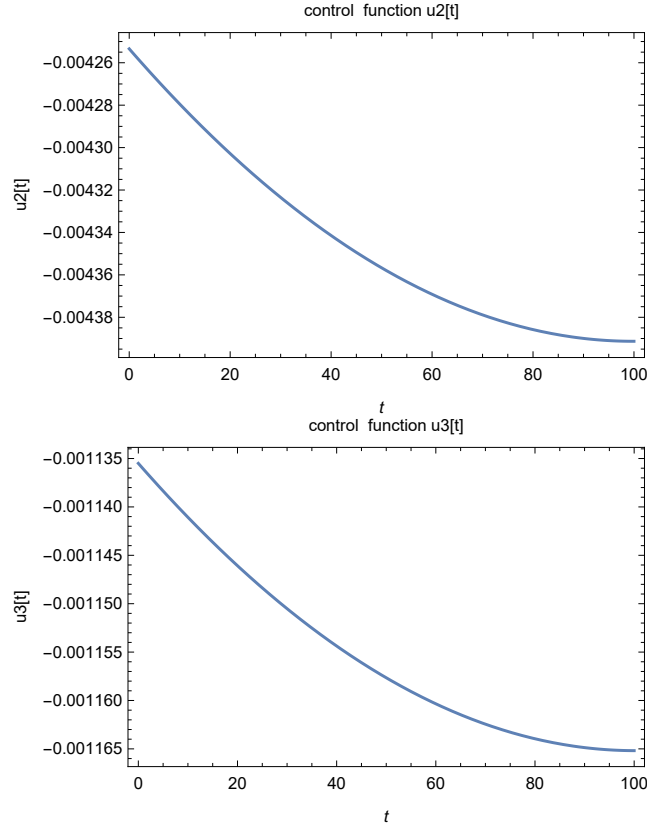
$$J = \frac{1}{2} \int_0^1 (x_1^2 + x_2^2 + u^2) dt.$$

FIGURE 3. Suboptimal state $x3(t)$ and control $u1(t)$ (Example 2.2)

By using PMP, we obtain the following equations:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2) - \lambda_2,$$
$$\dot{\lambda}_1 = -x_1 + \lambda_2 + 2x_1x_2\lambda_2,$$
$$\dot{\lambda}_2 = -x_2 - \lambda_1 - \lambda_2 + \lambda_2 x_1^2,$$

we consider the initial condition $x_1(0) = 1$, $x_2(0) = 0$, $\lambda_1(1) = \lambda_2(1) = 0$, and the optimal control is $u = -\lambda_2$. Similar to the previous examples, we gained $\alpha_1 = 2.352709654003546$ and $\alpha_2 = 0.48235132085840954$.

FIGURE 4. Suboptimal control $u2(t)$ and $u3(t)$ (Example 2.2)

Then

$$x_{1,1}(t) = 0,$$

$$x_{1,2}(t) = \frac{1}{2}(1-n)(-1+n)\,t^2(1+\alpha_2),$$

$$x_{2,1}(t) = (1-n)\,t(1+\alpha_2),$$

$$x_{2,2}(t) = (1-n)\,n\,t(1+\alpha_2) - \frac{1}{2}(1-n)(-1+n)\,t(2+t\alpha_1+2\alpha_2),$$

$$\vdots$$

$$\lambda_{1,1}(t) = (1-n)(t-t\alpha_2),$$

$$\lambda_{1,2}(t) = (1-n)n(t-t\alpha_2) + \frac{1}{2}(1-n)(-1+n)t(2(-1+\alpha_2)+t(\alpha_1+2\alpha_2(1+\alpha_2))),$$

$$\lambda_{2,1}(t) = (1-n)t\,\alpha_1,$$

$$\lambda_{2,2}(t) = (1-n)n\,t\,\alpha_1 - (1-n)(-1+n)t(t+\alpha_1),$$

$$\vdots$$

Table 5 presents the numerical results for J_N and the relative error of objective value as obtained by applying the proposed method. In Table 6, the maximum absolute error of the proposed method is compared with that of other methods. The obtained numerical solutions for $m = 8$ and $n = 2.7385$ are depicted in Figures 5 to 6.

TABLE 5. Numerical results for various iterations (Example 2.3)

N	J_N	$\frac{J_N - J_{N-1}}{J_N}$
1	3.94696	-
2	2.53263	$5.584431994 \times 10^{-1}$
3	1.56757	$6.156407688 \times 10^{-1}$
4	1.24656	$2.575166859 \times 10^{-1}$
5	1.27434	$2.17995198 \times 10^{-2}$
6	1.02859	$2.389192973 \times 10^{-1}$
7	1.06416	$3.34254247 \times 10^{-2}$
8	1.18127	$9.91390622 \times 10^{-2}$

TABLE 6. Minimum performance index value of J and final state error of the proposed method and other methods (Example 2.3)

Method	Objective value	Max error
Proposed method (m=8)	1.18127	$2.17384 * 10^{-6}$
SHAM Chebyshev(m=15, N=50, h=−0.5)	1.0472	$4.2749 * 10^{-4}$
SHAM Legendre (m=15, N=50, h=−0.5)	1.0472	$4.2749 * 10^{-4}$
DT (m=15)	1.0478	$4.4380 * 10^{-4}$

3. CONCLUSION

In this paper, we implemented the q-HAM to solve NOCPs. This method has been applied for the first time for NOCPs and can also be adapted to solving OCP with integral constraints and fractional problems. The q-HAM is a semi-exact method for solving linear and nonlinear equations which does not need small/large parameters. The q-HAM contains an auxiliary parameter n that provides us a convenient way to guarantee the convergence of solution series so that it is valid even if non-linearity becomes rather strong. The accuracy and efficiency of q-HAM are presented with some examples and the results of solving OCP by q-HAM are compared with other methods. Comparing the proposed method reveals that the accuracy of the q-HAM is better than
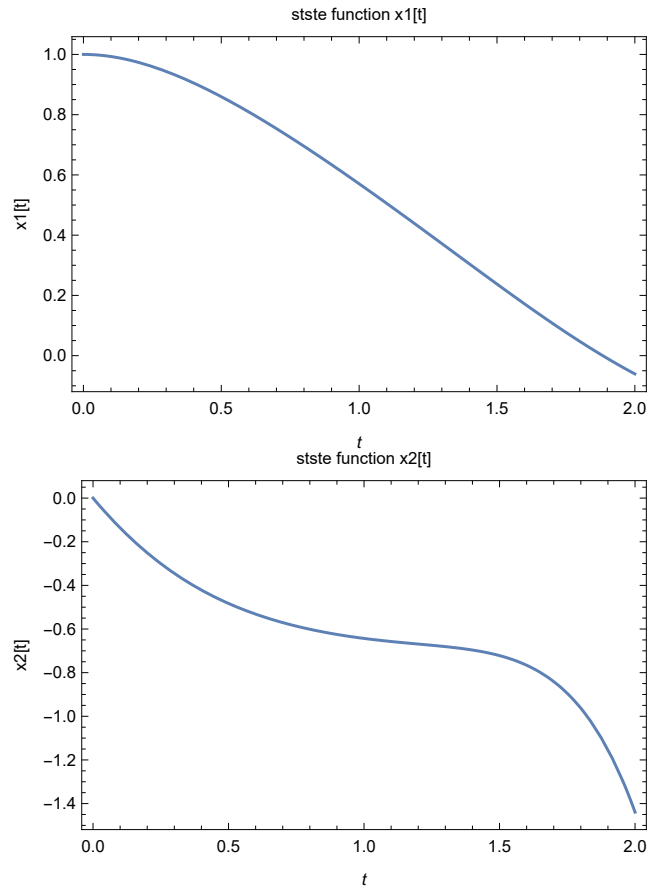
stste function x1[t]

stste function x2[t]

FIGURE 5.   Suboptimal state $x1(t)$ and $x2(t)$ (Example 2.3)

other methods. For example, as you can see in the table of example 3, the q-HAM has performed better than the HAM and has a faster convergence. Also, the q-HAM software programs are easily written with any software. Mathematica software was used for the current study.

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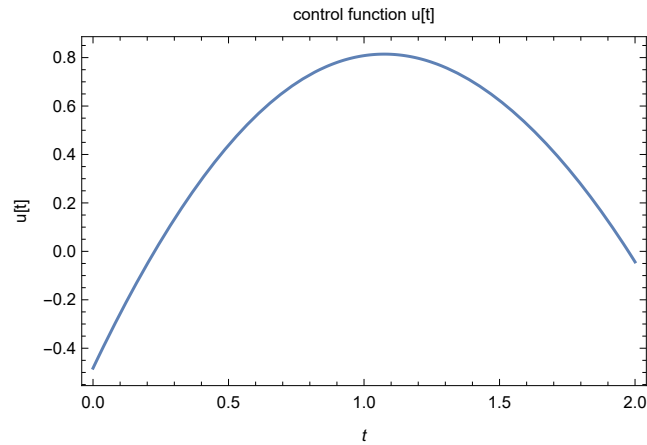
FIGURE 6.   Suboptimal control $u(t)$ (Example 2.3)

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