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New Ostrowski type inequalities for differentiable harmonically quasi-convex functions via fractional integral

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Abstract. In this paper, we prove a new integral identity, and then we establish some new Ostrowski's inequalities for fonctions whose first derivatives are harmonically quasi-convex via Riemann-Liouville fractional integrals. Keywords: integral inequality, Ostrowski inequality, harmonically quasi-convex functions.

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1. Introduction

Ostrowski [19] proved the following integral inequality

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$, be a differentiable mapping in the interior I° of I, and $a, b \in I^\circ$ with $a < b$.*

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If $|f'| \leq M$ *for all* $x \in [a,b]$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq (b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] M.$$

The above inequalities has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature, one can mention [1–6, 9–18, 20], and references therein.

İşcan [7], gave the following Ostrowski's inequalities for harmonically quasi-convex functions

Theorem 1.2. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I$ *with* $a < b$ *and* $f \in L([a,b])$. *If* $|f'|^q$ $(q \geq 1)$ *is harmonically quasi-convex on* $[a,b]$, *then for all* $x \in [a,b]$, *we have*

$$\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \right|$$

$$\leq \frac{ab}{b-a}\left\{ (x-a)^2 \left(C_1(a,x,\theta,\rho)\sup\left\{|f'(x)|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}} + (b-x)^2\left(C_2(b,x,\theta,\rho)\sup\left\{|f'(x)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \right\},$$

where

$$\begin{cases} C_1(a,x,\theta,\rho) = \frac{B(\rho+1,1)}{x^{2\theta}} \times {}_2F_1\left(2\theta, \rho+1; \rho+2; 1-\frac{a}{x}\right) \\ C_2(b,x,\theta,\rho) = \frac{B(1,\rho+1)}{x^{2\theta}} \times {}_2F_1\left(2\theta, 1; \rho+2; 1-\frac{x}{b}\right) \end{cases}$$

where $B(.,.)$ *and* ${}_2F_1(.,.;.;.)$ *are Beta and hypergeometric functions.*

Theorem 1.3. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I$ *with* $a < b$ *and* $f \in L([a,b])$. *If* $|f'|^q$ $(q \geq 1)$ *is harmonically quasi-convex on* $[a,b]$, *then for all* $x \in [a,b]$, *we have*

$$\left| f(x) - \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \right|$$

$$\leq \frac{ab}{b-a}\left\{ C_3(a,x)(x-a)^2 \left(\sup\left\{|f'(x)|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}} + C_3(b,x)(b-x)^2\left(\sup\left\{|f'(x)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \right\},$$

where

$$C_3(i,x) = \frac{1}{x-i}\left(\frac{1}{i} - \frac{\ln x - \ln i}{x-i}\right).$$

Motivated by the above results, in this paper, we prove a new integral identity, and then we establish some new Ostrowski's inequalities for fonctions whose first derivatives are harmonically quasi-convex via Riemann-Liouville fractional integrals.

2. Preliminaries

In this sections we recall some definitions and lemma

Definition 2.1. [22] A function $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \sup\{f(x), f(y)\}$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. [8] Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function and $I^0_{a^+}f(x) = I^0_{b^-}f(x) = f(x)$.

We also recall that the hypergeometric function is defined as follows

$$_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

where $c > b > 0, |z| < 1$, and B is the Euler beta function given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \; .$$

Lemma 2.3. *[21] For any $0 \leq a < b$ in $\mathbb{R}$ and $0 < p \leq 1$, we have*

$$|a^p - b^p| \leq (b-a)^p \, .$$

3. Main results

Lemma 3.1. *Let $f : [a,b] \subset (0,+\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L([a,b])$, then the following equality holds*

$$\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^-}(f\circ h)\left(\tfrac{1}{b}\right) + I^{\delta}_{\left(\frac{1}{b}\right)^+}(f\circ h)\left(\tfrac{1}{a}\right)\right) - f(x) \quad (3.1)$$

$$= \frac{ab(b-a)}{2}\left(\int_0^1 \frac{k}{((1-t)a+tb)^2} f'\left(\frac{ab}{(1-t)a+tb}\right)dt + \int_0^1 \frac{\left(t^{\delta}-(1-t)^{\delta}\right)}{((1-t)a+tb)^2} f'\left(\frac{ab}{(1-t)a+tb}\right)dt\right)$$

where $h(x) = \frac{1}{x}$ *and*

$$k = \begin{cases} 1 \ \textit{if} \ t \in \left[0, \frac{a(b-x)}{x(b-a)}\right] \\ -1 \ \textit{if} \ t \in \left[\frac{a(b-x)}{x(b-a)}, 1\right] \end{cases}, \tag{3.2}$$

for all $x \in [a,b]$.

Proof. Let

$$I_1 = \int_0^1 \frac{k}{((1-t)a+tb)^2} f'\left(\frac{ab}{(1-t)a+tb}\right) dt,$$

and

$$I_2 = \int_0^1 \frac{\left(t^\delta - (1-t)^\delta\right)}{((1-t)a+tb)^2} f'\left(\frac{ab}{(1-t)a+tb}\right) dt.$$

Clearly,

$$\begin{aligned} I_1 &= \int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^2} f'\left(\frac{ab}{(1-t)a+tb}\right) dt - \int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^2} f'\left(\frac{ab}{(1-t)a+tb}\right) dt \\ &= \frac{1}{ab(a-b)} f\left(\frac{ab}{(1-t)a+tb}\right)\Bigg|_{t=0}^{t=\frac{a(b-x)}{x(b-a)}} - \frac{1}{ab(a-b)} f\left(\frac{ab}{(1-t)a+tb}\right)\Bigg|_{t=\frac{a(b-x)}{x(b-a)}}^{t=1} \\ &= \frac{1}{ab(b-a)}\left(f(b) + f(a)\right) - \frac{2}{ab(b-a)} f(x). \end{aligned} \tag{3.3}$$

By integration by parts, I_2 gives

$$\begin{aligned} I_2 &= \frac{\left(t^\delta - (1-t)^\delta\right)}{ab(a-b)} f\left(\frac{ab}{(1-t)a+tb}\right)\Bigg|_{t=0}^{t=1} \\ &\quad - \frac{\delta}{ab(a-b)} \int_0^1 \left(t^{\delta-1} + (1-t)^{\delta-1}\right) f\left(\frac{ab}{(1-t)a+tb}\right) dt \\ &= -\frac{1}{ab(b-a)}\left(f(a) + f(b)\right) \\ &\quad - \frac{\delta}{ab(a-b)} \left(\int_0^1 t^{\delta-1} f\left(\frac{ab}{(1-t)a+tb}\right) dt + \int_0^1 (1-t)^{\delta-1} f\left(\frac{ab}{(1-t)a+tb}\right) dt\right). \end{aligned} \tag{3.4}$$

Making the change of variable $\frac{(1-t)a+tb}{ab} = u$, (3.4) becomes

$$I_2 = -\frac{1}{ab(b-a)}\left(f(a) + f(b)\right) - \frac{\Gamma(\delta+1)}{ab(a-b)}\left(\frac{ab}{b-a}\right)^\delta$$

$$\times\left(\frac{1}{\Gamma(\delta)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(u-\frac{1}{b}\right)^{\delta-1}f\left(\frac{1}{u}\right)du+\frac{1}{\Gamma(\delta)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-u\right)^{\delta-1}f\left(\frac{1}{u}\right)du\right)$$

$$=-\frac{1}{ab(b-a)}\left(f(a)+f(b)\right)$$

$$+\frac{\Gamma(\delta+1)}{ab(b-a)}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^{-}}(f\circ h)\left(\frac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^{+}}(f\circ h)\left(\frac{1}{a}\right)\right)\quad(3.5)$$

Summing (3.3) and (3.5), and then multiplying the resulting equality by $\frac{ab(b-a)}{2}$, we get the desired result. □

Theorem 3.2. *Let $f:[a,b]\subset(0,+\infty)\to\mathbb{R}$ be differentiable mapping on (a,b) with $a<b$, such that $f\in L([a,b])$. If $|f'|$ is harmonically quasi-convex, then for all $x\in[a,b]$ the following fractional inequality holds*

$$\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^{-}}(f\circ h)\left(\frac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^{+}}(f\circ h)\left(\frac{1}{a}\right)\right)-f(x)\right|$$

$$\leq\frac{ab(b-a)}{2}\left(\frac{b-x}{ab(b-a)}\max\left\{|f'(x)|,|f'(b)|\right\}+\frac{x-a}{ab(b-a)}\max\left\{|f'(a)|,|f'(x)|\right\}\right.$$

$$+\frac{1}{2a^2(\delta+1)}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,|f'(b)|\right\}{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)$$

$$\left.+\frac{1}{2b^2(\delta+1)}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,|f'(a)|\right\}{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right),$$

where ${}_2F_1(.,.;.;.)$ is the hypergeometric function.

Proof. Using Lemma 3.1 and properties of modulus, and then by Applying Lemma 2.3 and the harmonic quasi-convexity of $|f'|$ to the resulting inequality, we get

$$\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^{-}}(f\circ h)\left(\frac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^{+}}(f\circ h)\left(\frac{1}{a}\right)\right)-f(x)\right|$$

$$\leq\frac{ab(b-a)}{2}\left(\int_{0}^{\frac{a(b-x)}{x(b-a)}}\frac{1}{((1-t)a+tb)^2}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|dt\right.$$

$$+\int_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{1}{((1-t)a+tb)^2}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|dt$$

$$\left.+\int_{0}^{\frac{1}{2}}\frac{\left((1-t)^{\delta}-t^{\delta}\right)}{((1-t)a+tb)^2}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|dt+\int_{\frac{1}{2}}^{1}\frac{\left(t^{\delta}-(1-t)^{\delta}\right)}{((1-t)a+tb)^2}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|dt\right)$$

$$\leq \frac{ab(b-a)}{2}\left(\max\left\{\left|f'(x)\right|,\left|f'(b)\right|\right\}\int_0^{\frac{a(b-x)}{x(b-a)}}\frac{1}{((1-t)a+tb)^2}dt\right.$$

$$+\max\left\{\left|f'(a)\right|,\left|f'(x)\right|\right\}\int_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{1}{((1-t)a+tb)^2}dt$$

$$+\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,\left|f'(b)\right|\right\}\int_0^{\frac{1}{2}}\frac{(1-2t)^\delta}{((1-t)a+tb)^2}dt$$

$$\left.+\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,\left|f'(a)\right|\right\}\int_{\frac{1}{2}}^{1}\frac{(2t-1)^\delta}{((1-t)a+tb)^2}dt\right)$$

$$= \frac{ab(b-a)}{2}\left(\frac{1}{b-a}\max\left\{\left|f'(x)\right|,\left|f'(b)\right|\right\}\int_0^{\frac{a(b-x)}{x(b-a)}}\frac{b-a}{(a+t(b-a))^2}dt\right.$$

$$+\frac{1}{b-a}\max\left\{\left|f'(a)\right|,\left|f'(x)\right|\right\}\int_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{b-a}{(a+t(b-a))^2}dt$$

$$+\frac{1}{2a^2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,\left|f'(b)\right|\right\}\int_0^1\frac{(1-t)^\delta}{\left(1-\frac{1}{2}t\left(1-\frac{b}{a}\right)\right)^2}dt$$

$$\left.+\frac{1}{2b^2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,\left|f'(a)\right|\right\}\int_0^1\frac{(1-t)^\delta}{\left(1-\frac{1}{2}t\left(1-\frac{a}{b}\right)\right)^2}dt\right)$$

$$= \frac{ab(b-a)}{2}\left(\frac{b-x}{ab(b-a)}\max\left\{\left|f'(x)\right|,\left|f'(b)\right|\right\}+\frac{x-a}{ab(b-a)}\max\left\{\left|f'(a)\right|,\left|f'(x)\right|\right\}\right.$$

$$+\frac{1}{2a^2(\delta+1)}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,\left|f'(b)\right|\right\}{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)$$

$$\left.+\frac{1}{2b^2(\delta+1)}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|,\left|f'(a)\right|\right\}{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right).$$

The proof is completed. □

Theorem 3.3. *Let $f:[a,b]\subset(0,+\infty)\to\mathbb{R}$ be differentiable mapping on (a,b) with $a<b$, such that $f\in L([a,b])$. If $|f'|^q$ is harmonically quasi-convex where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following fractional inequality*

$$\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^\delta\left(I^\delta_{\left(\frac{1}{a}\right)^-}(f\circ h)\left(\frac{1}{b}\right)+I^\delta_{\left(\frac{1}{b}\right)^+}(f\circ h)\left(\frac{1}{a}\right)\right)-f(x)\right|$$

$$\leq \frac{ab(b-a)}{2}\left(\left(\frac{b^{2p-1}-x^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}}\right)^{\frac{1}{p}}\left(\frac{a(b-x)}{x(b-a)}\max\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{x^{2p-1}-a^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}}\right)^{\frac{1}{p}}\left(\frac{b(x-a)}{x(b-a)}\max\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$+\left(\frac{{}_2F_1\left(2p,1;\delta p+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^2(\delta p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{{}_2F_1\left(2p,1;\delta p+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^2(\delta p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right)$$

holds for all $x\in[a,b]$.

Proof. From Lemma 3.1, properties of modulus, and Hölder inequality, we have

$$\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^-}(f\circ h)\left(\frac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^+}(f\circ h)\left(\frac{1}{a}\right)\right)-f(x)\right|$$

$$\leq\frac{ab(b-a)}{2}\left(\left(\int\limits_0^{\frac{a(b-x)}{x(b-a)}}\frac{1}{((1-t)a+tb)^{2p}}dt\right)^{\frac{1}{p}}\left(\int\limits_0^{\frac{a(b-x)}{x(b-a)}}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int\limits_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{1}{((1-t)a+tb)^{2p}}dt\right)^{\frac{1}{p}}\left(\int\limits_{\frac{a(b-x)}{x(b-a)}}^{1}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_0^{\frac{1}{2}}\frac{\left((1-t)^{\delta}-t^{\delta}\right)^p}{((1-t)a+tb)^{2p}}dt\right)^{\frac{1}{p}}\left(\int\limits_0^{\frac{1}{2}}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int\limits_{\frac{1}{2}}^{1}\frac{\left(t^{\delta}-(1-t)^{\delta}\right)^p}{((1-t)a+tb)^{2p}}dt\right)^{\frac{1}{p}}\left(\int\limits_{\frac{1}{2}}^{1}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^q dt\right)^{\frac{1}{q}}\right). \qquad (3.6)$$

Now, using Lemma 2.3, and the harmonic quasi-convexity of $|f'|^q$, we obtain

$$\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^-}(f\circ h)\left(\frac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^+}(f\circ h)\left(\frac{1}{a}\right)\right)-f(x)\right|$$

$$\leq\frac{ab(b-a)}{2}\left(\left(\int\limits_0^{\frac{a(b-x)}{x(b-a)}}\frac{1}{(a+t(b-a))^{2p}}dt\right)^{\frac{1}{p}}\left(\frac{a(b-x)}{x(b-a)}\max\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int\limits_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{1}{(a+t(b-a))^{2p}}dt\right)^{\frac{1}{p}}\left(\frac{b(x-a)}{x(b-a)}\max\left\{|f'(x)|^{q},|f'(a)|^{q}\right\}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{2a^{2}}\int\limits_{0}^{1}\frac{(1-t)^{\delta p}}{\left(1-\frac{1}{2}t\left(1-\frac{b}{a}\right)\right)^{2p}}dt\right)^{\frac{1}{p}}\left(\frac{1}{2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{2b^{2}}\int\limits_{0}^{1}\frac{(1-t)^{\delta p}}{\left(1-\frac{1}{2}t\left(1-\frac{a}{b}\right)\right)^{2p}}dt\right)^{\frac{1}{p}}\left(\frac{1}{2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},|f'(a)|^{q}\right\}\right)^{\frac{1}{q}}\right)$$

$$=\frac{ab(b-a)}{2}\left(\left(\frac{b^{2p-1}-x^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}}\right)^{\frac{1}{p}}\left(\frac{a(b-x)}{x(b-a)}\max\left\{|f'(x)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{x^{2p-1}-a^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}}\right)^{\frac{1}{p}}\left(\frac{b(x-a)}{x(b-a)}\max\left\{|f'(x)|^{q},|f'(a)|^{q}\right\}\right)^{\frac{1}{q}}$$

$$+\left(\frac{{}_2F_1\left(2p,1;\delta p+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^{2}(\delta p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{{}_2F_1\left(2p,1;\delta p+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^{2}(\delta p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},|f'(a)|^{q}\right\}\right)^{\frac{1}{q}}\right),$$

which is the desired result. □

Theorem 3.4. *Let $f:[a,b]\subset(0,+\infty)\to\mathbb{R}$ be differentiable mapping on (a,b) with $a<b$, such that $f\in L([a,b])$. If $|f'|^{q}$, $q\geq 1$ is harmonically quasi-convex, then for all $x\in[a,b]$ the following fractional inequality holds*

$$\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^{-}}(f\circ h)\left(\frac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^{+}}(f\circ h)\left(\frac{1}{a}\right)\right)-f(x)\right|$$

$$\leq\frac{ab(b-a)}{2}\left(\frac{b-x}{(b-a)ab}\left(\max\left\{|f'(x)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\right.$$

$$+\frac{x-a}{(b-a)ab}\left(\max\left\{|f'(x)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}$$

$$+\frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^{2}(\delta+1)}\left(\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}$$

$$\left.+\frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^{2}(\delta+1)}\left(\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},|f'(a)|^{q}\right\}\right)^{\frac{1}{q}}\right),$$

where ${}_2F_1(.,.;.;.)$ is the hypergeometric function.

Proof. From Lemma 3.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^{-}}\left(f\circ h\right)\left(\tfrac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^{+}}\left(f\circ h\right)\left(\tfrac{1}{a}\right)\right)-f\left(x\right)\right| \\
&\leq\frac{ab(b-a)}{2}\left(\left(\int_{0}^{\frac{a(b-x)}{x(b-a)}}\frac{1}{((1-t)a+tb)^{2}}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{a(b-x)}{x(b-a)}}\frac{1}{((1-t)a+tb)^{2}}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{1}{((1-t)a+tb)^{2}}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a(b-x)}{x(b-a)}}^{1}\frac{1}{((1-t)a+tb)^{2}}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{0}^{\frac{1}{2}}\frac{(1-t)^{\delta}-t^{\delta}}{((1-t)a+tb)^{2}}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}\frac{(1-t)^{\delta}-t^{\delta}}{((1-t)a+tb)^{2}}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\frac{t^{\delta}-(1-t)^{\delta}}{((1-t)a+tb)^{2}}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\frac{t^{\delta}-(1-t)^{\delta}}{((1-t)a+tb)^{2}}\left|f'\left(\frac{ab}{(1-t)a+tb}\right)\right|^{q}dt\right)^{\frac{1}{q}}\right). \qquad (3.7)
\end{aligned}$$

Now, using Lemma 2.3, harmonic quasi-convexity of $|f'|^{q}$, we deduces

$$\begin{aligned}
&\left|\frac{\Gamma(\delta+1)}{2}\left(\frac{ab}{b-a}\right)^{\delta}\left(I^{\delta}_{\left(\frac{1}{a}\right)^{-}}\left(f\circ h\right)\left(\tfrac{1}{b}\right)+I^{\delta}_{\left(\frac{1}{b}\right)^{+}}\left(f\circ h\right)\left(\tfrac{1}{a}\right)\right)-f\left(x\right)\right| \\
&\leq\frac{ab(b-a)}{2}\left(\frac{b-x}{(b-a)ab}\left(\max\left\{\left|f'\left(x\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \\
&\quad+\frac{x-a}{(b-a)ab}\left(\max\left\{\left|f'\left(x\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right\}\right)^{\frac{1}{q}} \\
&\quad+\frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^{2}(\delta+1)}\left(\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right\}\right)^{\frac{1}{q}} \\
&\quad\left.+\frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^{2}(\delta+1)}\left(\max\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|^{q},\left|f'\left(a\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right),
\end{aligned}$$

which is the desired result. □

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