Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **xx**(x)(201x), xx-xx

Examining The Planar Kinematic Based on Dual Quaternions

Vahide Bulut [1]
[1] Department of Engineering Sciences, Izmir Katip Celebi University, Cigli, Izmir, 35620, Turkey

Abstract. Kinematics studies the motion of a rigid object, such as displacement, velocity, acceleration, etc. A general planar motion can be defined as a combination of translation and rotation. Planar motion is widely used in many fields. Since most mobile robots move on flat terrain, many grippers and kinematic linkages use planar motion. The dual quaternion is the generalization of the quaternion and is used in various fields. In this paper, we introduce a new approach to planar motions by using the dual quaternion to study the pole points and pole trajectories, the triple coordinate system and the canonical system, and find the Euler-Savary equation.

Keywords: Dual quaternion, planar kinematic, velocity, pole trajectory, Euler-Savary equation.

2000 Mathematics subject classification: 70B10; Secondary 53A17.

1. Introduction

A rigid body can be expressed as a model of an object that does not deform or change shape. This object has the property that the distance between each pair of points of a rigid body is constant. If all points of a rigid body move along paths equidistant from a fixed plane, then this rigid body is said to have planar motion. Dual quaternions are of great importance in many fields such as computer graphics, robotics, etc., as

[1]Corresponding author: vahide.bulut@ikcu.edu.tr
Received: xx Month 201x
Revised: xx Month 201x
Accepted: xx Month 201x

they provide a simpler and more robust solution for these areas. They combine translation and rotation in a single state. Dual quaternions are studied in computer graphics [7], [4], inverse kinematics [10] and robotics [3], [13]. In [1] we represented a point and its velocities in planar kinematics in terms of dual quaternions. In this study, we give the pole points and pole trajectories, the triple coordinate system, the canonical system, and the Euler-Savary equation in planar kinematics based on dual quaternions. Our contribution is a novel approach to planar kinematics with dual quaternions.

2. Dual Quaternions

Quaternions represent the rotation of a rigid body about an axis. Quaternions, especially unit quaternions, are widely used because they provide better opportunities for interpolation of key images. Moreover, they do not suffer from singularity problems [5], [6] [8], [14].

A quaternion in R^4 is a 4-tuple and a quaternion is defined as.

$$q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$$

with a, b, c, d real numbers, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the standard orthonormal basis in R^3 and $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$.

Dual quaternions, which are an extension of real quaternions, were discovered by William Kingdon Clifford in 1873 [2]. They are used for both rotations and translations. A dual quaternion can be represented as follows.

$$\hat{Q} = Q + \varepsilon Q^* \tag{2.1}$$

where

$$Q = q_r + \vec{q} \quad and \quad Q^* = q_r^* + \vec{q}^*$$

and $\varepsilon^2 = 0$. Alternatively, a dual quaternion whose four terms are dual numbers can be interpreted as follows.

$$\hat{Q} = \hat{q_r} + \hat{q_x}\boldsymbol{i} + \hat{q_y}\boldsymbol{j} + \hat{q_z}\boldsymbol{k} \tag{2.2}$$

or

$$\hat{Q} = s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + \varepsilon\left(s_\varepsilon + x_\varepsilon\boldsymbol{i} + y_\varepsilon\boldsymbol{j} + z_\varepsilon\boldsymbol{k}\right).$$

When $\hat{q_r} = 0$, a dual quaternion is transformed into a dual vector. Dual numbers and dual vectors are special cases of dual quaternions [12]. If two dual quaternions can be taken as

$$\hat{Q}_1 = Q_1 + \varepsilon Q_1^* \quad and \quad \hat{Q}_2 = Q_2 + \varepsilon Q_2^*.$$

then the dual quaternion multiplication ($\otimes$) can be written as

$$\hat{Q}_1 \otimes \hat{Q}_2 = Q_1.Q_2 + \varepsilon\left(Q_1.Q_2^* + Q_2.Q_1^*\right).$$

The dual conjugate $\overline{\hat{Q}}$ and the dual quaternion norm are defined as

$$\overline{\hat{Q}} = Q - \varepsilon Q^*$$

and

$$||\hat{Q}|| = ||Q|| + \varepsilon \frac{< Q, Q^* >}{||Q||}.$$

A dual quaternion satisfying the conditions $||\hat{Q}|| = 1$ and . $= 0$ is called an unit dual quaternion. A second conjugation operator for a dual quaternion is given as

$$\overline{\hat{Q}}^* = (s, -x, -y, -z, -s_\varepsilon, x_\varepsilon, y_\varepsilon, z_\varepsilon).$$

Transformations represented by dual quaternions can be combined to form a dual quaternion. Suppose that $\hat{Q}$ and $\hat{P}$ are two transformation dual quaternions and Q_v is a position vector dual quaternion. The combined transformation C can be applied to Q_v as

$$\hat{Q}'_v = \hat{P} \otimes \left(\hat{Q} \otimes Q_v \otimes \overline{\hat{Q}}^*\right) \otimes \overline{\hat{P}}^* = \left(\hat{P} \otimes \hat{Q}\right) \otimes (Q_v) \otimes \left(\overline{\hat{Q}}^* \otimes \overline{\hat{P}}^*\right) \quad (2.3)$$

or

$$\hat{C} = \hat{P} \otimes \hat{Q} \Rightarrow \hat{Q}'_v = \hat{C} \otimes Q_v \otimes \overline{\hat{C}}^*.$$

The pure rotation about the vector $\boldsymbol{n}$ with angle θ and the pure translation can be written as follows using the unit dual quaternion.

$$\hat{Q}_R = \left[\cos\left(\frac{\theta}{2}\right),\, n_x \sin\left(\frac{\theta}{2}\right),\, n_y \sin\left(\frac{\theta}{2}\right),\, n_z \sin\left(\frac{\theta}{2}\right)\right] [0,\, 0,\, 0,\, 0], \quad (2.4)$$

and

$$\hat{Q}_T = [1,\, 0,\, 0,\, 0] \left[0,\, \frac{t_x}{2},\, \frac{t_y}{2},\, \frac{t_z}{2}\right]. \quad (2.5)$$

Combining the rotation and translation transformations into a single unit quaternion to represent rotation followed by translation, we obtain

$$\begin{aligned} \hat{Q} = & \quad \hat{Q}_T \otimes \hat{Q}_R \\ = & \quad \left(1 + \tfrac{\varepsilon}{2}\left[t_x \boldsymbol{i} + t_y \boldsymbol{j} + t_z \boldsymbol{k}\right]\right) \otimes \hat{Q}_R \\ = & \quad R + \varepsilon \tfrac{TR}{2} \end{aligned} \quad (2.6)$$

We can write its inverse as [9], [8]

$$\left(R + \varepsilon \frac{TR}{2}\right)^{-1} = R^* - \varepsilon \frac{R^* T}{2}. \quad (2.7)$$

3. Planar Kinematic using Dual Quaternions

Let be two planes $\boldsymbol{M}$ (moving) and $\boldsymbol{F}$ (fixed), and let us define the orthonormal coordinate systems in these planes as M:$\{O_m; \boldsymbol{e_{1m}}, \boldsymbol{e_{2m}}\}$ and F:$\{O_f; \boldsymbol{e_{1f}}, \boldsymbol{e_{2f}}\}$ (see Figure (1)).

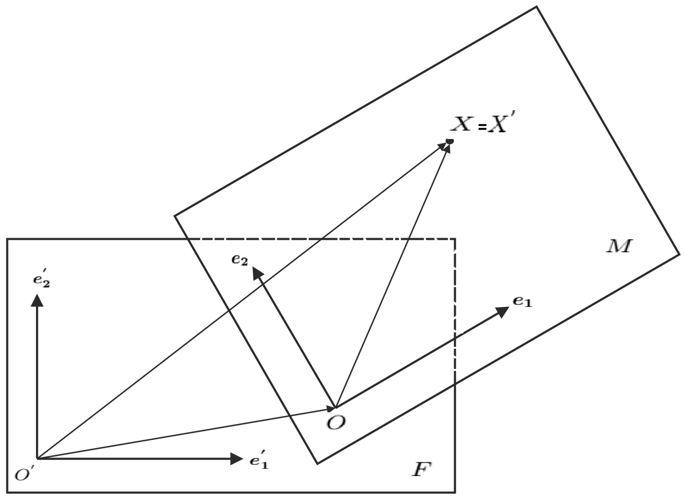

Figure 1. Planar kinematic of two planes $\boldsymbol{M}$ and $\boldsymbol{F}$

First, we can assume that these two frames coincide in their points of origin (O_m) and (O_f) (see Figure (2)) and we can take the same point $\boldsymbol{X}(x_1, x_2)$ in both frames. Next, we rotate and translate the moving

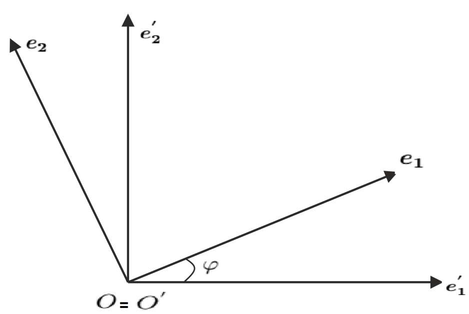

Figure 2. Planar kinematik of two frames M and F

frame using dual quaternions and denote the point $\boldsymbol{X}(x_1, x_2)$ as $\boldsymbol{X}_m$ in the moving frame and $\boldsymbol{X}_f$ in the fixed frame. We can define the point $\boldsymbol{X}(x_1, x_2)$ in dual quaternion form as

$$\hat{Q}_X = [1 + \varepsilon (x_1,\, x_2,\, 0)]\,. \tag{3.1}$$

We can apply the equation (2.6) to Q_X, then we obtain the final position of the point $\boldsymbol{X}\,(x_1, x_2)$ according to the fixed frame F with the following equation:

$$\begin{aligned}\hat{Q}'_X = \hat{Q} \otimes Q_X \otimes \overline{\hat{Q}}^* = \quad & (1,\,0,\,0,\,0) \\ & +\tfrac{\varepsilon}{2}\,(0,\,2x_1\cos\varphi - 2x_2\sin\varphi + 2d_1, \\ & 2x_1\sin\varphi + 2x_2\cos\varphi + 2d_2,\,0)\end{aligned} \tag{3.2}$$

where $(d_1,\,d_2)$ denotes the coordinates of the origin (O_m) of the moving frame M in the fixed frame F and φ denotes the angle of rotation of M relative to F, i.e. i.e., φ is the angle between vectors $\boldsymbol{e_{1m}}$ and $\boldsymbol{e_{1f}}$ and is called the angle of rotation. Finally, we can write the point $\boldsymbol{X}\,(x_1,\,x_2)$ in the fixed frame F as

$$\begin{aligned}\boldsymbol{X_f} = \quad & (x_1\cos\varphi - x_2\sin\varphi + d_1)\,\boldsymbol{e_{1f}} \\ & + (x_1\sin\varphi + x_2\cos\varphi + d_2)\,\boldsymbol{e_{2f}}\end{aligned} \tag{3.3}$$

On the other hand, we can use the following equations obtained via dual quaternions,

$$\begin{aligned}\boldsymbol{e_{1f}} = \quad & \boldsymbol{e_{1m}}\cos\varphi - \boldsymbol{e_{2m}}\sin\varphi \\ \boldsymbol{e_{2f}} = \quad & \boldsymbol{e_{1m}}\sin\varphi + \boldsymbol{e_{2m}}\cos\varphi\end{aligned}$$

we can define the point $\boldsymbol{X_f}$ in the moving frame M with the following equation:

$$\begin{aligned}\boldsymbol{X_f} = \quad & (x_1 + d_1\cos\varphi + d_2\sin\varphi)\,\boldsymbol{e_{1m}} \\ & + (x_2 - d_1\sin\varphi + d_2\cos\varphi)\,\boldsymbol{e_{2m}}\end{aligned} \tag{3.4}$$

We can also express the point $\boldsymbol{X_m}$ in the moving frame M as

$$\boldsymbol{X_m} = x_1\boldsymbol{e_{1m}} + x_2\boldsymbol{e_{2m}} \tag{3.5}$$

3.1. **Evaluation of Velocities.** We determined the relative, absolute, and drift velocities in [1] respectively as follows:

$$\mathbf{v}_r = \dot{\mathbf{X}} = \dot{x}_1\boldsymbol{e_{1m}} + \dot{x}_2\boldsymbol{e_{2m}}, \tag{3.6}$$

$$\begin{aligned}\boldsymbol{v}_a = \quad & \dot{x}_1\boldsymbol{e_{1m}} + \dot{x}_2\boldsymbol{e_{2m}} \\ & + \left(\dot{d}_1\cos\varphi + \dot{d}_2\sin\varphi - x_2\dot{\varphi}\right)\boldsymbol{e_{1m}} \\ & \left(-\dot{d}_1\sin\varphi + \dot{d}_2\cos\varphi + x_1\dot{\varphi}\right)\boldsymbol{e_{2m}},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \boldsymbol{v_d} &= \left(\dot{d}_1 \cos\varphi + \dot{d}_2 \sin\varphi - x_2\dot{\varphi}\right) \boldsymbol{e_{1m}} \\ &+ \left(-\dot{d}_1 \sin\varphi + \dot{d}_2 \cos\varphi + x_1\dot{\varphi}\right) \boldsymbol{e_{2m}}. \end{aligned} \tag{3.8}$$

where we denote the derivative by dot.

3.2. **Pole Points and Pole Trajectories.** First, the following theorem can be expressed for pole points.

Theorem 1: There is only one point (pole point) that remains stationary in both planes at any instant of the motion with a non-zero angular velocity.

Let us determine the pole point $\boldsymbol{P}(p_1, p_2)$ that satisfies the equation $\boldsymbol{v_d} = 0$. Using the equation (3.8) for $\boldsymbol{v_d} = 0$, we can write

$$\dot{d}_1 \cos\varphi + \dot{d}_2 \sin\varphi - x_2\dot{\varphi} = 0$$

$$-\dot{d}_1 \sin\varphi + \dot{d}_2 \cos\varphi + x_1\dot{\varphi} = 0.$$

Then we can find the pole point $\boldsymbol{P}(p_1, p_2)$ using the unit dual quaternion as follows.

$$\begin{aligned} p_1 &= \frac{\dot{d}_1 \sin\varphi - \dot{d}_2 \cos\varphi}{\dot{\varphi}} \\ p_2 &= \frac{\dot{d}_1 \cos\varphi + \dot{d}_2 \sin\varphi}{\dot{\varphi}} \end{aligned} \tag{3.9}$$

The drift velocity of point X can be written using the pole point $\boldsymbol{P}(p_1, p_2)$ with the following equation:

$$\boldsymbol{v_d} = \left[-\boldsymbol{e_1}(x_2 - p_2) + \boldsymbol{e_2}(x_1 - p_1)\right]\dot{\varphi}. \tag{3.10}$$

Therefore, the following results and theorems based on dual quaternions can be written as in [11] as follows:

i) $\boldsymbol{PXv_d} = 0$, i.e. $\boldsymbol{PX} \perp \boldsymbol{v_d}$.

ii) $|\boldsymbol{v_d}| = \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2} = |\boldsymbol{PX}|\,\dot{\varphi}$.

Theorem 2: Each point $\boldsymbol{X}$ of the moving plane M performs a rotational motion with pole point P-centered and angular velocity $(\dot{\varphi})$ at each instant.

Theorem 3: A one-parameter planar motion is the rotation of the moving plane M about the instantaneous rotation pole P with angular velocity $\dot{\varphi}$.

Theorem 4: In a planar motion with one parameter, the points $\boldsymbol{X}$ of the moving plane M draw paths on the fixed plane F. The geometric position of the pole point P in the moving plane M is called the trajectory of the moving pole $(\boldsymbol{p_m})$, and in the fixed plane F the trajectory of the fixed pole $(\boldsymbol{p_f})$.

Theorem 5: In a one-parameter planar motion, the curve of the moving pole $(\boldsymbol{p_m})$ rolls without sliding on the curve of the fixed pole $(\boldsymbol{p_f})$.

4. Triple Coordinate System and Planar Kinematic Based on Dual Quaternions

We have studied planar kinematics using dual quaternions with moving and fixed planes M and F, respectively. In this section we take three planes M (moving), F (fixed) and the other plane A which is moving plane according to planes M and F. Let us represent the orthonormal coordinate systems in these planes as $M:\{O_m;\boldsymbol{e_{1m}},\boldsymbol{e_{2m}}\}$, $F:\{O_f;\boldsymbol{e_{1f}},\boldsymbol{e_{2f}}\}$ and $A:\left\{\bar{O};\bar{\boldsymbol{e}}_{\boldsymbol{1}},\bar{\boldsymbol{e}}_{\boldsymbol{2}}\right\}$. We assume that the motion will be examined according to the frame $A:\left\{\bar{O};\bar{\boldsymbol{e}}_{\boldsymbol{1}},\bar{\boldsymbol{e}}_{\boldsymbol{2}}\right\}$. Next, the following equations can be defined as follows.

$$\begin{aligned}\boldsymbol{O_m\bar{O}}=\boldsymbol{a_m}= &\quad a_{m1}\bar{\boldsymbol{e}}_{\boldsymbol{1}}+a_{m2}\bar{\boldsymbol{e}}_{\boldsymbol{2}}\\ \boldsymbol{O_f\bar{O}}=\boldsymbol{a_f}= &\quad a_{f1}\bar{\boldsymbol{e}}_{\boldsymbol{1}}+a_{f2}\bar{\boldsymbol{e}}_{\boldsymbol{2}}\end{aligned}$$

and

$$\begin{array}{ll} d\theta_m=\omega_m & d\theta_f=\omega_f\\ da_{m1}-a_{m2}d\theta_m=\mu_{m1} & da_{f1}-a_{f2}d\theta_f=\mu_{f1}\\ da_{m2}-a_{m1}d\theta_m=\mu_{m2} & da_{f2}-a_{f1}d\theta_f=\mu_{f2}\end{array}\tag{4.1}$$

where d is used to denote the differential form. Therefore, we can obtain the linear differential forms, called Pfaff forms, as follows.

i) The motion A with respect to M (A/M):

$$d_m\bar{\boldsymbol{e}}_{\boldsymbol{1}}=\bar{\boldsymbol{e}}_{\boldsymbol{2}}\omega_m,\quad d_m\bar{\boldsymbol{e}}_{\boldsymbol{2}}=-\bar{\boldsymbol{e}}_{\boldsymbol{1}}\omega_m,\quad d\boldsymbol{a_m}=\bar{\boldsymbol{e}}_{\boldsymbol{1}}\mu_{m1}+\bar{\boldsymbol{e}}_{\boldsymbol{2}}\mu_{m2}\tag{4.2}$$

ii) The motion A with respect to F (A/F):

$$d_m\bar{\boldsymbol{e}}_{\boldsymbol{1}}=\bar{\boldsymbol{e}}_{\boldsymbol{2}}\omega_m,\quad d_f\bar{\boldsymbol{e}}_{\boldsymbol{2}}=-\bar{\boldsymbol{e}}_{\boldsymbol{1}}\omega_f,\quad d\boldsymbol{a_f}=\bar{\boldsymbol{e}}_{\boldsymbol{1}}\mu_{f1}+\bar{\boldsymbol{e}}_{\boldsymbol{2}}\mu_{f2}\tag{4.3}$$

where d_m and d_f are used to denote differential forms with respect to M and F, respectively. Let us determine the velocities of the point $\boldsymbol{X}$ with respect to the frame A. Using the following equations below via dual quaternions

$$\begin{aligned}\boldsymbol{e_{1f}} &= \bar{\boldsymbol{e}}_{\boldsymbol{1}}\cos\theta_f-\bar{\boldsymbol{e}}_{\boldsymbol{2}}\sin\theta_f\\ \boldsymbol{e_{2f}} &= \bar{\boldsymbol{e}}_{\boldsymbol{1}}\sin\theta_f+\bar{\boldsymbol{e}}_{\boldsymbol{2}}\cos\theta_f\end{aligned}$$

and

$$\boldsymbol{e_{1m}} = \bar{\boldsymbol{e}}_{\boldsymbol{1}} \cos\psi_m - \bar{\boldsymbol{e}}_{\boldsymbol{2}} \sin\psi_m$$

$$\boldsymbol{e_{2m}} = \bar{\boldsymbol{e}}_{\boldsymbol{1}} \sin\psi_m + \bar{\boldsymbol{e}}_{\boldsymbol{2}} \cos\psi_m$$

we can write the point $\boldsymbol{X}$ in the moving plane M with respect to the basis vectors of the frame A as

$$\begin{aligned} \boldsymbol{X_m} &= \left(x_1 + a_{m1}\cos\psi_m + a_{m2}\sin\psi_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ \left(x_2 - a_{m1}\sin\psi_m + a_{m2}\cos\psi_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{2}}. \end{aligned} \tag{4.4}$$

The differential of the point $\boldsymbol{X}$ about to moving frame M can be written with the following equation:

$$\begin{aligned} d\boldsymbol{X_m} &= d_m\left(x_1 + d_m a_{m1}\cos\psi_m + d_m a_{m2}\sin\psi_m - x_2\omega_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ d_m\left(x_2 - d_m a_{m1}\sin\psi_m + d_m a_{m2}\cos\psi_m + x_1\omega_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{2}}. \end{aligned} \tag{4.5}$$

Then we can write the relative velocity vector of the point $\boldsymbol{X}$ with respect to the moving frame M as $\boldsymbol{v_r} = \dfrac{d\boldsymbol{X_m}}{dt}$. If $\boldsymbol{v_r} = 0$ or $d\boldsymbol{X_m} = 0$, then the condition of fixing the point X in the moving frame M as

$$\begin{aligned} d_m x_1 &= -d_m a_{m1}\cos\psi_m - d_m a_{m2}\sin\psi_m + x_2\omega_m \\ d_m x_2 &= d_m a_{m1}\sin\psi_m - d_m a_{m2}\cos\psi_m - x_1\omega_m. \end{aligned} \tag{4.6}$$

Using the equations in (4.1), the equation (4.6) can be expressed as

$$\begin{aligned} d_m x_1 &= -\left(\mu_{m1} + a_{m2}\omega_m\right)\cos\psi_m \\ &- \left(\mu_{m2} - a_{m1}\omega_m\right)\sin\psi_m + x_2\omega_m \\ d_m x_2 &= \left(\mu_{m1} + a_{m2}\omega_m\right)\sin\psi_m \\ &- \left(\mu_{m2} - a_{m1}\omega_m\right)\cos\psi_m - x_1\omega_m. \end{aligned} \tag{4.7}$$

On the other hand, we can write the point $\boldsymbol{X}$ in the fixed plane F with respect to the basis vectors of the frame A as

$$\begin{aligned} \boldsymbol{X_f} &= \left(x_1 + a_{f1}\cos\theta_f + a_{f2}\sin\theta_f\right)\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ \left(x_2 - a_{f1}\sin\theta_f + a_{f2}\cos\theta_f\right)\bar{\boldsymbol{e}}_{\boldsymbol{2}}. \end{aligned} \tag{4.8}$$

The differential of the point $\boldsymbol{X}$ about to fixed frame F can be expressed by the following equation:

$$\begin{aligned} d\boldsymbol{X_f} &= \left(d_f x_1 + d_f a_{f1}\cos\theta_f + d_f a_{f2}\sin\theta_f - x_2\omega_f\right)\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ \left(d_f x_2 - d_f a_{f1}\sin\theta_f + d_f a_{f2}\cos\theta_f + x_1\omega_f\right)\bar{\boldsymbol{e}}_{\boldsymbol{2}}. \end{aligned} \tag{4.9}$$

Then we can write the absolute velocity vector of the point $\boldsymbol{X}$ about to fixed frame F with $v_a = \dfrac{d\boldsymbol{X_f}}{dt}$. If $v_a = 0$ or $d\boldsymbol{X_f} = 0$, then the condition of fixing the point $\boldsymbol{X}$ in the fixed frame F is as follows.

$$\begin{aligned} d_f x_1 &= -d_f a_{f1} \cos\theta_f - d_f a_{f2} \sin\theta_f + x_2 \omega_f \\ d_f x_2 &= d_f a_{f1} \sin\theta_f - d_f a_{f2} \cos\theta_f - x_1 \omega_f . \end{aligned} \tag{4.10}$$

Using the equations in (4.1), the equation (4.10) can be written using the following equation:

$$\begin{aligned} d_f x_1 &= (-\mu_{f1} + a_{f2}\omega_f) \cos\theta_f + (\mu_{f2} - a_{f1}\omega_f) \sin\theta_f + x_2 \omega_f \\ d_f x_2 &= (\mu_{f1} - a_{f2}\omega_f) \sin\theta_f + (\mu_{f2} - a_{f1}\omega_f) \cos\theta_f - x_1 \omega_f . \end{aligned} \tag{4.11}$$

We can define the vector of the drift velocity of the point $\boldsymbol{X}$ if the point $\boldsymbol{X}$ remains fixed in the moving plane M according to the fixed plane F with $v_d = \dfrac{d_d\boldsymbol{X}}{dt}$. Using equations (4.5) and (4.9), we can express the $d_d\boldsymbol{X}$ as

$$\begin{aligned} d_d\boldsymbol{X} &= [(\mu_{f1}\cos\theta_f - \mu_{m1}\cos\psi_m) - a_{m2}(\omega_f\cos\theta_f - \omega_m\cos\psi_m) \\ &+ (\mu_{f2}\sin\theta_f - \mu_{m2}\sin\psi_m) - a_{m1}(\omega_f\sin\theta_f - \omega_m\sin\psi_m) \\ &- x_2(\omega_f - \omega_m)]\,\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ [-(\mu_{f1}\sin\theta_f - \mu_{m1}\sin\psi_m) + a_{m2}(\omega_f\sin\theta_f - \omega_m\sin\psi_m) \\ &+ (\mu_{f2}\cos\theta_f - \mu_{m2}\cos\psi_m) - a_{m1}(\omega_f\cos\theta_f - \omega_m\cos\psi_m) \\ &+ x_1(\omega_f - \omega_m)]\,\bar{\boldsymbol{e}}_{\boldsymbol{2}} \end{aligned} \tag{4.12}$$

Let us now determine the pole point $\boldsymbol{P}(p_1, p_2)$ from $d_d\boldsymbol{X} = 0$ as

$$\begin{aligned} \boldsymbol{p_1} &= \frac{(\mu_{f1}\sin\theta_f - \mu_{m1}\sin\psi_m) - a_{m2}(\omega_f\sin\theta_f - \omega_m\sin\psi_m)}{(\omega_f - \omega_m)} \\ &- \frac{(\mu_{f2}\cos\theta_f - \mu_{m2}\cos\psi_m) - a_{m1}(\omega_f\cos\theta_f - \omega_m\cos\psi_m)}{(\omega_f - \omega_m)} \\ \boldsymbol{p_2} &= \frac{(\mu_{f1}\cos\theta_f - \mu_{m1}\cos\psi_m) - a_{m2}(\omega_f\cos\theta_f - \omega_m\cos\psi_m)}{(\omega_f - \omega_m)} \\ &- \frac{(\mu_{f2}\sin\theta_f - \mu_{m2}\sin\psi_m) - a_{m1}(\omega_f\sin\theta_f - \omega_m\sin\psi_m)}{(\omega_f - \omega_m)} \end{aligned} \tag{4.13}$$

If the point $\boldsymbol{X}$ is a fixed point in the plane A, since $d_m x_1 = d_m x_2 = 0$ with respect to the frame M, we can define the pole point $\boldsymbol{R}(r_1, r_2)$ as follows:

$$\begin{aligned} r_1 &= \frac{(\mu_{m1} + a_{m2}\omega_m)\sin\psi_m - (\mu_{m2} - a_{m1}\omega_m)\cos\psi_m}{\omega_m} \\ r_2 &= \frac{(\mu_{m1} + a_{m2}\omega_m)\cos\psi_m + (\mu_{m2} - a_{m1}\omega_m)\sin\psi_m}{\omega_m} \end{aligned} \tag{4.14}$$

If the point $\boldsymbol{X}$ is a fixed point in the plane A, since $d_f x_1 = d_f x_2 = 0$ with respect to the frame F, we can define the pole point $\boldsymbol{S}(s_1, s_2)$ as follows:

$$\begin{aligned} s_1 &= \frac{(\mu_{f1} - a_{m2}\omega_f)\sin\theta_f - (\mu_{f2} - a_{f1}\omega_f)\cos\theta_m}{\omega_f} \\ s_2 &= \frac{(\mu_{f1} - a_{f2}\omega_f)\cos\theta_f + (\mu_{f2} - a_{f1}\omega_f)\sin\theta_m}{\omega_f} \end{aligned} \tag{4.15}$$

4.1. **Canonical System.** In this section, we will use the system $A : \left\{\bar{O}; \bar{\boldsymbol{e}}_{\boldsymbol{1}}, \bar{\boldsymbol{e}}_{\boldsymbol{2}}\right\}$ as a special system with the following restrictions:

i) The origin point $\bar{O}$ of the frame A coincides with the pole point P, since $A = P$.

ii) $A : \left\{\bar{O}; \bar{\boldsymbol{e}}_{\boldsymbol{1}}\right\}$ axis coincides with the common pole tangent vector of the pole curves $(\boldsymbol{p_m})$ and $(\boldsymbol{p_f})$. From the first constraint it follows that $p_1 = p_2 = 0$ is written, then the relations $\mu_{m1} = \mu_{f1}$ and $\mu_{m2} = \mu_{f2}$ are true. From the second constraint the relation $\mu_{m2} = \mu_{f2} = 0$ can be written. The differential of the point $\boldsymbol{X}$ with respect to the moving frame M and the fixed frame F can be written with the following equations:

$$\begin{aligned} d\boldsymbol{X_m} &= \left(d_m x_1 + (\mu + a_{m2}\omega_m)\cos\psi_m - a_{m1}\omega_m \sin\psi_m - x_2\omega_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ \left(d_m x_2 - (\mu + a_{m2}\omega_m)\sin\psi_m + a_{m1}\omega_m\cos\psi_m - x_1\omega_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{2}} \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} d\boldsymbol{X_f} &= \left(d_f x_1 + (\mu - a_{m2}\omega_m)\cos\theta_f - a_{m1}\omega_m \sin\theta_f - x_2\omega_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+ \left(d_f x_2 - (\mu - a_{m2}\omega_m)\sin\theta_f + a_{m1}\omega_m\cos\theta_f - x_1\omega_m\right)\bar{\boldsymbol{e}}_{\boldsymbol{2}} \end{aligned} \tag{4.17}$$

So we can express the condition of fixing the point $\boldsymbol{X}$ in the moving frame M and in the fixed frame F as follows:

$$\begin{aligned} d_m x_1 &= -\left(\mu + a_{m2}\omega_m\right)\cos\psi_m + a_{m1}\omega_m\sin\psi_m + x_2\omega_m \\ d_m x_2 &= \left(\mu + a_{m2}\omega_m\right)\sin\psi_m - a_{m1}\omega_m\cos\psi_m + x_1\omega_m \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} d_f x_1 &= -\left(\mu - a_{m2}\omega_f\right)\cos\theta_f + a_{m1}\omega_m\sin\theta_f + x_2\omega_f \\ d_f x_2 &= \left(\mu - a_{m2}\omega_f\right)\sin\theta_f - a_{m1}\omega_m\cos\theta_f + x_1\omega_f \end{aligned} \tag{4.19}$$

In addition, the following equation can be written as follows:

$$\begin{aligned} d_d\boldsymbol{X} &= \left[-a_{m2}\left(\omega_f\cos\theta_f + \omega_m\sin\psi_m\right) - a_{m1}\left(\omega_f\sin\theta_f - \omega_m\cos\psi_m\right)\right. \\ &\left.-x_2\left(\omega_f - \omega_m\right)\right]\bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ &+\left[a_{m2}\left(\omega_f\sin\theta_f + \omega_m\sin\psi_m\right) - a_{m1}\left(\omega_f\cos\theta_f - \omega_m\cos\psi_m\right)\right. \\ &\left.+x_1\left(\omega_f - \omega_m\right)\right]\bar{\boldsymbol{e}}_{\boldsymbol{2}} \end{aligned} \tag{4.20}$$

4.2. **The Curvature of The Trajectory Curves.** Let us determine the curvatures of the trajectories in the fixed plane F drawn from the points of the moving plane M. We will use the canonical system. The points $\boldsymbol{X_f}$, $\boldsymbol{X_m}$ and the pole point $\boldsymbol{P}$ located on the instantaneous trajectory normal belong to the $\boldsymbol{X_m}$ at every moment t. Therefore, the following vectors have the same direction through the point $\boldsymbol{P}$ as:

$$\begin{aligned} \boldsymbol{PX_m} &= x_{1m}\bar{\boldsymbol{e}}_{\boldsymbol{1}} + x_{2m}\bar{\boldsymbol{e}}_{\boldsymbol{2}} \\ \boldsymbol{PX_f} &= x_{1f}\bar{\boldsymbol{e}}_{\boldsymbol{1}} + x_{2f}\bar{\boldsymbol{e}}_{\boldsymbol{2}} \end{aligned}$$

Therefore, the following relationships apply to the coordinates:

$$x_{1m} : x_{2m} = x_{1f} : x_{2f}$$

or

$$x_{1m}x_{2f} - x_{1f}x_{2m} = 0. \tag{4.21}$$

If we now take the differential of equation (4.21) and substitute equations (4.18) and (4.19), we obtain the following equation as:

$$\begin{aligned}
&\mu\left(-x_{2f}\cos\psi_m + x_{1m}\sin\theta_f + x_{2m}\cos\theta_f - x_{1f}\sin\psi_m\right)\\
&+a_{m2}\left(x_{2f}\omega_m\cos\psi_m - x_{1m}\omega_f\sin\theta_f - x_{2m}\omega_f\cos\theta_f + x_{1f}\omega_f\sin\psi_m\right)\\
&+a_{m1}\left(x_{2f}\omega_m\sin\psi_m + x_{1m}\omega_f\cos\theta_f - x_{2m}\omega_f\sin\theta_f - x_{1f}\omega_m\cos\psi_m\right)\\
&+\left(x_{1m}x_{1f} + x_{2m}x_{2f}\right)\omega_m - \left(x_{m1}^2 + x_{m2}^2\right)\omega_f = 0.
\end{aligned} \tag{4.22}$$

We can use polar coordinates as follows:

$$\begin{aligned}
x_{1m} &= \lambda_m\cos\alpha \qquad x_{2m} = \lambda_m\sin\alpha\\
x_{1f} &= \lambda_f\cos\alpha \qquad x_{2f} = \lambda_f\sin\alpha
\end{aligned}$$

Therefore, we can express the equation (4.22) as

$$\begin{aligned}
&\lambda_f\sin\alpha\left[(-\mu + a_{m2}\omega_m)\cos\psi_m + a_{m1}\omega_m\sin\psi_m\right]\\
&+\lambda_f\cos\alpha\left[(-\mu + a_{m2}\omega_f)\sin\psi_m - a_{m1}\omega_m\cos\psi_m\right]\\
&+\lambda_m\sin\alpha\left[(\mu - a_{m2}\omega_f)\cos\theta_f - a_{m1}\omega_f\sin\theta_f\right]\\
&+\lambda_m\cos\alpha\left[(\mu - a_{m2}\omega_f)\sin\theta_f + a_{m1}\omega_f\cos\theta_f\right] = 0.
\end{aligned} \tag{4.23}$$

So, using the above equation, we can determine the Euler-Savary formula as follows:

$$\left[\frac{1}{\lambda_m}(*_1) + \frac{1}{\lambda_f}(*_3)\right]\sin\alpha + \left[\frac{1}{\lambda_m}(*_2) + \frac{1}{\lambda_f}(*_4)\right]\cos\alpha = \frac{1}{\lambda_f}\left(\lambda_m\omega_f - \lambda_f\omega_m\right) \tag{4.24}$$

where

$$\begin{aligned}
(*_1) &= (-\mu + a_{m2}\omega_m)\cos\psi_m + a_{m1}\omega_m\sin\psi_m\\
(*_2) &= (-\mu + a_{m2}\omega_f)\sin\psi_m - a_{m1}\omega_m\cos\psi_m\\
(*_3) &= (\mu - a_{m2}\omega_f)\cos\theta_f - a_{m1}\omega_f\sin\theta_f\\
(*_4) &= (\mu - a_{m2}\omega_f)\sin\theta_f + a_{m1}\omega_f\cos\theta_f.
\end{aligned}$$

5. Conclusion

In this study, we have developed a new approach by using dual quaternions to study the pole points and pole trajectories, the triple coordinate system, the canonical system, and the Euler-Savary equation in planar kinematics. Thus, we have given a new perspective to planar motion.

References

[1] V. Bulut. On the planar kinematic using dual quaternions. In *The International Conference on Materials Science, Machine and Automotive Engineerings and Technology in Cesme/IZMIR (IMSMATEC)*, 2018.

[2] W. Clifford. Preliminary sketch of bi-quaternions. *Proc. London Math. Soc.*, 1873.

[3] K. Daniilidis. Hand-eye calibration using dual quaternions. *International Journal of Robotics Research*, 18:286–298, 1999.

[4] Q. J. Ge and B. Ravani. Computer aided geometric design of motion interpolants. *ASME Journal of Mechanical Design*, 116(3):756–762, 1994.

[5] W. R. Hamilton. Xxii. on quaternions; or on a new system of imaginaries in algebra. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 1846.

[6] W. R. Hamilton. *Elements of quaternions.* Vol. I & II, Newyork Chelsea, 1869.

[7] B. Juttler. Visualization of moving objects using dual quaternion curves. *Computers and Graphics*, 18(3):315–326, 1994.

[8] Zara J. Kavan L., Collins S. and O'Sullivan C. Skinning with dual quaternions. In *In Proceedgings of ACM SIGGRAPH Symposium on Interactive 3D Graphics and Games*, 2007.

[9] B. Kenwright. A beginners guide to dual-quaternions: What they are, how they work and how to use them for 3d character hierarchies. In *In The 20th International Conference on Computer Graphics, Visualization and Computer Vision*, 2012.

[10] C. Luciano and P. Banerjee. Avatar kinematics modeling for telecollaborative virtual environments. In *Proceedings of the 32nd conference on Winter simulation, Society for Computer Simulation International*, pages 1533–1538, 2000.

[11] H.R. Muller. Kinematische geometrie. *Jahresber, DMV*, 72:143–164, 1970.

[12] E. Pennestri and P. Valentini. Dual quaternions as a tool for rigid body motion analysis: A tutorial with an application to biomechanics. *Archive of Mechanical Engineering*, 57(2):187–205, 2010.

[13] A. Perez and J. M. Mccarthy. Dual quaternion synthesis of constrained robotic systems. *Journal of Mechanical Design*, 126:425–435, 2004.

[14] E. Salamin. Application of quaternions to computation with rotations. Technical report, AI Lab, Stanford Univ., 1979.