

---

---

## Improvement of the Grüss type inequalities for positive linear maps on $C^*$ -algebras

F. Golfarshchi<sup>1</sup> A. A. Khalilzadeh<sup>2</sup> F. Moradlou<sup>3</sup>

<sup>1</sup> Department of Multimedia, Tabriz Islamic Art University, Tabriz, Iran

<sup>2,3</sup> Department of Mathematics, Sahand University of Technology, Tabriz, Iran

ABSTRACT. Assume that  $A$  and  $B$  are unital  $C^*$ -algebras and  $\varphi : A \rightarrow B$  is a unital positive linear map. We show that if  $B$  is commutative, then for all  $x, y \in A$  and  $\alpha, \beta \in \mathbb{C}$

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq [\varphi(|x^* - \alpha 1_A|^2)]^{\frac{1}{2}} [\varphi(|y - \beta 1_A|^2)]^{\frac{1}{2}} - |\varphi(x^* - \alpha 1_A)| |\varphi(y - \beta 1_A)|.$$

Furthermore, we prove that if  $z \in A$  with  $|z| = 1$  and  $\lambda, \mu \in \mathbb{C}$  are such that  $\operatorname{Re}(\varphi((x^* - \beta z^*)(\alpha z - x))) \geq 0$  and  $\operatorname{Re}(\varphi((y^* - \bar{\mu} z^*)(\lambda z - y))) \geq 0$ , then

$$|\varphi(x^* y) - \varphi(x^* z)\varphi(z^* y)| \leq \frac{1}{4} |\beta - \alpha| |\mu - \alpha| - [\operatorname{Re}(\varphi((x^* - \beta z^*)(\alpha z - x)))]^{\frac{1}{2}} [\operatorname{Re}(\varphi((y^* - \bar{\mu} z^*)(\lambda z - y)))]^{\frac{1}{2}}.$$

The presented bounds for the Grüss type inequalities on  $C^*$ -algebras improve the other ones in the literature under mild conditions.

As an application, using our results, we give some inequalities in  $L^\infty([a, b])$ , which refine the other ones in the literature.

---

<sup>1</sup>Corresponding author: [f.golfarshchi@tabriziau.ac.ir](mailto:f.golfarshchi@tabriziau.ac.ir)


Received: 11 June 2022

Revised: 20 August 2022

Accepted: 10 September 2022

**How to Cite:** Golfarshchi, Fatemeh; Khalilzadeh, Ali Asghar; Moradlou, Feridoon. Improvement of the Grüss type inequalities for positive linear maps on  $C^*$ -algebras, *Casp.J. Math. Sci.*, **12**(1)(2023), 81-93.

This work is licensed under a Creative Commons Attribution 4.0 International License.

 Copyright © 2023 by University of Mazandaran. Submitted for possible open access publication under the terms and conditions of the Creative Commons Attribution(CC BY) license(<https://creativecommons.org/licenses/by/4.0/>)

Keywords:  $C^*$ -algebras, Grüss type inequalities, Positive linear map.

2000 Mathematics subject classification: 47A63, 47C15, 26D15, 47A30

## 1. INTRODUCTION

In 1935, Grüss [13] proved the following complement of Chebyshev's inequality:

$$\left| \frac{1}{b-a} \int_a^b fg(x)dx - \frac{1}{(b-a)} \int_a^b f(x)dx \frac{1}{(b-a)} \int_a^b g(x)dx \right| \leq \frac{1}{4} |(M-m)| |(N-n)|,$$

where  $f$  and  $g$  are real integrable functions on  $[a, b]$  such that there exist  $m, n, M, N \in \mathbb{R}$  for which  $m \leq f(x) \leq M$  and  $n \leq g(x) \leq N$  for all  $x \in [a, b]$ . It is well-known that the constant  $\frac{1}{4}$  can not be replaced by a smaller one and is derived for  $f(x) = g(x) = \operatorname{sgn}(x - (a+b)/2)$ . This inequality is known as Grüss inequality in the literature and attracted the attention of many mathematicians, for more information about Grüss inequality see [18, chapter X]. The goal of these researches is to investigate and to generalize the various types of the Grüss inequality in the different area of mathematics and to obtain improved bounds for these inequalities by assuming suitable conditions, for more details see [9, 10, 22, 23]. Moreover, Grüss type inequalities have some important applications in integral arithmetic mean, difference equations, coding theory and statistics [1, 8, 17].

Recently, Dragomir [9] generalized the Grüss inequality in the setting of inner product spaces and also Ilašević and Varošaneć [14] presented some results about the Grüss type inequalities in the framework of inner product modules. Also, many mathematicians are interested to study the Grüss type inequalities for positive linear maps. In 2000, Bhatia and Davis [2], presented a reverse to the so-called Kadison inequality [15]. More precisely, for a positive unital linear map  $\varphi$  between  $C^*$ -algebras, they proved that

$$\varphi(x^2) - \varphi(x)^2 \leq \frac{1}{4}(M - m)$$

for all self-adjoint element  $x$  with  $m \leq x \leq M$ . Moreover, Bhatia and Sharma [3] proved the following extension of the above inequality:

$$\varphi(x^*x) - \varphi(x)^*\varphi(x) \leq \inf_{\lambda \in \mathbb{C}} \|x - \lambda\| \quad (1.1)$$

for all  $x$ . Moslehian and Rajić [19] extended (1.1) and presented the following inequality for unital  $n$ -positive ( $n \geq 3$ ) linear map  $\varphi$ :

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \inf_{\lambda \in \mathbb{C}} \|x - \lambda\| \inf_{\mu \in \mathbb{C}} \|y - \mu\|, \quad (1.2)$$

for all  $x, y$ .

Ghazanfari and Dragomir [11] introduced a simple formulation of the Grüss type inequality in inner product  $C^*$ -modules. They also presented some generalization of the Grüss type inequalities in inner product modules. Also, Ghazanfari [12] investigated Grüss type inequality for vector-valued functions in Hilbert  $C^*$ -modules.

Very recently, Dadkhah and Moslehian [7, Theorem 3.5] for  $n$ -positive ( $n \geq 3$ ) linear map  $\varphi : A \rightarrow B$  between two  $C^*$ -algebras  $A$  and  $B$  presented the following refined Grüss type inequality:

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \|\varphi(|x^* - \alpha|^2)\|^{\frac{1}{2}} \|\varphi(|y - \beta|^2)\|^{\frac{1}{2}} \quad (1.3)$$

for all  $x, y \in A$  and for all  $\alpha, \beta \in \mathbb{C}$ . Also they proved that the inequality (1.2) is true for all unital  $n$ -positive linear maps and

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \frac{1}{4} |\lambda - \mu| |\alpha - \beta|, \quad (1.4)$$

for all  $x, y$  and all  $\alpha, \beta \in \mathbb{C}$ .

In this paper, motivated and inspired by Dadkhah and Moslehian [7], assuming a mild condition, we present two improved bounds for the left sides of the inequalities (1.2) and (1.4). As an application, we give sharpened Grüss inequalities in  $C^*$ -algebra  $L^\infty([a, b])$ .

## 2. PRELIMINARIES

Let  $A$  and  $B$  be unital  $C^*$ -algebras. An element  $a \in A$  is said to be positive and is denoted by  $a \geq 0$ , if  $a$  is self-adjoint and  $\sigma(a) \subseteq \mathbb{R}^+$ . If in addition  $a$  is also invertible, then it is called strictly positive and is denoted by  $a > 0$ . The set of self-adjoint and positive elements of  $A$  are denoted by  $A_{sa}$  and  $A^+$ , respectively. It is well known that  $|a|^2 = a^*a$  and  $Re(a) = \frac{a+a^*}{2}$ . The linear map  $\varphi : A \rightarrow B$  is said to be unital if  $\varphi(1_A) = 1_B$ . Furthermore, if  $\varphi(xy) = \varphi(x)\varphi(y)$  and  $\varphi(x^*) = \varphi(x)^*$  for all  $x, y \in A$ , then  $\varphi$  is called a  $*$ -homomorphism. For any positive integer  $n$ , we define  $\varphi_n : M_n(A) \rightarrow M_n(B)$  by  $\varphi_n((a_{ij})_{n \times n}) = (\varphi(a_{ij}))_{n \times n}$ , where  $M_n(A)$  denotes the  $C^*$ -algebra of all  $n \times n$  matrices with entries in  $A$ . The map  $\varphi$  is called positive if  $\varphi(a) \geq 0$  for all  $a \in A^+$ , and  $n$ -positive if  $\varphi_n$  is positive. Moreover,  $\varphi$  is called completely positive if  $\varphi$  is  $n$ -positive for all  $n$ . It worth noting that, every positive linear map is not completely positive, for example if  $\varphi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  is defined by  $\varphi(A) = A^t$ , then  $\varphi$  is positive but is not completely positive. Also, every  $*$ -homomorphism on a  $*$ -algebras is completely positive [4, Example II.6.9.3], but the converse is not true in general, for example if  $\varphi : M_2(\mathbb{C}) \rightarrow \mathbb{C}$  is defined by  $\varphi((a_{ij})_{2 \times 2}) = \sum_{i=1}^2 a_{ii}$ , then  $\varphi$  is completely positive but  $\varphi$  is not a homomorphism because  $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^2 = 9$

$$\text{and } \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^2 = 5.$$

It should be noted that, if  $B$  is commutative, using Theorem V.35.4 of [5], Theorem 3.9 of [21] and Theorem 3.11 of [21], we can conclude that every unital positive linear map from  $A$  to a commutative  $C^*$ -algebra  $B$ , is completely positive.

**Lemma 2.1.** *Assume that  $A$  is a commutative unital  $C^*$ -algebra and  $a, b, c, d \in A_{sa}$ . The following statements hold:*

- i) *If  $b$  is strictly positive and  $a \leq cb^{-1}$ , then  $ab \leq c$ .*
- ii) *If  $0 \leq a \leq b$  and  $0 \leq c \leq d$ , then  $ac \leq bd$ .*
- iii)  *$(a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2$ .*
- iv) *If  $a, b, c, d \in A^+$ , then  $ac + bd \leq (a^2 + b^2)^{\frac{1}{2}}(c^2 + d^2)^{\frac{1}{2}}$ .*

### 3. GRÜSS TYPE INEQUALITIES ON $C^*$ -ALGEBRAS

In this section, we improve the bounds of the Grüss type inequalities for a positive linear map  $\varphi : A \rightarrow B$  between two  $C^*$ -algebras. More precisely, assuming the commutativity of  $B$ , we obtain sharpened results for the Grüss type inequalities. For this purpose some elementary results have been proven.

**Lemma 3.1.** *Assume that  $A$  and  $B$  are unital  $C^*$ -algebras,  $B$  is commutative and  $\varphi : A \rightarrow B$  is a unital positive linear map. Then*

$$|\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2 \leq [\varphi(|x|^2) - |\varphi(z^*x)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2],$$

for all  $x, y, z \in A$  with  $|z| = 1$ .

*Proof.* Let  $x, y, z \in A$  with  $|z| = 1$ . Utilizing [7, Lemma, 3.4], we deduce

$$\begin{bmatrix} \varphi(|x|^2) - |\varphi(z^*x)|^2 & \varphi(x^*y) - \varphi(z^*y)\varphi(x^*z) \\ \varphi(y^*x) - \varphi(z^*x)\varphi(y^*z) & \varphi(|y|^2) - |\varphi(z^*y)|^2 \end{bmatrix} \geq 0.$$

Without lose of generality, we may suppose that  $\varphi(|y|^2) - |\varphi(z^*y)|^2 > 0$ , then using [6, Lemma, 2.1], we obtain

$$\begin{aligned} \varphi(|x|^2) - |\varphi(z^*x)|^2 &\geq [\varphi(x^*y) - \varphi(z^*y)\varphi(x^*z)] [\varphi(|y|^2) - |\varphi(z^*y)|^2]^{-1} \\ &\quad \times [\varphi(y^*x) - \varphi(z^*x)\varphi(y^*z)]. \end{aligned}$$

On the other hand Lemma 2.1 implies that

$$[\varphi(|x|^2) - |\varphi(z^*x)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2] \geq |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2.$$

The positivity of  $\varphi(|y|^2) - |\varphi(z^*y)|^2$  implies that  $\varphi(|y|^2) - |\varphi(z^*y)|^2 + \frac{1}{n} > 0$  for all  $n \in \mathbb{N}$ . Also since

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \geq 0,$$

we have

$$\begin{bmatrix} \varphi(|x|^2) - |\varphi(z^*x)|^2 & \varphi(x^*y) - \varphi(z^*y)\varphi(x^*z) \\ \varphi(y^*x) - \varphi(z^*x)\varphi(y^*z) & \varphi(|y|^2) - |\varphi(y)|^2 + \frac{1}{n} \end{bmatrix} \geq 0.$$

Therefore

$$[\varphi(|x|^2) - |\varphi(z^*x)|^2] \left[ \varphi(|y|^2) - |\varphi(z^*y)|^2 + \frac{1}{n} \right] \geq |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2. \quad (3.1)$$

Taking the limits as  $n \rightarrow \infty$  in (3.1), we deduce

$$[\varphi(|x|^2) - |\varphi(z^*x)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2] \geq |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2.$$

□

**Theorem 3.2.** *Assume that  $A$  and  $B$  are unital  $C^*$ -algebras,  $B$  is commutative and  $\varphi : A \rightarrow B$  is a unital positive linear map. Then*

$$|\varphi(xy) - \varphi(xz)\varphi(z^*y)| \leq [\varphi(|x^* - \alpha z|^2)]^{\frac{1}{2}} [\varphi(|y - \beta z|^2)]^{\frac{1}{2}} \quad (3.2)$$

for all  $x, y, z \in A$  with  $|z| = 1$  and for all  $\alpha, \beta \in \mathbb{C}$ ,

*Proof.* Let  $x, z \in A$  with  $|z| = 1$  and  $\alpha, \beta \in \mathbb{C}$ . Utilizing Lemma 3.1, we deduce

$$|\varphi(xy) - \varphi(xz)\varphi(z^*y)|^2 \leq [\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2].$$

Also, using [7, Lemma 3.1], we can derive that

$$0 \leq \varphi(|x^*|^2) - |\varphi(z^*x^*)|^2 \leq \varphi(|x^* - \alpha z|^2)$$

and

$$0 \leq \varphi(|y|^2) - |\varphi(z^*y)|^2 \leq \varphi(|y - \beta z|^2),$$

so, it follows from Lemma 2.1 that

$$[\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2] \leq \varphi(|x^* - \alpha z|^2)\varphi(|y - \beta z|^2).$$

Since  $B$  is commutative, by using [4, Proposition II.3.1.2], we can conclude that

$$[\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2] \geq 0.$$

Applying Theorem 2.2.6 of [20], we get

$$\begin{aligned} & [\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2]^{\frac{1}{2}} [\varphi(|y|^2) - |\varphi(z^*y)|^2]^{\frac{1}{2}} \\ & \leq [\varphi(|x^* - \alpha z|^2)]^{\frac{1}{2}} [\varphi(|y - \beta z|^2)]^{\frac{1}{2}}. \end{aligned}$$

Thus

$$|\varphi(xy) - \varphi(xz)\varphi(z^*y)| \leq [\varphi(|x^* - \alpha z|^2)]^{\frac{1}{2}} [\varphi(|y - \beta z|^2)]^{\frac{1}{2}}.$$

□

**Theorem 3.3.** *Assume that  $A$  and  $B$  are unital  $C^*$ -algebras,  $B$  is commutative and  $\varphi : A \rightarrow B$  is a unital positive linear map. Then*

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq [\varphi(|x^* - \alpha 1_A|^2)]^{\frac{1}{2}} [\varphi(|y - \beta 1_A|^2)]^{\frac{1}{2}} - |\varphi(x^* - \alpha 1_A)| |\varphi(y - \beta 1_A)|,$$

for all  $x, y \in A$  and for all  $\alpha, \beta \in \mathbb{C}$ .

*Proof.* Letting  $x, y \in A$ ,  $\alpha, \beta \in \mathbb{C}$  and  $z = 1_A$  in Lemma 3.1, we get

$$|\varphi(xy) - \varphi(x)\varphi(y)|^2 \leq [\varphi(|x^*|^2) - |\varphi(x^*)|^2] [\varphi(|y|^2) - |\varphi(y)|^2].$$

Also, it is readily seen that

$$\begin{cases} \varphi(|x^* - \alpha 1_A|^2) - |\varphi(x^* - \alpha 1_A)|^2 = \varphi(|x^*|^2) - |\varphi(x^*)|^2, \\ \varphi(|y - \beta 1_A|^2) - |\varphi(y - \beta 1_A)|^2 = \varphi(|y|^2) - |\varphi(y)|^2, \end{cases}$$

so

$$|\varphi(xy) - \varphi(x)\varphi(y)|^2 \leq [\varphi(|x^* - \alpha|^2) - |\varphi(x^* - \alpha)|^2] [\varphi(|y - \beta|^2) - |\varphi(y - \beta)|^2].$$

Thus using Lemma 2.1, we can conclude that

$$|\varphi(xy) - \varphi(x)\varphi(y)|^2 \leq \left[ (\varphi(|x^* - \alpha|^2))^{\frac{1}{2}} (\varphi(|y - \beta|^2))^{\frac{1}{2}} - |\varphi(x^* - \alpha)| - |\varphi(y - \beta)| \right]^2.$$

But

$$(\varphi(|x^* - \alpha|^2))^{\frac{1}{2}} (\varphi(|y - \beta|^2))^{\frac{1}{2}} - |\varphi(x^* - \alpha)| |\varphi(y - \beta)| \geq 0,$$

because it is self-adjoint and if

$$\lambda \in \sigma\left( \left[ (\varphi(|x^* - \alpha|^2))^{\frac{1}{2}} (\varphi(|y - \beta|^2))^{\frac{1}{2}} - |\varphi(x^* - \alpha)| |\varphi(y - \beta)| \right] \right),$$

then there exist  $\gamma \in \sigma((\varphi(|x^* - \alpha|^2))^{\frac{1}{2}})$ ,  $\mu \in \sigma((\varphi(|y - \beta|^2))^{\frac{1}{2}})$ ,  $\zeta \in \sigma(|\varphi(x^* - \alpha)|)$  and  $\eta \in \sigma(|\varphi(y - \beta)|)$ , such that  $\lambda = \gamma\mu - \zeta\eta$ . Since  $\varphi(|x^* - \alpha 1_A|^2) \geq |\varphi(x^* - \alpha 1_A)|^2$  and  $\varphi(|y - \beta|^2) \geq |\varphi(y - \beta)|^2$ , we can conclude that  $\gamma \geq \zeta \geq 0$  and  $\mu \geq \eta \geq 0$ . Thus  $\gamma\mu \geq \zeta\mu \geq \zeta\eta$ , so  $\lambda \geq 0$ . Therefore

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq [\varphi(|x^* - \alpha 1_A|^2)]^{\frac{1}{2}} [\varphi(|y - \beta 1_A|^2)]^{\frac{1}{2}} - |\varphi(x^* - \alpha 1_A)| |\varphi(y - \beta 1_A)|.$$

□

Now, we prove some lemmas to obtain another Grüss type inequality for a positive linear map by considering a mild conditions.

**Lemma 3.4.** *Suppose that  $A$  is a unital  $C^*$ -algebras and  $a, b \in A$ . Then*

$$|a + b|^2 \geq 4\operatorname{Re}(a^*b).$$

*Proof.* Since  $(a - b)^*(a - b) \geq 0$ , we can conclude that  $a^*a + b^*b \geq a^*b + b^*a$ . Hence

$$|a+b|^2 = a^*a + b^*b + a^*b + b^*a \geq 2(a^*b + b^*a) = 4\left(\frac{a^*b + b^*a}{2}\right) = 4\operatorname{Re}(a^*b).$$

□

**Lemma 3.5.** *Suppose that  $A$  and  $B$  are unital  $C^*$ -algebras and  $\varphi : A \rightarrow B$  is a unital positive linear map. Then*

$$\varphi(|x|^2) - |\varphi(z^*x)|^2 = \operatorname{Re}((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})) - \operatorname{Re}(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))),$$

for all  $x, y, z \in A$  with  $|z| = 1$  and for all  $\alpha, \beta \in \mathbb{C}$ .

*Proof.* Let  $x, y, z \in A$  with  $|z| = 1$  and suppose that  $\alpha, \beta \in \mathbb{C}$ . Then

$$\begin{aligned} & \operatorname{Re}((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})) - \operatorname{Re}(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) \\ &= \frac{1}{2} [(\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta}) + ((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta}))^*] \\ &\quad - \frac{1}{2} [(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) + (\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)))^*] \\ &= \varphi(|x|^2) - |\varphi(z^*x)|^2. \end{aligned}$$

□

Using [7, corollary 3.8], we can easily conclude the following theorem.

**Theorem 3.6.** *Suppose that  $A$  and  $B$  are unital  $C^*$ -algebras. Assume that  $\varphi : A \rightarrow B$  is a unital positive linear map. If for  $x, y, z \in A$  with  $|z| = 1$  and for  $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ ,*

$$\left\{ \begin{array}{l} \operatorname{Re}(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) \geq 0, \\ \quad \& \\ \operatorname{Re}(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))) \geq 0, \end{array} \right. \quad (3.3)$$

then

$$|\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)| \leq \frac{1}{4}|\alpha - \beta||\lambda - \mu|. \quad (3.4)$$

Now, utilizing the above theorem, we preset a sharper bound for the left side of (3.4), which is different the other ones in the literature.

**Theorem 3.7.** *Suppose that  $A$  and  $B$  are unital  $C^*$ -algebras and  $B$  is commutative. Assume that  $\varphi : A \rightarrow B$  is a unital positive linear map. If for  $x, y, z \in A$  with  $|z| = 1$  and for  $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ ,*

$$\left\{ \begin{array}{l} \operatorname{Re}(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) \geq 0 \\ \quad \& \\ \operatorname{Re}(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))) \geq 0, \end{array} \right. \quad (3.5)$$

then

$$\begin{aligned} & |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)| \\ & \leq \frac{1}{4}|\beta - \alpha||\lambda - \mu| \\ & \quad - [Re(\varphi((\alpha z - x)(x^* - \bar{\beta}z^*)))]^{\frac{1}{2}} [Re(\varphi((\lambda z - y)(y^* - \bar{\mu}z^*)))]^{\frac{1}{2}}. \end{aligned}$$

*Proof.* Assume that  $x, y, z \in A$  with  $|z| = 1$  and assume that  $\alpha, \beta, \lambda, \mu \in \mathbb{C}$  such that the conditions (3.5) are satisfied. Using [7, Lemma 3.1], we can derive

$$|\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2 \leq [\varphi(|x|^2) - |\varphi(z^*x)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2]. \quad (3.6)$$

Also it follows from Lemma 3.5 that

$$\left\{ \begin{array}{l} [\varphi(|x|^2) - |\varphi(z^*x)|^2] = \\ Re((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})) - Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))), \\ \quad \& \\ [\varphi(|y|^2) - |\varphi(z^*y)|^2] = \\ Re((\lambda - \varphi(z^*y))(\varphi(y^*z) - \bar{\mu})) - Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))). \end{array} \right.$$

Furthermore, using Lemma 3.4, we can conclude that

$$\left\{ \begin{array}{l} Re((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})) \leq \frac{1}{4}|\bar{\alpha} - \bar{\beta}|^2 = \frac{1}{4}|\beta - \alpha|^2 \\ \quad \& \\ Re((\lambda - \varphi(z^*y))(\varphi(y^*z) - \bar{\mu})) \leq \frac{1}{4}|\bar{\lambda} - \bar{\mu}|^2 = \frac{1}{4}|\mu - \lambda|^2. \end{array} \right.$$

Thus, utilizing Lemma 2.1, we have

$$\begin{aligned} & [\varphi(|x|^2) - |\varphi(z^*x)|^2] [\varphi(|y|^2) - |\varphi(z^*y)|^2] \leq \\ & \left[ \frac{1}{4}|\beta - \alpha||\lambda - \mu| - [Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)))]^{\frac{1}{2}} [Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y)))]^{\frac{1}{2}} \right]^2, \end{aligned}$$

so, it follows from (3.6) that

$$\begin{aligned} & |\varphi(xy^*) - \varphi(x^*z)\varphi(z^*y)|^2 \leq \\ & \left[ \frac{1}{4}|\beta - \alpha||\lambda - \mu| - [Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)))]^{\frac{1}{2}} [Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y)))]^{\frac{1}{2}} \right]^2. \end{aligned}$$

Now we prove that

$$\frac{1}{4}|\beta - \alpha||\lambda - \mu| - [Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)))]^{\frac{1}{2}} [Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y)))]^{\frac{1}{2}} \geq 0.$$

Evidently,

$$\frac{1}{4}|\beta - \alpha||\lambda - \mu| - [Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)))]^{\frac{1}{2}} [(Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y)))]^{\frac{1}{2}}$$



is self-adjoint. On the other hand, let

$$\zeta \in \sigma\left(\frac{1}{4}|\beta - \alpha||\lambda - \mu| - [Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))]^{\frac{1}{2}} [Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))]^{\frac{1}{2}})\right),$$

so there exist  $p \in \sigma(|\beta - \alpha|)$ ,  $q \in \sigma(|\lambda - \mu|)$ ,  $m \in \sigma([Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))]^{\frac{1}{2}})$ , and  $n \in \sigma([Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))]^{\frac{1}{2}})$  such that  $\zeta = \frac{1}{4}pq - mn$ . Since

$$Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)) \leq \frac{1}{4}|\beta - \alpha|^2,$$

we can conclude that

$$[Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))]^{\frac{1}{2}} \leq \frac{1}{2}|\beta - \alpha|,$$

so  $\frac{1}{2}p - m \geq 0$ . Similarly, we can prove  $\frac{1}{2}q - n \geq 0$ . Thus  $\zeta \geq 0$ .

Therefore

$$\begin{aligned} & |\varphi(xy^*) - \varphi(x^*z)\varphi(z^*y)| \\ & \leq \frac{1}{4}|\beta - \alpha||\lambda - \mu| \\ & \quad - [Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))]^{\frac{1}{2}} [Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))]^{\frac{1}{2}}. \end{aligned}$$

□

**Corollary 3.8.** *Suppose that  $A$  and  $B$  are unital  $C^*$ -algebras,  $B$  is commutative, and  $\varphi : A \rightarrow B$  is a unital positive linear map. If for  $x, y \in A$  and  $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ ,*

$$Re(\varphi((x - \beta)(\bar{\alpha} - x^*))) \geq 0 \quad \& \quad Re(\varphi((y^* - \bar{\mu})(\lambda - y))) \geq 0,$$

then

$$\begin{aligned} & |\varphi(xy) - \varphi(x)\varphi(y)| \\ & \leq \frac{1}{4}|\beta - \alpha||\mu - \lambda| \\ & \quad - [Re(\varphi((x - \beta)(\bar{\alpha} - x^*))]^{\frac{1}{2}} [Re(\varphi((y^* - \bar{\mu})(\lambda - y))]^{\frac{1}{2}}. \end{aligned}$$

*Proof.* Letting  $z = 1_A$  in Theorem 3.7, we can conclude the desired results. □

#### 4. GRÜSS INEQUALITY IN $L^\infty([a, b])$ .

Now, using the obtained results in pervious section, we get refined bounded for Gruss inequality in  $L^\infty([a, b])$ .

*Remark 4.1.* Let  $A = L^\infty([a, b])$  which is the usual  $C^*$ -algebra of essentially bounded functionals defined on  $[a, b]$ . Assume that  $\varphi : A \rightarrow \mathbb{C}$  is defined by  $\varphi(h) = \frac{1}{b-a} \int_a^b h(x)dx$ . Let  $m, n, M$  and  $N$  be positive real

numbers and  $\alpha = \frac{m+M}{2}$  and  $\beta = \frac{n+N}{2}$ . Suppose that  $f, g \in L^\infty([a, b])$  which satisfy in the following conditions

$$\begin{cases} m \leq f(x) \leq M, \\ n \leq g(x) \leq N, \end{cases}$$

for all  $x \in [a, b]$ . It is easily seen that  $\varphi$  is positive unital linear map. Using Theorem 3.3, we can derive

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b (fg)(x)dx - \frac{1}{(b-a)2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \\ & \leq \left[ \frac{1}{b-a} \int_a^b \left( \left| \bar{f} - \frac{m+M}{2} \right|^2 \right)(x)dx \right]^{\frac{1}{2}} \left[ \frac{1}{b-a} \int_a^b \left( \left| g - \frac{n+N}{2} \right|^2 \right)(x)dx \right]^{\frac{1}{2}} \\ & \quad - \left| \frac{1}{b-a} \int_a^b \left( \bar{f} - \frac{m+M}{2} \right)(x)dx \right| \left| \frac{1}{b-a} \int_a^b \left( g - \frac{n+N}{2} \right)(x)dx \right|. \end{aligned}$$

Therefore

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b (fg)(x)dx - \frac{1}{(b-a)2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \\ & \leq \frac{1}{4}(M-m)(N-n) \\ & \quad - \left| \frac{1}{b-a} \int_a^b \left( f - \frac{m+M}{2} \right)(x)dx \right| \left| \frac{1}{b-a} \int_a^b \left( g - \frac{n+N}{2} \right)(x)dx \right|. \end{aligned} \tag{4.1}$$

It is readily seen that

$$Re(\varphi((\bar{f} - \bar{m})(M - f))) = \frac{1}{b-a} \int_a^b (f - m)(M - f)(x)dx \geq 0$$

and

$$Re(\varphi((\bar{g} - \bar{n})(N - g))) = \frac{1}{b-a} \int_a^b (g - n)(N - g)(x)dx \geq 0,$$

so, it follows from Corollary 3.8 that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b (fg)(x)dx - \frac{1}{(b-a)2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \\ & \leq \frac{1}{4}(M-m)(N-n) \\ & \quad - \frac{1}{b-a} \left[ \int_a^b (f - m)(M - f)(x)dx \right]^{\frac{1}{2}} \left[ \int_a^b (g - n)(N - g)(x)dx \right]^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

Now, we give two inequalities which are sharper than the Grüss inequality obtained by Mercer [16, Theorem 1.1].

*Remark 4.2.* Assume that  $f, g \in L^\infty([a, b])$  which satisfy the conditions  $0 \leq f(x), g(x) \leq 1$  for all  $x \in [a, b]$  and  $S(x) = \max\{f(x), g(x)\}$  and  $T(x) = \min\{f(x), g(x)\}$ . Suppose that  $P = \{x : f(x) \geq g(x)\}$  and  $Q = \{x : f(x) < g(x)\}$ . It is easy to see that  $\int_a^b (fg)(x)dx = \int_a^b (ST)(x)dx$ . Also

$$\begin{aligned} & \int_a^b f(x)dx \int_a^b g(x)dx - \int_a^b S(x)dx \int_a^b T(x)dx \\ &= \left[ \int_P f(x)dx + \int_Q f(x)dx \right] \left[ \int_P g(x)dx + \int_Q g(x)dx \right] \\ & \quad - \left[ \int_P S(x)dx + \int_Q S(x)dx \right] \left[ \int_P T(x)dx + \int_Q T(x)dx \right] \\ &= \int_P (f - g)(x)dx \int_Q (g - f)(x)dx, \end{aligned}$$

so

$$\begin{aligned} & \frac{1}{b-a} \int_a^b (fg)(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \\ & \quad + \frac{1}{(b-a)^2} \int_P (f - g)(x)dx \int_Q (g - f)(x)dx \quad (4.3) \\ &= \frac{1}{b-a} \int_a^b (ST)(x)dx - \frac{1}{(b-a)^2} \int_a^b S(x)dx \int_a^b T(x)dx. \end{aligned}$$

Using inequality (4.1), we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b (ST)(x)dx - \frac{1}{(b-a)^2} \int_a^b S(x)dx \int_a^b T(x)dx \\ & \leq \frac{1}{4} - \frac{1}{(b-a)^2} \left| \int_a^b (S - \frac{1}{2})(x)dx \right| \left| \int_a^b (T - \frac{1}{2})(x)dx \right| \quad (4.4) \end{aligned}$$

and also utilizing inequality (4.2), we get

$$\begin{aligned} & \frac{1}{b-a} \int_a^b (ST)(x)dx - \frac{1}{(b-a)^2} \int_a^b S(x)dx \int_a^b T(x)dx \\ & \leq \frac{1}{4} - \frac{1}{b-a} \left[ \int_a^b (S)(1 - S)(x)dx \right]^{\frac{1}{2}} \left[ \int_a^b (T)(1 - T)(x)dx \right]^{\frac{1}{2}}. \quad (4.5) \end{aligned}$$

Therefore, using the inequalities (4.3) and (4.4), we can conclude

$$\begin{aligned} & \frac{1}{b-a} \int_a^b fg(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \\ & \leq \frac{1}{4} - \frac{1}{(b-a)^2} \left[ \int_P (f-g)(x)dx \right] \left[ \int_Q (g-f)(x)dx \right] \\ & \quad - \frac{1}{(b-a)^2} \left| \int_a^b (S - \frac{1}{2})(x)dx \right| \left| \int_a^b (T - \frac{1}{2})(x)dx \right|, \end{aligned}$$

and also utilizing the inequalities (4.3) and (4.5), we can deduce

$$\begin{aligned} & \frac{1}{b-a} \int_a^b fg(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \\ & \leq \frac{1}{4} - \frac{1}{(b-a)^2} \left[ \int_P (f-g)(x)dx \right] \left[ \int_Q (g-f)(x)dx \right] \\ & \quad - \frac{1}{b-a} \left[ \int_a^b (S)(1-S)(x)dx \right]^{\frac{1}{2}} \left[ \int_a^b (T)(1-T)(x)dx \right]^{\frac{1}{2}}. \end{aligned}$$

#### REFERENCES

- [1] E. Akin, S. Ashyüce, A. F. Güvenilir and B. Kaymakçalan, Discrete Grüss type inequality on fractional calculus, *J. Inequa. and Appl.***174**(1), (2015), 1-7.
- [2] R. Bhatia, C. Davis, A better bound on the variance, *Am. Math. Monthly.***107** (2000), 353-357.
- [3] R. Bhatia, R. Sharama, Some inequalities for positive linear maps, *Linear Algebra Appl.* **436** (2012), 1562-1571.
- [4] B. Blackadar, Theory of  $C^*$ -algebras and Von Neumann algebras, Springer-Verlag, 2006.
- [5] F.F. Bonsall, J. Duncan, Complete Normed algebras, Berlin New york. Springer Verlag, 1973.
- [6] M.-D. Choi, Some assorted inequalities for positive linear maps on  $C^*$ -Algebras, *J. Operator Theory.* **4** (1980), 271-285.
- [7] A. Dadkhah, M. S. Moslehian, Grüss type inequalities for positive linear maps on  $C^*$ -algebras, *Linear Multilinear Algebra.* **65** (7) (2017), 1386-1401.
- [8] S. S. Dragomir, S. Wang, An inequality of Ostrowski Grüss type and its applications to the estimation of error bounds for some means and for some numerical quadrature rules, *Comput. Math. Appl.* **33** (1997), 15-20.
- [9] S. S. Dragomir, Some Grüss type inequality in inner product spaces, *J. Inequal. Pure Appl. Math.* **4**(2003), 1-10.
- [10] A. M. Fink, A treatise on Grüss inequality, In "Analytic and Geometric Inequalities and Applications" (T. M. Rassias and H. M. Srivastava, eds). 93-113, Kluwer Academic Publishers, Dordrecht, 1999.
- [11] A. G. Ghazanfari, S. S. Dragomir, Bessel and Grüss type inequalities in inner product modules over Banach  $*$ -algebra, *Linear Algebra Appl.* **434** (2011), 16 pages.

- [12] A. G. Ghazanfari, Grüss type inequality for vector-valued functions in Hilbert  $C^*$ -modules, *J. Inequal. Appl.* **16** (2014), 10 pages.
- [13] G. Grüss, Über das Maximum des absoluten Betrages von  $\frac{1}{b-a} \int_a^b fg(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx$ , *Math. Z.* **39** (1935), 215-226.
- [14] D. Ilišević, S. Varošanec, Grüss type inequalities in inner product modules, *Amer. Math. Soc.* **133** (2005) 3271-3280.
- [15] R. V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. Math.* **56** (1952), 494-503.
- [16] A. McD. Mercer, An improvement of the Grüss inequality, *J. Inequal. Pure Appl. Math.* **6** (2005), 1-4.
- [17] N. Minculete, L. Ciurdariu, A generalized form of Grüss type inequality and other integral inequalities, *J. Inequal. and Appl.* **119** (1) (2014), 119-214.
- [18] D. S. Mitrinovic, J. E. Pecaric, A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic, Dordrecht, 1993.
- [19] M. S. Moslehian, R. Rajić, A Grüss inequality for  $n$ -positive linear maps, *Linear Algebra Appl.* **433** (2010), 1555-1560.
- [20] J. Murphy,  $C^*$ -Algebras and Operator Theory, Academic Press, 1990.
- [21] V. Paulsen, Completely bounded maps and Operator algebras, Cambridge University Press, 2002.
- [22] I. Perić, R. Rajić, Grüss inequality for completely bounded maps, *Linear Algebra Appl.* **390** (2004), 287-292.
- [23] P. F. Renaud, A Matrix formulation of Grüss inequality, *Linear Algebra Appl.* **335** (2001), 95-100.