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β-Ideal topological groups

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Abstract. In this paper, we introduce and study a class of topologized groups called β-$\mathcal{I}$-topological groups.

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1. Introduction

If $(G, \star)$ is a group, and τ is a topology on G, then we say that $(G, \star, \tau)$ is a topologized group. Given a topologized group G, a question arises about interactions and relations between algebraic and topological structures: which topological properties are satisfied by the multiplication mapping $m : G \times G \to G$, $(x, y) \to x \star y$, and the inverse mapping $i : G \to G$, $x \to x^{-1}$. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [3] and Vaidyanathaswamy, [4]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of

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subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the set of all subsets of X, is called the local function [4] of A with respect to τ and $\mathcal{I}$ and is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^\star(\cdot)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\tau, \mathcal{I})$. When there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. In this paper, we introduce and study a new class of topologized groups called β-$\mathcal{I}$-topological groups.

2. Preliminaries

Throughout this paper $(G, \star, \tau)$, or simply G, will denote a group $(G, \star)$ endowed with the topologies τ on G. The identity element of G is denoted by e, or e_G when it is necessary, the operation $\star : G \times G \to G$, $(x, y) \to x \star y$, is called the multiplication mapping and sometimes denoted by m, and the inverse mapping $i : G \to G$, $x \to x^{-1}$ is denoted by i. For a subset A of a topological space (X, τ), $cl(A)$ and $Int(A)$ denote the closure of A and the interior of A in (X, τ), respectively. A topological space is extremely disconnected if closure of an open set is open. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-open [2] if $S \subset cl(Int(cl^*(S)))$. The complement of a β-$\mathcal{I}$-open set is called β-$\mathcal{I}$-closed [2]. The intersection of all β-$\mathcal{I}$-closed sets containing S is called the β-$\mathcal{I}$-closure of S and is denoted by ${}_{\beta\mathcal{I}}cl(S)$. The β-$\mathcal{I}$-interior of S is defined by the union of all β-$\mathcal{I}$-open sets contained in S and is denoted by ${}_{\beta\mathcal{I}}Int(S)$. The family of all β-$\mathcal{I}$-open (resp. β-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\beta\mathcal{I}O(X)$ (resp. $\beta\mathcal{I}C(X)$). The family of all β-$\mathcal{I}$-open (resp. β-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\beta\mathcal{I}O(X, x)$ (resp. $\beta\mathcal{I}C(X, x)$).

Definition 2.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The family Ω of β-$\mathcal{I}$-open sets is called a β-$\mathcal{I}$-base if and only if for each β-$\mathcal{I}$-open set is a union of members of Ω.

Definition 2.2. A subset $M(x)$ of an ideal topological space $(X, \tau, \mathcal{I})$ is called a β-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists a β-$\mathcal{I}$-open set S such that $x \in S \subset M(x)$.

Definition 2.3. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is said to be:

(1) β-$\mathcal{I}$continuous if $f^{-1}(V) \in \beta\mathcal{I}O(X)$ for every $V \in \sigma$.
(2) β-$\mathcal{I}$-irresolute if $f^{-1}(V) \in \beta\mathcal{I}O(X)$ for every $V \in \beta\mathcal{I}O(Y)$.
(3) β-$\mathcal{I}$-open if $f(U) \in \beta\mathcal{I}O(Y)$ for every $U \in \beta\mathcal{I}O(X)$.

(4) β-$\mathcal{I}$-homeomorphism if f is bijective, β-$\mathcal{I}$-irresolute and β-$\mathcal{I}$-open.

Lemma 2.4. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is a β-$\mathcal{I}$-homeomorphism, then:*

(1) ${}_{\beta\mathcal{I}}cl(f(A)) = f({}_{\beta\mathcal{I}}cl(A))$ *for all $A \subset X$;*

(2) ${}_{\beta\mathcal{I}}Int(f(A)) = f({}_{\beta\mathcal{I}}Int(A))$ *for all $A \subset X$.*

3. On β-$\mathcal{I}$-topological groups

Definition 3.1. A topologized group $(G, *, \tau, \mathcal{I})$ is called an ideal topological group if for each $x, y \in G$ and each neighbourhood W of $x * y^{-1}$ in G there exist $\mathcal{I}$ - open neighbourhoods U of x and V of y such that $U * V^{-1} \subseteq W$.

Definition 3.2. $(G, \circ, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-topological group if $(G, \circ)$ is a group, $(G, \tau, \mathcal{I})$ is an ideal topological space and left translation $L_x : G \to G$ for all $x \in G$ and right translation $R_x : G \to G$ for all $x \in G$ are β-$\mathcal{I}$-continuous and the mapping of inversion $i : G \to G$ defined by $i(x) = x^{-1}$ is β-$\mathcal{I}$-continuous on G.

Example 3.3. Any group with the discrete topology, or indiscrete topology, is a topological group, hence β-$\mathcal{I}$-topological group.

Example 3.4. The set $G = \{-1, 1\}$ is a group under usual multiplication. Let topology and ideal on G be $\tau = \{\emptyset, G, \{1\}\}$, $\mathcal{I} = \mathcal{P}(G)$. Then open sets of G are the only β - $\mathcal{I}$ - open sets and so $(G, \circ, \tau, \mathcal{I})$ is not a β-$\mathcal{I}$-topological group.

Example 3.5. The set $\mathbb{Z}_n = \{0, 1, 2, \dots n-1\}$ is a group under addition modulo n. Let topology and ideal on $\mathbb{Z}_n$ be $\tau = \emptyset, \{0\}, \mathbb{Z}_n$, $\mathcal{I} = \{\mathcal{P}(\mathbb{Z}_n) \setminus \mathcal{M}\}$ where $\mathcal{M} = \{M \subseteq \mathbb{Z}_n : 0 \in M\}$. Here $M^* = \{0\} \forall M \in \mathcal{M}$ and so $\{0\}$ is β - $\mathcal{I}$ - open but $\{x \in \mathbb{Z}_n : x \neq 0\}$ is not β - $\mathcal{I}$ - open. Thus, $(\mathbb{Z}_n, \oplus, \tau, \mathcal{I})$ is not a β-$\mathcal{I}$-topological group.

In addition, every ideal topological group is β-$\mathcal{I}$-topological group but converse need not be true by the following example.

Example 3.6. Consider the addition modulo group $\mathbb{Z}_2$ with discrete topology and an ideal $\mathcal{I} = \{\emptyset, 0\}$. Then $\{0\}^* = \emptyset, \{1\}^* = \{0, 1\}^* = \{0, 1\}$. Thus the collection of $\mathcal{I}$ - open sets and β - $\mathcal{I}$ - open sets are $\mathcal{P}(\mathbb{Z}_2) \setminus 0$ and $\mathcal{P}(\mathbb{Z}_2)$. Hence $(\mathbb{Z}_2, \tau, \mathcal{I})$ is β - Ideal topological group but not an Ideal topological group.

Theorem 3.7. *Let $(G, \circ, \tau, \mathcal{I})$ be a β-$\mathcal{I}$-topological group and β_e be the base at identity element e of G. Then:*

(1) *for every $U \in \beta_e$, there is an element $V \in \beta\mathcal{I}O(G, e)$ such that $V^{-1} \subset U$.*

(2) *for every $U \in \beta_e$, there is an element $V \circ x \subset U$, and $x \circ V \subset U$ for each $x \in U$.*

Proof. (1) Since $(G, \circ, \tau, \mathcal{I})$ is a β-$\mathcal{I}$-topological group, for every $U \in \beta_e$ there exists $V \in \beta\mathcal{I}O(G, e)$ such that $i(V) = V^{-1} \in U$ because the inverse mapping $i : G \to G$ is β-$\mathcal{I}$-continuous.
(2) Since $(G, \circ, \tau, \mathcal{I})$ is a β-$\mathcal{I}$-topological group, for each $U \in \tau$ containing x, there exists $V \in \beta\mathcal{I}O(G, e)$ such that $R_x(V) = V \circ x \subset U$. □

Lemma 3.8. *Let A be a subset of a β-$\mathcal{I}$-topological group $(G, \circ, \tau, \mathcal{I})$. Then ${}_{\beta\mathcal{I}}cl(A^{-1}) \subset cl(A^{-1})$.*

Proof. Let $x \in ({}_{\beta\mathcal{I}}cl(A))^{-1}$ and $U \in \tau$ containing x. Then, U^{-1} is a β-$\mathcal{I}$-open neighbourhood of x^{-1}. Since $x^{-1} \in_{\beta\mathcal{I}} cl(A)$, $U^{-1} \cap A \neq \emptyset$. This implies that $U \cap A^{-1} \neq \emptyset$. That is, $x \in cl(A^{-1})$ and so $({}_{\beta\mathcal{I}}cl(A))^{-1} \subset cl(A^{-1})$. □

Theorem 3.9. *Let $(G, \circ, \tau, \mathcal{I})$ be a β-$\mathcal{I}$-topological group. If U is β-$\mathcal{I}$-open set in $(G, \circ, \tau, \mathcal{I})$, then U^{-1} is β-$\mathcal{I}$-open in $(G, \circ, \tau^{-1}, \mathcal{I}^{-1})$.*

Proof. The proof follows from the respective definitions. □

We denote that $\mathcal{I}^{-1} = \{I^{-1} : I \in \mathcal{I}\}$. It is easily verify that $\mathcal{I}^{-1}$ is an ideal on X.

Theorem 3.10. *If $(G, \circ, \tau, \mathcal{I})$ is a β-$\mathcal{I}$-topological group, then $(G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ is also a β-$\mathcal{I}$-topological group.*

Proof. Since $(G, \circ)$ is a group and $(G, \tau, \mathcal{I})$ is an ideal topological space, $(G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ is an ideal topological group. We need to prove that: $i : (G, \circ, \tau^{-1}, \mathcal{I}^{-1}) \to (G, \circ, \tau^{-1}, \mathcal{I}^{-1})$, and $L_x : (G, \circ, \tau^{-1}, \mathcal{I}^{-1}) \to (G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ and $R_x : (G, \tau^{-1}) \to (G, \tau^{-1})$ are β-$\mathcal{I}$-continuous mappings. First, we show that L_x is β-$\mathcal{I}$-continuous. For this, let $V \in \tau^{-1}$. Then $V^{-1} = U \in \tau$. Since $(G, \circ, \tau, \mathcal{I})$ is β-$\mathcal{I}$-topological group, the left (right) translation is β-$\mathcal{I}$-continuous. Hence $L_x^{\leftarrow}(U) \in \beta\mathcal{I}O(G, \tau, \mathcal{I})$, that is, $(U \circ x^{-1})^{-1} \in \beta\mathcal{I}O(G, \tau^{-1}, \mathcal{I}^{-1})$, that is, $U \circ x^{-1} = V^{-1} \circ x^{-1} = (x \circ V)^{-1} = L_x^{\leftarrow}(V) \in \beta\mathcal{I}O(G, \tau^{-1}, \mathcal{I}^{-1})$. This proves that $L_x : (G, \circ, \tau^{-1}, \mathcal{I}^{-1}) \to (G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ is β-$\mathcal{I}$-continuous for every $x \in G$. Similarly, we can prove that right translation $R_x : (G, \circ, \tau^{-1}, \mathcal{I}^{-1}) \to (G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ is β-$\mathcal{I}$-continuous. Trivially $i : (G, \circ, \tau^{-1}, \mathcal{I}^{-1}) \to (G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ is continuous and hence β-$\mathcal{I}$-continuous. Hence $(G, \circ, \tau^{-1}, \mathcal{I}^{-1})$ is also a β-$\mathcal{I}$-topological group. □

Theorem 3.11. *If H is a discrete subgroup of a β-$\mathcal{I}$-topological group $(G, \circ, \tau^{-1}, \mathcal{I}^{-1})$, then ${}_{\beta\mathcal{I}}cl(H)$ is a subgroup of G.*

Proof. Let $x, y \in_{\beta\mathcal{I}} cl(H)$. If U and V are respective τ^{-1}-open neighbourhoods of x and y, then $L_{x^{-1}}(U) = x^{-1} \circ U$ and $L_{y^{-1}}(U) = y^{-1} \circ U$

are β-$\mathcal{I}$-open neighbourhoods of e. Since H is a discrete subgroup of a β-$\mathcal{I}$-topological group G, $x^{-1} \circ U \cap H \neq \emptyset$ and $y^{-1} \circ U \cap H \neq \emptyset$. Therefore, $(x \circ y^{-1} \circ x^{-1} \circ U \cap x \circ y^{-1} \circ H) \cup (x \circ y^{-1} \circ y^{-1} \circ V \cap x \circ y^{-1} \circ H) \neq \emptyset$. That is, $W \cap x^{-1} \circ y^{-1} \circ H \neq \emptyset$, where $W = x \circ y^{-1} \circ x^{-1} \circ U \cup x \circ y^{-1} \circ y^{-1} \circ V$ is a β-$\mathcal{I}$-open neighbourhood of $x \circ y^{-1}$. Thus, for each $x, y \in_{\beta\mathcal{I}} cl(H)$ implies that $x \circ y^{-1} \in_{\beta\mathcal{I}} cl(H)$. Hence ${}_{\beta\mathcal{I}}cl(H)$ is a subgroup of G. □

Corollary 3.12. *If H is a discrete subgroup of a β-$\mathcal{I}$-topological group $(G, \circ, \tau, \mathcal{I})$, then $cl(H)$ is a subgroup of G.*

Theorem 3.13. *Let $(G, \circ, \tau, \mathcal{I})$ be a β-$\mathcal{I}$-topological group. If A is open in G, then $A \circ B$ and $B \circ A$ are β-$\mathcal{I}$-open in $(G, \circ, \tau, \mathcal{I})$ for any subset B of G.*

Proof. Let $x \in B$ and $z \in A \circ x$ we show that z is β-$\mathcal{I}$-interior point of $A \circ x$. Let $z = y \circ x$ for some $y \in A = A \circ x \circ x^{-1}$. This implies that $y = z \circ x^{-1}$. Now $R_{x^{-1}} : G \to G$ is β-$\mathcal{I}$-continuous, that is, for every open set containing $R_{x^{-1}}(z) = z \circ x^{-1} = y$, there exists a β-$\mathcal{I}$-open set M_z containing z such that $R_{x^{-1}}(M_z) \subset A$. Now we have that $M_z \circ x^{-1} \subset A$ or $M_z \subset A \circ x$. This implies z is β-$\mathcal{I}$-interior point of $A \circ x$. Thus $A \circ x$ is β-$\mathcal{I}$-open. It follows that $A \circ B = \underset{x \in B}{\cup} A \circ x$ is β-$\mathcal{I}$-open in $(G, \circ, \tau, \mathcal{I})$. Similarly we can prove that for every open set A of G and arbitrary subset B of G, $B \circ A$ is β-$\mathcal{I}$-open in a β-$\mathcal{I}$-topological group $(G, \circ, \tau, \mathcal{I})$. □

Proposition 3.14. *Let $(G, \circ, \tau, \mathcal{I})$ be a β-$\mathcal{I}$-topological group. If C is closed in G, then for any $a \in G$, $a \circ C$ and $C \circ a$ are β - $\mathcal{I}$ - closed.*

Proof. Let $x \in {}_{\beta\mathcal{I}}cl(a \circ C)$, $b = a^{-1} \circ x$ and D be an open neighbourhood of b. Then by Definition 3.1, there exist an $\beta - \mathcal{I}$ - open set F of x in G such that $a^{-1} \circ F \subset D$. Since $x \in {}_{\beta\mathcal{I}}cl(a \circ C)$ we have $F \cap \mathsf{a} \circ C \neq \emptyset$. Let $c \in F \cap a \circ C$, then $a^{-1} \circ c \in C \cap a^{-1} \circ F \subseteq C \cap D$ which implies $C \cap D \neq \emptyset$. Thus b is a limit point of C. Since C is closed we have $\mathsf{b} \in C$. Now $x = a \circ b$ and so $x \in a \circ C$. By the above argument, ${}_{\beta\mathcal{I}}cl(a \circ C) \subseteq a \circ C$ and since $a \circ C \subseteq_{\beta\mathcal{I}} cl(a \circ C)$ is trivial we have $a \circ C = {}_{\beta\mathcal{I}}cl(a \circ C)$. Hence $a \circ C$ is β - $\mathcal{I}$ - closed. Proof of $C \circ a$ is similar. □

Theorem 3.15. *Let $(G, \circ, \tau, \mathcal{I})$ be a β-$\mathcal{I}$-topological group. Then each left (right) translation $L_x : G \to G$, $R_x : G \to G$ is a β-$\mathcal{I}$-homeomorphism.*

Proof. Since $(G, \circ, \tau, \mathcal{I})$ is β-$\mathcal{I}$-topological group, $L_x : G \to G$ is β-$\mathcal{I}$-continuous. So it is enough to show that $L_x : G \to G$ is β-$\mathcal{I}$-open. Let V be an open set in G. Then by Theorem 3.13, $L_g(V) = g \circ V \in \beta\mathcal{I}O(G)$. Hence $L_x : G \to G$ is a β-$\mathcal{I}$-open mapping. □

Theorem 3.16. *Suppose that a subgroup H of a β-$\mathcal{I}$-topological group $(G, \circ, \tau, \mathcal{I})$ contains a nonempty open subset of G. Then H is β-$\mathcal{I}$-open in G.*

Proof. By Theorem 3.15 for every $g \in H$, $R_g : G \to G$ is β-$\mathcal{I}$-homeomorphism. Let $U \in \tau$ and $U \subset H$, then for every $g \in H$, the set $R_g(U) = U \circ g$ is β-$\mathcal{I}$-open in $(G, \circ, \tau, \mathcal{I})$. Now $H = \cup\{U \circ g : g \in H\}$ is β-$\mathcal{I}$-open in G being the union of β-$\mathcal{I}$-open sets of G. □

Definition 3.17. A topological space $(G, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-homogeneous if for all $x, y \in G$, there is a β-$\mathcal{I}$-homeomorphism f of the space G onto itself such that $f(x) = y$.

Theorem 3.18. *If $(G, \circ, \tau, \mathcal{I})$ is a β-$\mathcal{I}$-topological group, then every open subgroup of G is also β-$\mathcal{I}$-closed.*

Proof. Since $(G, \circ, \tau, \mathcal{I})$ is a β-$\mathcal{I}$-topological group and H is an open subgroup of G, then any left or right translation $x \circ H$ or $H \circ x$ is β-$\mathcal{I}$-open for each $x \in G$. So the set $Y = \{x \circ H : x \in G\}$ of all left cosets of H in G forms a partition of G. Thus Y is a β-$\mathcal{I}$-open covering of G by disjoint β-$\mathcal{I}$-open sets of G. This gives $G \backslash H$ is union of β-$\mathcal{I}$-open sets and hence β-$\mathcal{I}$-open. This proves that H is β-$\mathcal{I}$-closed. □

Corollary 3.19. *Every β-$\mathcal{I}$-topological group is a β-$\mathcal{I}$-homogeneous space.*

Proof. Let us take elements x and y in $(G, \circ, \tau, \mathcal{I})$ and put $z = x^{-1} \circ y$. Since $R_x : G \to G$ is a β-$\mathcal{I}$-homeomorphism of $(G, \circ, \tau, \mathcal{I})$ and $R_z(x) = x \circ z = x \circ (x^{-1} \circ y) = e \circ y = y$, $(G, \circ, \tau, \mathcal{I})$ is β-$\mathcal{I}$-homogeneous space. □

Lemma 3.20. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is β-$\mathcal{I}$-continuous and H is an open subset of X, then $f_H : (H, \tau|H, \mathcal{I}|H) \to (Y, \sigma|H)$ is β-$\mathcal{I}$-continuous.*

Theorem 3.21. *Every open subgroup H of a β-$\mathcal{I}$-topological group $(G, \circ, \tau, \mathcal{I})$ is also a β-$\mathcal{I}$-topological group (called β-$\mathcal{I}$-topological subgroup) of G.*

Proof. Let $(G, \circ, \tau, \mathcal{I})$ be a β-$\mathcal{I}$-topological group and H an open subgroup of G. We need to prove that $(H, \circ, \tau|H, \mathcal{I}|H)$ is a β-$\mathcal{I}$-topological group. For this, we show that $i : H \to H$, $L_x : H \to H$ and $R_x : H \to H$ are β-$\mathcal{I}$-continuous with respect to the relative topology. Since H is an open subgroup of G, by Lemma 3.20, $i_H : (H, \tau|H, \mathcal{I}|H) \to (Y, \sigma|H)$, $L_H : (H, \tau|H, \mathcal{I}|H) \to (Y, \sigma|H)$ and $R_H : (H, \tau|H, \mathcal{I}|H) \to (Y, \sigma|H)$ are β-$\mathcal{I}$-continuous. This proves that $(H, \circ, \tau|H, \mathcal{I}|H)$ is a β-$\mathcal{I}$-topological group. □

Theorem 3.22. *Let $f : (G, \circ, \tau_G, \mathcal{I}_G) \to (H, \circ, \tau_H, \mathcal{I}_H)$ be a homomorphism of β-$\mathcal{I}$-topological groups. If f is β-$\mathcal{I}$-irresolute at the neutral (identity) element e_G, then f is β-$\mathcal{I}$-continuous on G.*

Proof. Let $x \in G$ be an arbitrary element. Suppose that W is an open neighbourhood of $y = f(x) \in H$. Since the left translation in H is a β-$\mathcal{I}$-continuous mapping, there is a β-$\mathcal{I}$-open neighbourhood V of the neutral element e_H of H such that $L_y(V) = y \circ V \subset W$. Since f is β-$\mathcal{I}$-irresolute at e_G, therefore, $f(U) \subset V$ for some β-$\mathcal{I}$-open neighbourhood U of e_G, in G. Since $f(U) \subset V$, now $y \circ f(U) \subset y \circ V \subset W$. This implies that $(x \circ U) \subset W$. By the fact that $(G, \circ, \tau_G, \mathcal{I}_G)$ is a β-$\mathcal{I}$-topological group, thus $x \circ U$ is β-$\mathcal{I}$-open in G. This proves that f is β-$\mathcal{I}$-continuous at x. Since x was the arbitrary element of G, therefore f is β-$\mathcal{I}$-continuous on G. □

We recall that, an Ideal topological space X is β - I - connected if X cannot be written as union of two disjoint non - empty β - I - open sets in X.

Theorem 3.23. *Let $(G, \circ, \tau, \mathcal{I})$ be a extremely disconnected β - $\mathcal{I}$deal topological group and M be a subgroup of G. If M and G/M are β - $\mathcal{I}$ - connected, then G is β - $\mathcal{I}$ - connected.*

Proof. Suppose G is not β - $\mathcal{I}$ - connected. Let us assume that $G = E \cup F$ where E and F are disjoint non - empty β - $\mathcal{I}$ - open sets. Since M is β - $\mathcal{I}$ - connected, each coset of M is either a subset of E or a subset of F. Thus, the relation

$$\begin{aligned} G/M &= \{a \circ M : a \circ M \subset E\} \cup \{a \circ M : a \circ M \subset F\} \\ &= \{a \circ M : a \in E\} \cup \{a \circ M : a \in F\} \end{aligned}$$

It expresses G/M as the union of disjoint non - empty β - $\mathcal{I}$ - open sets which is a contradiction to β - $\mathcal{I}$ - connectedness of G/M. Thus, G is β - $\mathcal{I}$ - connected. □

Theorem 3.24. *Let a β - $\mathcal{I}$deal topological group $(G, \circ, \tau, \mathcal{I})$ be β - $\mathcal{I}$ - connected and e be its identity element. If S is any β - $\mathcal{I}$ - open neighbourhood of e, then G is generated by S.*

Proof. Let S be a β - $\mathcal{I}$ - open neighbourhood of e. For each $n \in \mathbb{N}$, we denote S^n by the set of elements of the form $s_1.s_2....s_n$ where each $s_i \in S$. Let $T = \cup_{n=1}^{\infty} S^n$. If we prove T is β - $\mathcal{I}$ - open and β - $\mathcal{I}$ - closed, Since G is β - $\mathcal{I}$ - connected, we have $G = T$ and so G is generated by S. Since each S^n is β - $\mathcal{I}$ - open and arbitrary union of β - $\mathcal{I}$ - open sets is β - $\mathcal{I}$ - open, therefore T is β - $\mathcal{I}$ - open. Now we prove that T is β - $\mathcal{I}$ - closed. Let $a \in {}_{\beta\mathcal{I}}cl(T)$. Since $a \circ S^{-1}$ is a β - $\mathcal{I}$ - open neighbourhood of a, it must intersect T. Thus, let $b \in T \cap a \circ S^{-1}$. Since

$b \in a \circ S^{-1}$ then $b = a \circ s^{-1}$ for some $s \in S$. Since $b \in T$ then $b \in S^n$ for some $n \in \mathbb{N}$ which implies $b = s_1 s_2 ... s_n$ with each $s_i \in S$. Now, we have $a = s_1 s_2 ... s_n . s$ and so $a \in S^{n+1} \subseteq T$. Hence T is β - $\mathcal{I}$ - closed. Since G is β - $\mathcal{I}$ - connected and T is β - $\mathcal{I}$ - open and β - $\mathcal{I}$ - closed, we have $T = G$. Thus, G is generated by S. □

Theorem 3.25. *If $(G, \circ, \tau, \mathcal{I})$ is a β - $\mathcal{I}$ - connected, extremely disconnected β - $\mathcal{I}$deal topological group and H, a discrete invariant subgroup of G, then $H \subseteq Z(G)$, where $Z(G)$ denotes the center of G.*

Proof. Suppose $H = \{e\}$, then the result is trivial. Suppose H is non - trivial. Let $h \neq e \in H$. Since H is discrete, we can find an open set D of h in G such that $D \cap H = \{h\}$. Now, by definition of β - $\mathcal{I}$deal topological group, there exists a β - $\mathcal{I}$ - open neighbourhood E of e and a β - $\mathcal{I}$ - open neighbourhood $E \circ h$ of h in G such that $(E \circ h) \circ E^{-1} \subset D$. Let $b \in E$ be arbitrary. Since H is an invariant subgroup of G, we have $b \circ H = H \circ b$ which implies that $b \circ h \in H \circ b$ and so $b \circ h \circ b^{-1} \in H$. It is also clear that $b \circ h \circ b^{-1} \in E \circ h \circ E^{-1} \subset D$. Therefore, $b \circ h \circ b^{-1} \in D \cap H = \{h\}$ which implies $b \circ h \circ b^{-1} = h$. Thus, $b \circ h = h \circ b$ for each $b \in E$. Since the group G is β - $\mathcal{I}$ - connected, E^n with $n \in \mathbb{N}$ covers the group G. Thus, $a \in G$ can be written in the form $a = b_1 . b_2 ... b_n$ where $b_1, b_2, ..., b_n \in E$ and $n \in \mathbb{N}$. Since h commutes with every element of E, we have

$$a.h = b_1.b_2....b_n.h = b_1.b_2....h.b_n = \vdots = b_1.h.b_2....b_n = h.b_1.b_2....b_n = h.a$$

Hence $h \in H$ is in the center of G. Since h is an arbitrary element of G, we proved that the center of G contains H. □

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