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Application of The Sine-Gordon Expansion Method on Nonlinear Various Physical Models

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Abstract. In this paper, by utilizing the Sine-Gordan expansion method, soliton solutions of the higher-order improved Boussinesq equation, Kuramoto-Sivashinsky equation, and seventh-order Sawada-Kotera equation are obtained. Given partial differential equations are reduced to ordinary differential equations, by choosing the compatible wave transformation associated with the structure of the equation. Based on the solution of the Sine-Gordan equation, a polynomial system of equations is obtained according to the principle of homogeneous balancing. The solution of the outgoing system gives the parameters which are included by the solution. Plot3d and Plot2d graphics are given in detail. As a result, many different graphic models are obtained from soliton solutions of equations that play a very important role in mathematical physics and engineering.

Keywords: The Sine-Gordon Epansion Method, Travelling Wave Solution, Nonlinear Equations,Higher Order Boussinesq equation.

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1. Introduction

Nonlinear partial differential equations are encountered in modeling of many problems in physics and engineering. In recent studies, the integrability, the existence and uniqueness of their solutions, obtaining conservation laws, finding numerical and exact solutions of these equations have been discussed. Since there is no universal method for obtaining exact solutions, it has caused many researchers to work in this field. Using the functionality and ease of symbolic computing programs such as maple matlab, many powerful techniques developed such as improved $(G'/G)-$ expansion method [1], $exp(-\Omega(\xi))-$expansion function method [2], the generalized Riccati equation mapping method [3], modified trial equation method [4], homotopy perturbation method [5], Kudryashov method [6]- [9], Jacobi elliptic function method [10], lie symmetry analysis method [11]-[13] and so on. Most methods have been used to find different perspectives and to obtain different solutions. All these exact solution methods are effective and lead to assorted types of solutions such as trigonometric solutions, non-periodic solutions, soliton and solitary solutions. Especially the last two types are very important in nonlinear physical phenomena and engineering.

In this study, we considered the sine-gordan expansion method to find exact solutions of various physical models. Firstly, we examined the higher-order improved Boussinesq equation which were derived formally from 2D water wave problem. Schneider and Wayne introduced a class of Boussinesq equations to model the water wave problem with surface tension [14].

$$-u_{xxxxtt}+u_{xxtt}-u_{tt}+u_{xx}+\mu u_{xxxx}+(u)^2_{xx}=0 \tag{1.1}$$

They have showed that the equation (1.1) really can be expressed by two decoupled Kawahara equations for a degenerate case. In [15], Ak çağıl and Gözükızıl utulized the tanh-coth method and obtained some travelling wave solutions. The authors of [16] have also showed that the finite time blow up solutions by employing the improved convexity method. And in [17], Wang and Xue considered the Cauchy problem for sixth order Boussinesq equation and proved that the existence and uniqueness of the local solution by the contraction mapping theorem.

Another physical model is Kuramoto-Sivashinsky (KS) equation is presented [18] as

$$u_t+auu_x+bu_{xx}+ku_{xxxx}=0. \tag{1.2}$$

in which a, b and k are some arbitrary constants. It appears in many scientific areas such as phase turbulence systems [19], an example of spatio temporal chaos [20], flame front propagation [21] and so on. In

[22], approximate analytical solutions were found with homotopy analysis method and compared numerical solutions. Khater et al. [23], Chebyshev spectral collocation method were performed and obtained approximate numerical solutions.

As the last application, the seventh-order Sawada-Kotera equation was examined [24].

$$u_t+[63u^4+63(2u^2u_{xx}+u(u_x)^2)+21(uu_{xxxx}+(u_{xx})^2+u_xu_{xxx})+u_{xxxxxx}]_x = 0 \quad (1.3)$$

Higher order KdV type equations have many applications in fluid dynamics [25]-[29]. Jafari et al.[30] obtained approximate solutions by applying the Adomian decomposition method and He's variational iteration method for two different physical models of higher order KdV type equations. Wazwaz employed the tanh-coth method and Hirota's direct method and obtained multiple soliton solutions [31]. The authors of [32], performed Bell polynomial approach to investigate integrable properties and constructed Lax pair, Backlund Transformation, Hirota D-operators and infinite conservation laws.

2. Methodology of the Method

Firstly, we will give some preliminaries about the methodology of the Sine-Gordon expansion method [33]- [37]. This method is constructed on the well known Sine-Gordan equation.

$$u_{xx} - u_{tt} = m^2 sin(u) \quad (2.1)$$

in which $u = u(x,t)$ is definition as a arbitrary function and m is a constant. By taking $u(x,t) = U(\xi)$ where it is applied $\xi = kx + lt$ which is called wave transformation, (2.1) equation is reduced in one dimension form.

$$U'' = \frac{m^2}{k^2 - l^2} sin^2(\frac{U}{2}) \quad (2.2)$$

When multiply by U' and integrating once, we get a simpler equation.

$$[(\frac{U}{2})']^2 = \frac{m^2}{k^2 - l^2} sin^2(\frac{U}{2}) + K \quad (2.3)$$

Here K is an integration constant. If we get $K = 0$, $\frac{U}{2} = w(\xi)$ and $\frac{m^2}{k^2 - l^2} = \alpha^2$,we get a simpler equation as follows.

$$w' = \alpha \sin(w) \quad (2.4)$$

Solution of Eq. (2.4) with taking $\alpha = 1$ can be find as below

$$\sin(w) = \sin(w(\xi)) = \frac{2pe^{\xi}}{p^2e^{2\xi} + 1} \mid_{p=1} = sech(\xi) \quad (2.5)$$

$$\cos(w) = \cos(w(\xi)) = \frac{2pe^{\xi} - 1}{p^2e^{2\xi} + 1} \mid_{p=1} = tanh(\xi), \tag{2.6}$$

where $p \neq 0$ is the integration constant.

We suppose that a nonlinear differential equation of the polynomial form

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0. \tag{2.7}$$

Applying the compatible wave transformation, (3.2) can be converted to ODE

$$O(U, U', U'', U'''...) = 0 \tag{2.8}$$

Thence, the expected solution to (3.3) of the form

$$U(\xi) = \sum_{i=1}^{n} tanh^{i-1}(\xi)[B_i sech(\xi) + A_i tanh(\xi)] + A_0 \tag{2.9}$$

Thanks to (2.5) and (3.1) equations, we get equation (3.4) as following

$$U(\omega) = \sum_{i=1}^{n} cos^{i-1}(\omega)[B_i sin(\omega) + A_i cos(\omega)] + A_0 \tag{2.10}$$

Operate the homogeneous balace principle to obtain the value of n. In equation (2.10), necessary derivatives are taken and replaced in equation (3.3). In obtained expression, letting the coefficients of $\sin^p \cos^q$ of equal power to be all zero, we construct an algebraic system. When the system is solved by Maple, the values of unknowns A_i, B_i, k, l are found. Finally, one can easily construct the soliton solutions of Eq. (3.3).

3. Applications

3.1. **Higher Order Boussinesq Equation.** In this section, we start with the sixth-order Boussinesq equation,

$$-u_{xxxxtt} + u_{xxtt} - u_{tt} + u_{xx} + \mu u_{xxxx} + u_{xx}^2 = 0, \tag{3.1}$$

where $u(x,t)$ is an analytic function and μ is arbitrary constant.

If we use the wave transformation $\xi = x - ct$ into Eq.(3.1), it transformed into an ODE

$$-c^2u^{(vi)} + c^2u'''' - c^2u'' + u'' + \mu u'''' + 2(u')^2 + 2uu'' = 0, \tag{3.2}$$

where prime expresses differentiation with respect to ξ. If we balance between the highest order derivative term $u^{(vi)}$ and nonlinear term of the highest degree uu'' in Eq.(3.2), we obtain $n = 4$.

Thus, the exact solution of the related equation has the form,

$$\begin{aligned} u(\xi) &= A_0 + B_1 \sec h(\xi) + B_2 \sec h(\xi) \tanh(\xi) \\ &+ B_3 \sec h(\xi) \tanh^2(\xi) + B_4 \sec h(\xi) \tanh^3(\xi) \\ &+ A_1 \tanh(\xi) + A_2 \tanh^2(\xi) + A_3 \tanh^3(\xi) + A_4 \tanh^4(\xi), \end{aligned} \tag{3.3}$$

where $u(x,t) = U(\xi)$ satisfies Eq.(2.5) and $A_0, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ are undetermined constants.

Firstly, we substitute Eq.(3.3) into Eq.(3.2) with taking the necessary derivatives and employ Eq.(2.5), in the end we equate all coefficients of the functions $[\cos(\xi), \sin(\xi)]$ to zero and we get the following nonlinear algebraic system :

$\cos^{10}(\xi) : -60480 A_4 c^2 + 72 A_4^2 - 72 B_4^2,$

$\cos^9(\xi) : -20160 A_3 c^2 + 112 A_3 A_4 - 112 B_4 B_3,$

$\cos^9(\xi) \sin(\xi) : -60480 B_4 c^2 + 144 A_4 B_4,$

$\cos^8(\xi) : -5040 A_2 c^2 + 195720 A_4 c^2 + 84 A_2 A_4$
$+42 A_3^2 - 128 A_4^2 + \mu 840 A_4 - 84 B_2 B_4 - 42 B_3^2 + 170 B_4^2,$

$\cos^8(\xi) \sin(\xi) : -20160 B_3 c^2 + 112 A_3 B_4 + 112 A_3 B_4,$

$\cos^7(\xi) : -720 A_1 c^2 + 60120 A_3 c^2 + 60 A_1 A_4 + 60 A_2 A_3$
$-196 A_3 A_4 + \mu 360 A_3 - 60 B_1 B_4 - 60 B_2 B_3 + 256 B_3 B_4,$

$\cos^7(\xi) \sin(\xi) : -5040 B_2 c^2 - 168000 B_4 c^2$
$+84 A_2 B_4 + 84 A_3 B_3 + 84 A_4 B_2 - 226 A_4 B_4 + \mu 840 B_4,$

$\cos^6(\xi) : 13560 A_2 c^2 - 232820 A_4 c^2 + 40 A_0 A_4 + 40 A_1 A_3$
$+20 A_2^2 - 144 A_2 A_4 + \mu 120 A_2 - 72 A_3^2 + 56 A_4^2 - \mu 2080 A_4$
$-40 B_1 B_3 - 20 B_2^2 + 184 B_2 B_4 + 92 B_3^2 - 128 B_4^2 + 20 A_4,$

$\cos^6(\xi) \sin(\xi) : -720 B_1 c^2 + 50400 B_3 c^2 + 60 A_1 B_4 + 60 A_2 B_3$
$+60 A_3 B_2 - 170 A_3 B_4 + 60 A_4 B_1 - 170 A_4 B_3 + \mu 360 B_3,$

$\cos^5(\xi) : 1704 A_1 c^2 - 63660 A_3 c^2 + 24 A_0 A_3 + 24 A_1 A_2$
$-100 A_1 A_4 + \mu 24 A_1 - 100 A_2 A_3 + 84 A_3 A_4 - \mu 816 A_3$
$-24 B_1 B_2 + 124 B_1 B_4 + 124 B_2 B_3 - 184 B_3 B_4 + 12 A_3,$

$\cos^5(\xi) \sin(\xi) : 11040 B_2 c^2 - 163160 B_4 c^2 + 40 A_0 B_4$
$+40 A_1 B_3 + 40 A_2 B_2 - 122 A_2 B_4 + 40 A_3 B_1 - 122 A_3 B_3 - 122 A_4 B_2$
$+84 A_4 B_4 + \mu 120 B_2 - \mu 1720 B_4 + 20 B_4,$

$$
\begin{aligned}
&\cos^4(\xi): -12342A_2c^2 + 121280A_4c^2 + 12A_0A_2 - 64A_0A_4 + 6A_1^2 - 64A_1A_3\\
&-32A_2^2 + 60A_2A_4 - \mu 240A_2 + 30A_3^2 + \mu 1696A_4 - 6B_1^2 + 76B_1B_3 + 38B_2^2\\
&-124B_2B_4 - 62B_3^2 + 30B_4^2 + 6A_2 - 32A_4,\\
\\
&\cos^4(\xi)\sin(\xi): 1344B_1c^2 - 41832B_3c^2 + 24A_0B_3 + 24A_1B_2 - 82A_1B_4 + 12B_3\\
&+24A_2B_1 - 82A_2B_3 - 82A_3B_2 + 60A_3B_4 - 82A_4B_1 + 60A_4B_3 + \mu 24B_1 - \mu 648B_3,\\
\\
&\cos^3(\xi): -1274A_1c^2 + 27522A_3c^2 + 4A_0A_1 - 36A_0A_3 - 36A_1A_2 + 2A_1 - 18A_3\\
&+40A_1A_4 - \mu 40A_1 + 40A_2A_3 + \mu 576A_3 + 40B_1B_2 - 76B_1B_4 - 76B_2B_3 + 40B_3B_4,\\
\\
&\cos^3(\xi)\sin(\xi): -7452B_2c^2 + 63091B_4c^2 + 12A_0B_2 - 50A_0B_4 + \mu 1061B_4 + 6B_2 - 25B_4\\
&+12A_1B_1 - 50A_1B_3 - 50A_2B_2 + 40A_2B_4 - 50A_3B_1 + 40A_3B_3 + 40A_4B_2 - \mu 180B_2,\\
\\
&\cos^2(\xi): 4112A_2c^2 - 24684A_4c^2 - 16A_0A_2 + 24A_0A_4 - 8A_1^2 + 24A_1A_3 + 12A_2^2\\
&+\mu 136A_2 - \mu 480A_4 + 8B_1^2 - 40B_1B_3 - 20B_2^2 - 24B_2B_4 + 12B_3^2 - 8A_2 + 12A_4,\\
\\
&\cos^2(\xi)\sin(\xi): -692B_1c^2 + 12283B_3c^2 + 4A_0B_1 - 26A_0B_3 - 26A_1B_2 - 13B_3\\
&+24A_1B_4 - 26A_2B_1 + 24A_2B_3 + 24A_3B_2 + 24A_4B_1 - \mu 28B_1 + \mu 317B_3 + 2B_1,\\
\\
&\cos(\xi): 290A_1c^2 - 3822A_3c^2 - 4A_0A_1 + 12A_0A_3 + 12A_1A_2\\
&+\mu 16A_1 - \mu 120A_3 - 16B_1B_2 + 12B_1B_4 + 12B_2B_3 - 2A_1 + 6A_3,\\
\\
&\cos(\xi)\sin(\xi): 1451B_2c^2 - 7452B_4c^2 - 10A_0B_2 + 12A_0B_4 - 10A_1B_1 + 12A_1B_3\\
&+12A_2B_2 + 12A_3B_1 + \mu 61B_2 - \mu 180B_4 - 5B_2 + 6B_4,\\
\\
&\sin(\xi): 67B_1c^2 - 692B_3c^2 - 2A_0B_1 + 4A_0B_3 + 4A_1B_2\\
&+4A_2B_1 + \mu 5B_1 - \mu 28B_3 - B_1 + 2B_3,\\
\\
&constant: -290A_2c^2 + 984A_4c^2 + 4A_0A_2 + 2A_1^2 - \mu 16A_2\\
&+\mu 24A_4 - 2B_1^2 + 4B_1B_3 + 2B_2^2 + 2A_2.
\end{aligned}
,\qquad (3.4)
$$

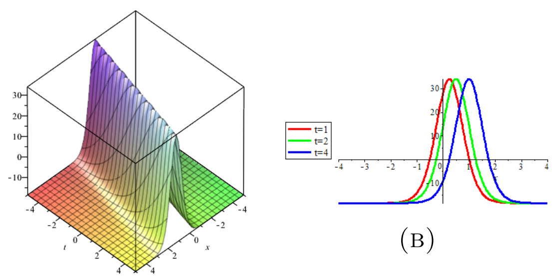
We solve the set of nonlinear algebraic equations with the help of Maple, the solutions of these algebraic equations are found to be in the following.

Case 1:

$$A_0 = \frac{1105}{2}c^2 - \frac{1}{2}, \quad A_1 = 0, \quad A_2 = -1680c^2, \quad A_3 = 0, \quad A_4 = 840c^2,$$
$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0, \quad B_4 = 0, \quad \mu = 51c^2, \quad c = \frac{1}{4}. \tag{3.5}$$

If we replace these results into (3.3) and insert the result into the transformation (2.2), we acquire the exact solitary wave solution of equation:

$$u_1(x,t) = -\frac{1}{2}\frac{575c^2cosh^4(x-ct) + cosh^4(x-ct) - 1680c^2}{cosh^4(x-ct)}. \tag{3.6}$$



(A) (B)

FIGURE 1. Figure 1 shows that the 3D and 2D soliton solution (3.6) for $c = \frac{1}{4}$, $-5 < x < 5$, $-5 < t < 5$.

Case 2:

$$\begin{aligned} &A_0 = -\frac{1247}{2}c^2 - 1368c^2(-\tfrac{1}{2} + \tfrac{1}{62}I\sqrt{31}) - \tfrac{1}{2}, \\ &A_1 = 0, \quad A_2 = 1680c^2(-\tfrac{1}{2} + \tfrac{1}{62}I\sqrt{31}), \quad A_3 = 0, \\ &A_4 = 840c^2, \quad B_1 = 0, \quad B_2 = 0, \quad B_3 = 0, \\ &B_4 = 0, \quad \mu = -105c^2 - 156c^2(-\tfrac{1}{2} + \tfrac{1}{62}I\sqrt{31}). \end{aligned} \tag{3.7}$$

If we replace these results into (3.3), we get the following solution:

$$u_2(x,t) = \frac{1}{62}I\left(\frac{312\sqrt{31}c^2cosh^4(x-ct) - 1680\sqrt{31}c^2cosh^2(x-ct) + 375c^2cosh^4(x-ct)}{cosh^4(x-ct)}\right)$$
$$+(\frac{52080c^2 - c^2 52080\cos h^2(x-ct) - 31cosh^4(x-ct)}{cosh^4(x-ct)}). \tag{3.8}$$

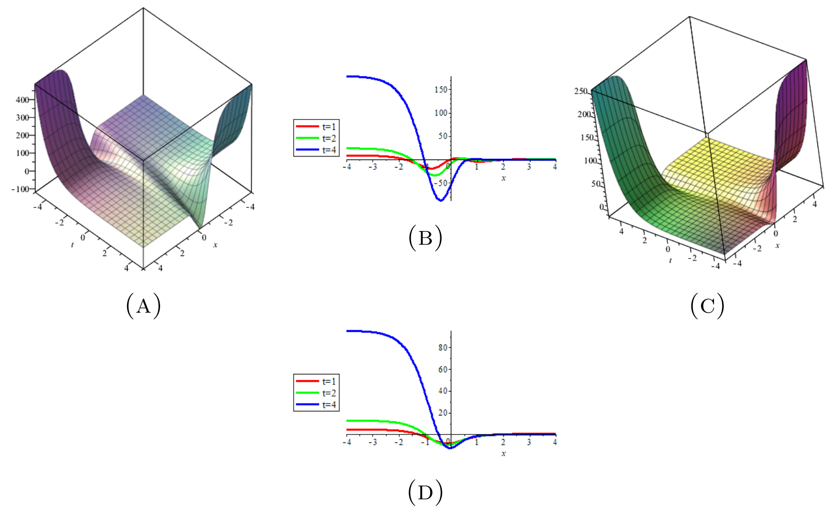


FIGURE 2. Figure 2 The 3D and 2D surfaces of solution (3.8) with respectively real part and imaginar part .

Case 3:

$$A_0 = \frac{385}{2}c^2 - \frac{1}{2}, \quad A_1 = 0, \quad A_2 = -630c^2, \quad A_3 = 0, \quad A_4 = 420c^2,$$
$$B_2 = -420Ic^2, \quad B_3 = 0, \quad B_4 = 420Ic^2, \quad \mu = 12c^2, \quad B_1 = 0. \tag{3.9}$$

Similar to the previous case, we get the following solitary wave solution:

$$u_3(x,t) = -\frac{1}{2}\left(\frac{35c^2cosh^4(x-ct) + cosh^4(x-ct)}{cosh^4(x-ct)}\right) \tag{3.10}$$

$$+\frac{420cosh^2(x-ct)c^2 - 840c^2}{cosh^4(x-ct)} + I\left(\frac{840c^2sinh(x-ct)}{cosh^4(x-ct)}\right) \tag{3.11}$$

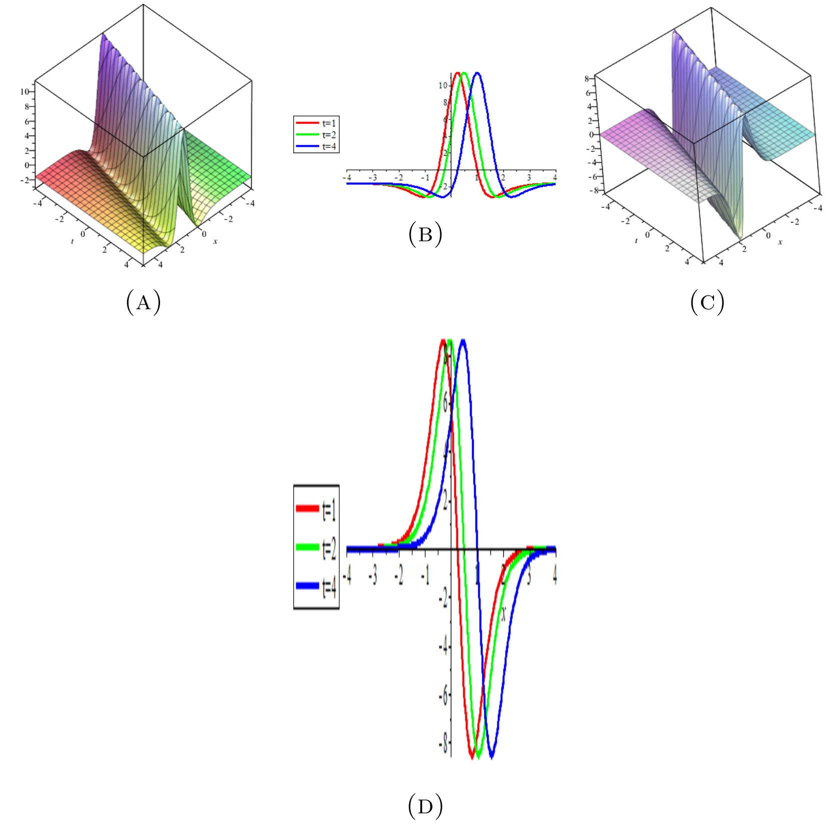
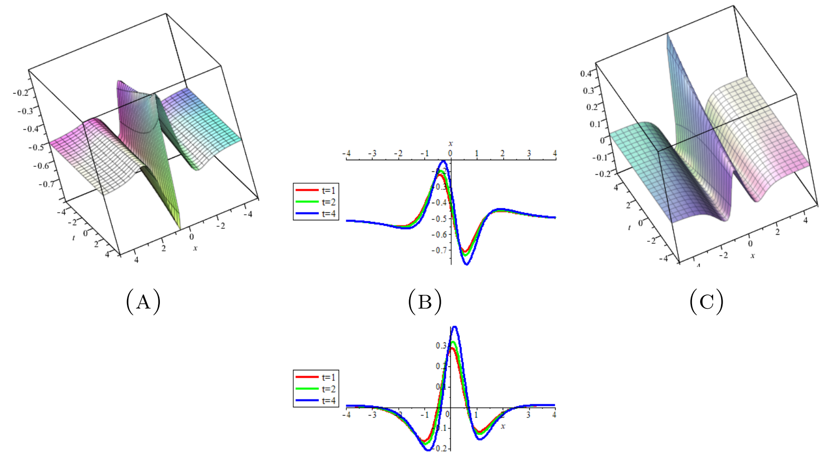


FIGURE 3. The 3D and 2D surfaces of solution (3.1) with respectively real part and imaginar part.

Case 4:

$$\begin{aligned}
&A_0 = -\tfrac{381}{4}(-\tfrac{3}{4} + \tfrac{1}{124}I\sqrt{31})\sqrt{-\tfrac{1}{11025}B_4^2} - \tfrac{353}{8}\sqrt{-\tfrac{1}{11025}B_4^2} - \tfrac{1}{2},\\
&A_1 = 0, \quad A_3 = 0, \quad B_1 = 0,\\
&A_2 = \tfrac{105}{2}(-\tfrac{5}{2} + \tfrac{1}{62}I\sqrt{31})\sqrt{-\tfrac{1}{11025}B_4^2},\\
&A_4 = 105\sqrt{-\tfrac{1}{11025}B_4^2}, \quad B_2 = B_4(-\tfrac{3}{4} + \tfrac{1}{124}I\sqrt{31}),\\
&B_3 = 0, \quad c = \tfrac{1}{2}(-\tfrac{1}{11025}B_4^2)^{\frac{1}{4}},\\
&\mu = -\tfrac{3}{2}\sqrt{-\tfrac{1}{11025}B_4^2(\tfrac{5}{4} + \tfrac{13}{124}I\sqrt{31})}, \ B_4 = 1.
\end{aligned} \tag{3.12}$$



(A) (B) (C)

(D)

FIGURE 4. The 3D and 2D surfaces of solution (3.13) with respectively real part and imaginar part.

Similar to the previous case, we get the following solitary wave solution:

$$u_4(\xi) = \left(\frac{1}{52080}\frac{39I\sqrt{-B_4^2}\sqrt{31}cosh^4(\xi) + 420IB_4sinh(\xi)\sqrt{31}cosh^2(\xi) + 527\sqrt{-B_4^2}cosh^4(\xi)}{cosh^4(\xi)}\right)$$

$$-\left(\frac{420I\sqrt{-31B_4^2}cosh^2(\xi)}{cosh^4(\xi)}\right)$$

$$+\left(\frac{1}{52080}\frac{13020B_4sinh(\xi)cosh^2(\xi) - 26040cosh^4(\xi) - 39060cosh^2(\xi)\sqrt{B_4^2}}{cosh^4(\xi)}\right)$$

$$-\left(\frac{52080B_4sinh(\xi)+52080\sqrt{B_4^2}}{cosh^4(\xi)}\right) \tag{3.13}$$

3.2. **Kuramoto-Sivashinsky Equation.** Secondly, we consider Kuramoto-Sivashinsky equation:

$$u_t + \alpha u u_x + b u_{xx} + k u_{xxxx} = 0, \tag{3.14}$$

In this subsection we are going to sustain the exact traveling wave solutions of the Kuramoto-Sivashinsky equation by using of the sine-Gordon method.

We look for the solutions of the Kuramoto-Sivashinsky equation by the method described in past section.We operate the travelling wave transformation of the following form,

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{3.15}$$

Eq.(3.14) is reduced into the following ordinary differential equations (ODE)

$$-cu' + \alpha u u' + b u'' + k u'''' = 0, \tag{3.16}$$

where prime denotes differentiation with respect to ξ. If we balance between the highest order nonlinear derivative u'''' and nonlinear term of the highest degree uu' in Eq.(3.16), we obtain $n = 3$.

Hence, the expected solution takes the form

$$\begin{aligned} u(\xi) &= A_0 + B_1 sech(\xi) + B_2 sech(\xi) tanh(\xi) + B_3 sech(\xi) tanh^2(\xi) \quad (3.17)\\ &\quad + A_1 tanh(\xi) + A_2 tanh^2(\xi) + A_3 tanh^3(\xi), \quad (3.18) \end{aligned}$$

where $u = u(\xi)$ satisfies Eq.(2.5) in which $A_0, A_1, A_2, A_3, B_1, B_2, B_3$ are undetermined constants.

We substitute Eq.(3.17) into Eq.(3.16) secondly utilize Eq.(2.5) finally we equate all coefficients of the functions $[\cos(\xi), \sin(\xi)]$ to zero and we get:

$\cos^7(\xi) : -3A_3^2 a + 3B_2^3 a + 360 A_3 k,$

$\cos^6(\xi) : -5A_2 A_3 a + 5B_2 B_3 a + 120 A_2 k,$

$\cos^6(\xi)\sin(\xi) : -6A_3 B_3 a + 360 B_3 k,$

$\cos^5(\xi) : -4A_1 A_3 a - 2A_2^2 a + 3A_3^2 a$
$+4B_1 B_3 a + 2B_2^2 a - 5B_3^2 a + 24 A_1 k + 12 A_3 b - 816 A_3 k,$

$$\cos^5(\xi)\sin(\xi) : -5A_2B_3a - 5A_3B_2a + 120B_2k,$$

$$\cos^4(\xi) : -3A_0A_3a - 3A_1A_2a + 5A_2A_3a + 3B_1B_2a - 8B_2B_3a + 6A_2b - 240A_2k + 3A_3c,$$

$$\cos^4(\xi)\sin(\xi) : -4A_1B_3a - 4A_2B_2a - 4A_3B_1a + 5A_3B_3a$$
$$+24B_1k + 12B_3b - 648B_3k,$$

$$\cos^3(\xi) : -2A_0A_2a - A_1^2a + 4A_1A_3a + 2A_2^2a + B_1^2a - 6B_1B_3a$$
$$-3B_2^2a + 2B_3^2a + 2A_1b - 40A_1k + 2A_2c - 18A_3b + 576A_3k,$$

$$\cos^3(\xi)\sin(\xi) : -3A_0B_3a - 3A_1B_2a - 3A_2B_1a + 4A_2B_3a$$
$$+4A_3B_2a + 6B_2b - 180B_2k + 3B_3c,$$

$$\cos^2(\xi) : -A_0A_1a + 3A_0A_3a + 3A_1A_2a - 4B_1B_2a + 3B_2B_3a + A_1c$$
$$-8A_2b + 136A_2k - 3A_3c,$$

$$\cos^2(\xi)\sin(\xi) : -2A_0B_2a - 2A_1B_1a + 3A_1B_3a$$
$$+3A_2B_2a + 3A_3B_1a + 2B_1b - 28B_1k + 2B_2c$$
$$-13B_3b + 317B_3k,$$

$$\cos(\xi) : 2A_0A_2a + A_1^2a - B_1^2a + 2B_1B_3a + B_2^2a - 2A_1b$$
$$+16A_1k - 2A_2c + 6A_3b - 120A_3k,$$

$$\cos(\xi)\sin(\xi) : -A_0B_1a + 2A_0B_3a + 2A_1B_2a + 2A_2B_1a + B_1c$$
$$-5B_2b + 61B_2k - 2B_3c,$$

$$\sin(\xi) : A_0B_2a + A_1B_1a - B_1b + 5B_1k - B_2c + 2B_3b - 28B_3k,$$

$$constant : \ A_0A_1a + B_1B_2a - A_1c + 2A_2b - 16A_2k. \tag{3.19}$$

We solved the set of nonlinear algebraic equations with the help of Maple, the solutions of these algebraic equations are found to be in the following.

Set 1:

$$A_0 = A_0, A_1 = -\frac{9}{11}A_3, A_2 = 0,, \tag{3.20}$$

$$B_1 = 0,\ B_2 = 0,\ B_3 = 0, \tag{3.21}$$

$$b = \frac{19}{330}A_3 a, \quad c = A_0 a, \tag{3.22}$$

$$k = \frac{1}{120}A_3 a. \tag{3.23}$$

Set 2:

$$A_1 = IB_1, \quad A_2 = 0, \quad A_3 = 0, \tag{3.24}$$

$$B_2 = 0, \quad B_3 = 0, \quad b = IaB_1, \tag{3.25}$$

$$c = A_0 a, \quad k = 0. \tag{3.26}$$

Set 3:

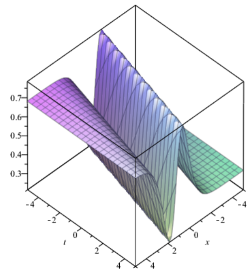
$$\begin{array}{l} A_1 = -\frac{3}{2}IB_3, \quad A_2 = 0, \quad A_3 = IB_3, \\ B_1 = -B_3, \quad B_2 = 0, \\ b = -\frac{19}{60}IaB_3, \quad c = A_0 a, \\ k = \frac{1}{60}IaB_3. \end{array} \tag{3.27}$$

Set 4:

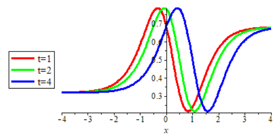
$$\begin{array}{l} A_1 = -\frac{21}{22}IB_3, \quad A_2 = 0, \quad A_3 = IB_3, \\ B_1 = -\frac{5}{11}B_3, \quad B_2 = 0, \\ b = \frac{19}{660}IaB_3, \quad c = A_0 a, \\ k = \frac{1}{60}IaB. \end{array} \tag{3.28}$$

If we substitute these results into (3.3) we attain the exact travelling wave solutions of equation:

$$u_1(x,t) = A_0 - \frac{9}{11}A_3 tanh(x - A_0 at) + A_3 tanh^3(x - A_0 at). \qquad (3.29)$$



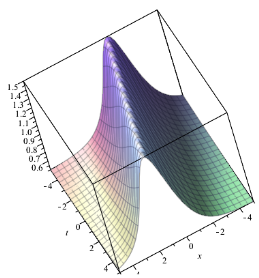
(A) Caption1



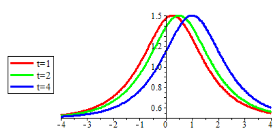
(B) Caption2

FIGURE 5. 3D and 2D soliton solution (3.29) for $A_0 = 1/2,\ a = 1/2,\ A_3 = 1$.

$$u_2(x,t) = \frac{IB_1 sinh(x - A_0 at) + A_0 cosh(x - A_0 at) + B_1}{cosh(x - A_0 at)}. \qquad (3.30)$$



(A)



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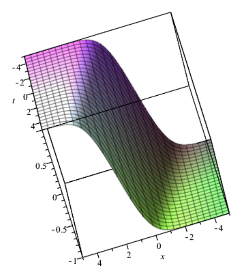
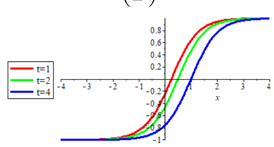


(C)



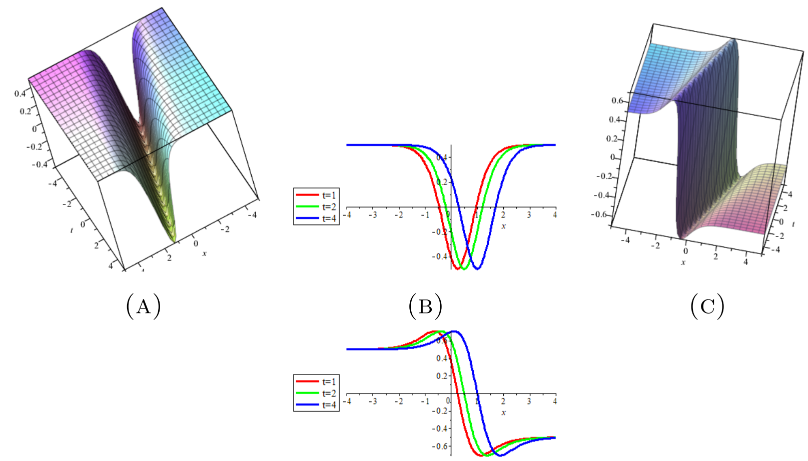
(D)

FIGURE 6. For $A_0 = 1/2,\ a = 1/2,\ B_1 = 1$ The 3D and 2D surfaces of solution (3.30) with respectively real part and imaginar part.

$$u_3(x,t) = \frac{1}{2}\left(\frac{2A_0cosh^3(x - A_0at) - -2B_3}{cosh^3(x - A_0at)}\right) \tag{3.31}$$

$$-I\frac{B_3sinh(x - A_0at)cosh^2(x - A_0at)}{cosh^3(x - A_0at)} \tag{3.32}$$

$$+I\frac{+2B_3sinh(x - A_0at)}{cosh^3(x - A_0at)} \tag{3.33}$$



(A) (B) (C)

(D)

FIGURE 7. For $A_0 = 1/2$, $a = 1/2$, $B_3 = 1$.The 3D and 2D surfaces of solution (3.33) with respectively real part and imanigar part.

$$u_4(x,t) = \frac{1}{22}\left(\frac{IB_3sinh(x - A_0at)cosh^2(x - A_0at) + 22A_0cosh^3(x - A_0at)}{cosh^3(x - A_0at)}\right)$$

$$+\frac{1}{22}\left(\frac{-22IB_3sinh(x - A_0at) + 12B_3cosh^2(x - A_0at) - 22B_3}{cosh^3(x - A_0at)}\right). \tag{3.34}$$

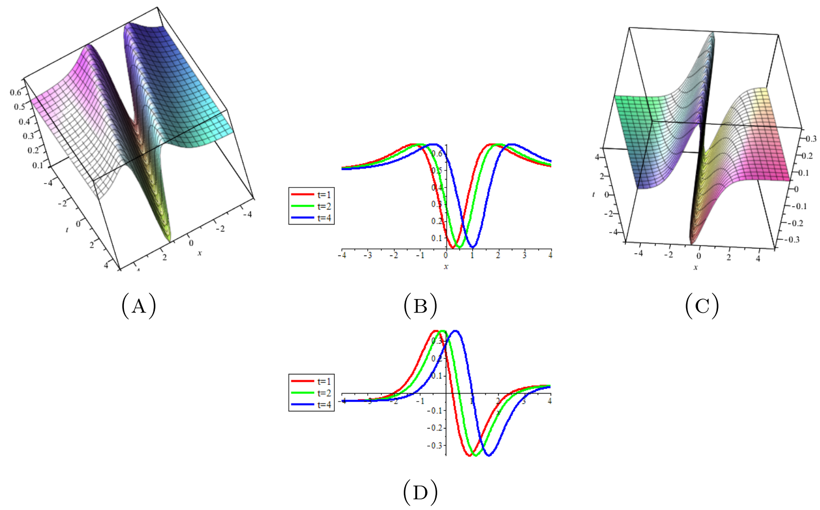


FIGURE 8. For $A_0 = 1/2,\ a = 1/2,\ B_3 = 1.$
The 3D and 2D surfaces of solution(3.34)
with respectively real part and imaginar part.

3.3. **Sawada-Kotera Equation.** As a last application of the related method, we consider Sawada-Kotera equation and is given as follows:

$$u_t+[63u^4+63(2u^2u_{xx}+uu_x^2)+21(uu_{xxxx}+u_{xx}^2+u_xu_{xxx})+u_{xxxxxx}]_x=0, \tag{3.35}$$

By substituting transformation $u(x,t)=u(\xi),\ \ \xi=x-ct$, Eq.(3.35) is transformed into an one dimension form

$$-cu'+252u^3u'+378uu''u'+126u^2u'''+63(u')^3+42u'u''''+21uu'''''+63u''u'''+u^{vii}=0, \tag{3.36}$$

where prime denotes differentiation with respect to ξ. The balance between u^{vii} and u^3u' gives $n=2$.

Thereby, the exact solution of the proposed equation has the form,

$$u(\xi)=A_0+B_1sech(\xi)+B_2sech(\xi)tanh(\xi)+A_1tanh(\xi)+A_2tanh^2(\xi), \tag{3.37}$$

where $u=u(\xi)$ satisfies Eq.(2.5) in which A_0, A_1, A_2, B_1, B_2 are undetermined constants.

As in other applications, the algebraic equation system construct and when it is solved, the following solution sets are obtained.

Set 1:

$$\begin{aligned} A_1 &= 0, \quad A_2 = -2, \quad B_1 = 0, && (3.38)\\ B_2 &= 0, \quad c = 252A_0^3 - 1008A_0^2 + 1344A_0 - 608. && (3.39) \end{aligned}$$

Set 2:

$$\begin{aligned} A_1 &= 0, \quad A_2 = -1, \quad B_1 = 0, && (3.40)\\ B_2 &= I, \quad c = 252A_0^3 - 630A_0^2 + 525A_0 - 146. && (3.41) \end{aligned}$$

Set 3:

$$\begin{aligned} A_0 &= \frac{8}{3}, \quad A_1 = 0, \quad A_2 = -4, && (3.42)\\ B_1 &= 0, \quad B_2 = 0, \quad c = -\frac{256}{3}. && (3.43) \end{aligned}$$

Set 4:

$$\begin{aligned} A_0 &= \frac{5}{3}, \quad A_1 = 0, \quad A_2 = -2, && (3.44)\\ B_1 &= 0, \quad B_2 = 2I, \quad c = -\frac{4}{3}. && (3.45) \end{aligned}$$

If we substitute these results into (3.3) we attain the exact travelling wave solutions of equation:

$$u_1(\xi) = \frac{A_0 cosh^2(\xi) - 2cosh^2(\xi) + 2}{cosh^2(\xi)}. \tag{3.46}$$

$$u_2(\xi) = \frac{Isinh(\xi) + A_0 cosh^2(\xi) + 1 - cosh^2(\xi)}{cosh^2(\xi)} \tag{3.47}$$

$$u_3(\xi) = -\frac{4}{3}\left(\frac{cosh^2(\xi) - 3}{cosh^2(\xi)}\right). \tag{3.48}$$

$$u_4(\xi) = \frac{1}{3}\left(\frac{6Isinh(\xi) - cosh^2(\xi) + 6}{cosh^2(\xi)}\right). \tag{3.49}$$

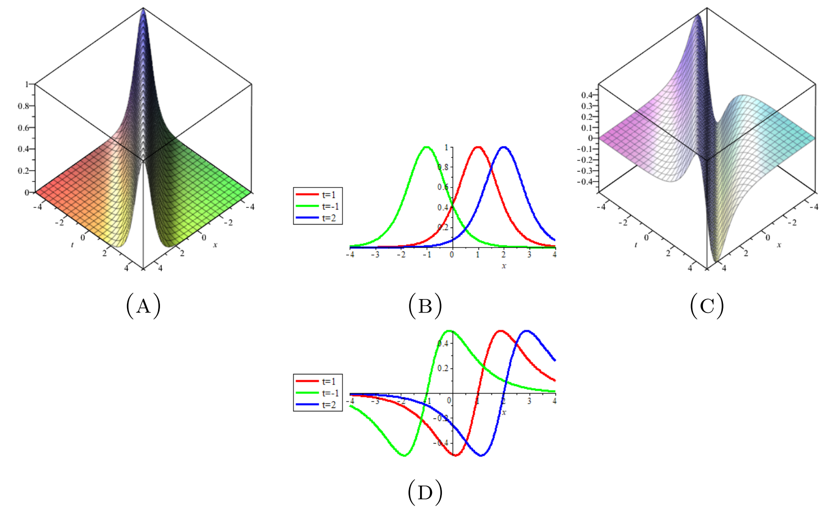


FIGURE 9. For $A_0 = 1$.
The 3D and 2D surfaces of solution(3.47)
with respectively real part and imaginar part.

4. Conclusion

In this article, we studied on acquiring the exact travelling wave solutions of (1+1)-dimensional nonlinear equations by introducing appropriate transformations and applying the sine-gordon expansion method. We employed this method to find the solutions of Higher-Order Boussinesq , Kuramoto-Sivashinsky and Sawada-Kotera Equations. We obtained the sets of nonlinear equations that can be solved by using Maple software.

The purpose of this study was constructed on reliable, beneficial and consistent treatment for the analysis of the above-mentioned equations. We can say that the sine-gordon expansion method is a very powerful and efficient mathematical tool to solve nonlinear equations. Furthermore, we believe that this method is also useful for a variety of other nonlinear equations that represent in mathematical, physics and other nonlinear sciences. The solutions acquired may be very beneficial in order to understand the mechanism of the intricate nonlinear physical phenomena of referred equations. The conclusions obtained will be useful for guiding future research works in the appropriate fields.

Declaration of Competing Interest

Authors have no interests to declare.

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